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ON THE PSEUDO-PROCESSES AND PSEUDO-DYNAMICAL SYSTEMS*

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Abstract. In this paper there are presented certain necessary and sufficient conditions of the extendability of the pseudo-processes in the sense of C. M. Dafermos and A. Pelczar. There are also described connections between the extendabilities of the pseudo-dynamical systems and of the pseudo-processes and connections between the pseudo-processes and the translation equation and the measurability and the continuity of the pseudo-processes.

Key words. Generalized dynamical systems, functional equations on the abstract structures.

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1. In accordance with Dafermos [1] and Pelczar [8], [9] let us introduce the following definitions:

Definition 1. A pseudo-dynamical system (A. Pelczar uses the term: pseudo-dynamical semi-system) is a triple (X, G, π), where X is a set, $(G, +)$ is an abelian semigroup having the neutral element 0, and the mapping $\pi\colon G \times X \to X$ such that

$$\pi(0, x) = x \qquad \text{for every } x \in X,$$
$$\pi(t, \pi(u, x)) = \pi(t + u, x) \qquad \text{for every } x \in X,\ t, u \in G.$$

The pseudo-dynamical system is called dynamical when $(G, +)$ is a topological semigroup, X is a topological space and π is a continuous function.

Definition 2. A pseudo-process is a quadruple (Y, G, H, μ), where Y is a set, $(G, +)$ is an abelian semigroup having the neutıal element 0, H is a subsemigroup of G such that $0 \in H$ and the mapping $\mu\colon G \times Y \times H \to Y$ is such that

$$\mu(t, y, 0) = y \qquad \text{for } y \in Y,\ t \in G,$$
$$\mu(t, y, u + v) = \mu(t + u, \mu(t, y, u), v) \qquad \text{for } t \in G,\ u, v \in H,\ y \in Y.$$

The pseudo-process is called process when Y is a metric space having the metric ϱ and for every u from H, the family of the functions $\{\mu(t, ., u)\}_{t \in G}$ is equicontinuous, that means:

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$$\bigwedge_{u\in H}\ \bigwedge_{y_1\in Y}\ \bigwedge_{\varepsilon>0}\ \bigvee_{\delta>0}\ \bigwedge_{y_2\in Y}\ \bigwedge_{t\in G}\ [\varrho(y_1, y_2) < \delta \Rightarrow \varrho(\mu(t, y_2, u), \mu(t, y_1, u)) < \varepsilon].$$

The process is continuous (uniformly continuous) when G is a topological semigroup and for every y from Y and u from H, the function $\mu(., y, u)$ is continuous (uniformly continuous).

It is known from [8] that:

Proposition 1. *If (Y, G, H, μ) is a pseudo-process, then (X, H, π), where $X = G \times Y$ and $\pi(u, (t, y)) := (t + u, \mu(t, y, u))$ for $t \in G$, $u \in H$, $y \in Y$, is a pseudo-dynamical system.*

2. Let (Y, G, H, μ) be a pseudo-process, thus (X, H, π) is the pseudo-dynamical system as in Proposition 1. In this case, it is true the following:

Proposition 2. a) *For arbitrary element $u_0 \in H - \{0\}$ such that*

$$\bigwedge_{t_1, t_2 \in G} (t_1 + u_0 = t_2 + u_0 \Rightarrow t_1 = t_2),$$

the mapping $\pi(u_0, .)$ is an injection on the set X if and only if the mapping $\mu(t, ., u_0)$ is an injection on the set Y for every $t \in G$.

b) *For an arbitrary element u_0 from $H - \{0\}$ being inverseable in G, the mapping $\pi(u_0, .)$ is a surjection of the set X on the set X if and only if the mapping $\mu(t, ., u_0)$ is a surjection of the set Y on the set Y for every $t \in G$.*

Proof. *Let* $\pi(u_0, .)$ be an injection on the set X. Assume $\mu(t, y_1, u_0) = \mu(t, y_2, u_0)$ for any $t \in G$. Hence $\pi(u_0, (t, y_1)) = \pi(u_0, (t, y_2))$, so $y_1 = y_2$. Now let us take that $\pi(u_0, (t_1, y_1)) = \pi(u_0, (t_2, y_2))$. Then $(t_1 + u_0, \mu(t_1, y_1, u_0)) = (t_2 + u_0, \mu(t_2, y_2, u_0))$ so $t_1 = t_2$, and $\mu(t_1, y_1, u_0) = \mu(t_2, y_2, u_0)$. Hence $\mu(t_2, y_1, u_0) = \mu(t_2, y_2, u_0)$, because $\mu(t_2, ., u_0)$ is the injection on the set Y, so $(t_1, y_1) = (t_2, y_2)$.

Let is assume at first for the proof of the condition b), that $\mu(t, ., u_0)$ is a surjection with every $t \in G$ and let $(t_0, y) \in X$. For $(t_0 - u_0, u_0) \in G \times H$ there exists such $y_1 \in Y$ that $\mu(t_0 - u_0, y_1, u_0) = y$. Hence, for a pair $(t_0, y) \in X$ there exists such a pair $(t_0 - u_0, y_1) \in X$ that $\pi(u_0, (t_0 - u_0, y_1)) = (t_0 - u_0 + u_0, \mu(t_0 - u_0, y_1, u_0)) = (t_0, y)$, so $\pi(u_0, .)$ is on surjection of the set X onto the set X. Now, let t be any element of the set G and $\pi(u_0, .)$ is a surjection of the set X onto the set X. For $(t + u_0, y) \in X$ there exists such $(t_1, y_1) \in X$ that $\pi(u_0, (t_1, y_1)) = (t + u_0, y)$ so $(t_1 + u_0, \mu(t_1, y_1, u_0)) = (t + u_0, y)$. Then $t_1 = t$ and $\mu(t_1, y_1, u_0) = y$, hence $\mu(t, y_1, u_0) = y$, which proofs that the mapping $\mu(t, ., u_0)$ is the surjection.

Now, let us refer to the remark in the paper [4], being always quoted by us in the terminology of the pseudo-dynamical systems: If H is a subsemigroup of non-negative elements of Archimedean group, then the condition:

– there exist such $u_0, u_1 \in H - \{0\}$ that $\pi(u_0, .)$ is the injection on the set $\pi(0, X)$

and $\pi(u_1, .)$ is a surjection on the set $\pi(0, X)$, is sufficient (and of course it is necessary) for the condition,

– for every $u \in H$ the function $\pi(u, .)$ will be a bijection of the set $\pi(0, X)$.

This remark can diminish the sufficient condition of the extendability of the pseudo-process which was formulated in the paper [2], in case when G is Archimedean group. First of all let us prove:

Proposition 3. *Let (Y, G, H, μ) be a pseudo-process in which H is a subsemigroup of non-negative elements of the Archimedean group G. Then the condition*

(1) *there exists such $u_1, u_2 \in H - \{0\}$ that for every $t \in G$ the mapping $\mu(t, ., u_1)$ is a surjection onto the set Y and the mapping $\mu(t, ., u_2)$ is an injection on the set Y, is sufficient (and of course it is necessary) in order to,*

(2) *for every $(t, u) \in G \times H$ the mapping $\mu(t, ., u)$ will be a bijection of the set Y.*

Proof. Let us consider the pseudo-dynamical system (X, H, π) given by the pseudo-process (Y, G, H, μ) according to Proposition 1. If μ satisfies the condition (1), on the ground of Proposition 2 there exist such $u_1, u_2 \in H - \{0\}$ that $\pi(u_1, .)$ is the surjection onto the set X and $\pi(u_2, .)$ is the injection on the set X. Moreover, $\pi(0, X) = X$, so the assumption of the above remark is satisfied. So, we obtain: for every $u \in H$ the mapping $\pi(u, .)$ is the bijection of the set X. Refering to Proposition 2 again we can obtain the assertion.

In Proposition 3 the condition (1) cannot be replaced by the following one:

– there exist such $(t_1, u_1), (t_2, u_2) \in G \times (H - \{0\})$ that $\mu(t_1, ., u_1)$ is the surjection onto the set Y and $\mu(t_2, ., u_2)$ is the injection on the set Y.

It is illustrated by **example 1.**

Let G be the additive group of real numbers R and H be the subsemigroup of its non-negative elements R^+. Moreover, let $Y = \langle 1, \infty)$. The mapping μ: $R \times \langle 1, \infty) \times R^+ \to \langle 1, \infty)$ is defined like this:

$$\mu(t, y, u) := y \,.\, 2^{[t+u] - [t]},$$

where $[x]$ is the entire of the number x.

The system $(\langle 1, \infty), R, R^+, \mu)$ is the pseudo-process, because

a) $[t + u] - [t] \geqq 0$, so $\mu(t, y, u) \geqq 1$,

b) $\mu(t, y, 0) = y \,.\, 2^{[t]-[t]} = y$,

c) $\mu(t + u, \mu(t, y, u), v) = \mu(t + u, y \,.\, 2^{[t+u]-[t]}, v) = y \,.\, 2^{[t+u]-[t]} 2^{[t+u+v]-[t+u]} = y \,.\, 2^{[t+u+v]-[t]} = \mu(t, y, u + v)$.

Besides, $\mu\left(1, y, \frac{1}{2}\right) = y \,.\, 2^{[1 + 1/2] - [1]} = y$, so $\mu\left(1, ., \frac{1}{2}\right)$ is the bijection of the set Y, but $\mu(t, ., u)$ is not the bijection of the set $\langle 1, \infty)$ with every $(t, u) \in G \times H$, because $\mu\left(\frac{3}{2}, \langle 1, \infty), \frac{1}{2}\right) = \langle 2, \infty)$.

The theorem 4 from the paper [2] and the Proposition 3 may formulate the following sufficient and necessary condition of the extendability of the pseudo-process.

Theorem 1. *If H is subsemigroup of non-negative elements of Archimedean group G then the pseudo-process (Y, G, H, μ) can be extended onto the set $G \times G$ if and only if the condition* (1) *is fulfilled.*

The assumption that G is Archimedean group, is important in the Proposition 3 and in Theorem 1. It can be showed by

Example 2. Let us consider the ordered group $(Z \times Z;\ +;\ \leqq)$, where Z is the set of integral numbers and $(a, b) + (c, d) := (a + c, b + d)$,

$$(a, b) \leqq (c, d) :\Leftrightarrow (a < c) \vee (a = c \wedge b \leqq d).$$

It is not Archimedean group because, the set $\{0\} \times Z$ is its non-trivial convex subgroup of $Z \times Z$.

The set $H := (Z^+ - \{0\}) \times Z \cup \{0\} \times Z^+$, where $Z^+ := Z \cap R^+$, is a subsemigroup of its non-negative elements. The mapping $\mu : Z \times Z \times \langle 1, \infty) \times H \to \langle 1, \infty)$ is defined like this:

$$\mu((a, b), y, (p, q)) := y \cdot 2^p.$$

The system $(\langle 1, \infty), Z \times Z, H, \mu)$ is the pseudo-process because:

$$\begin{aligned} &\mu((a, b) + (p, q), \mu((a, b), y, (p, q)), (r, s)) = \\ &= \mu((a, b) + (p, q), y \cdot 2^p, (r, s)) = y \cdot 2^{p+r} = \\ &= \mu((a, b), y, (p, q) + (r, s)); \mu((a, b), y, (0, 0)) = y \cdot 2^0 = y. \end{aligned}$$

For elements $((a, b), (0, q)) \in (Z \times Z) \times (\{0\} \times Z^+)$ the mapping $\mu((a, b), \cdot, (0, q))$ is the bijection of the set $\langle 1, \infty)$, but it does not satisfy (2). Hence, on the ground of Theorem 4 from the paper [2], the pseudo-process $(<1, \infty), Z \times Z, H, \mu)$ is not extendable.

The simple modification of the above example allows to give an example, which proves, that in Proposition 3 and in Theorem 1 the assumption:

- H is a subsemigroup of **non-negative** elements of Archimedean group G cannot be replaced by the following one,
- H is such a subsemigroup of the Archimedean group G, that

$$H \cup (-H) = G \qquad \text{and} \qquad H \cap (-H) = \{0\}.$$

Hence it follows, that this assumption is essential in the Proposition 3.

Example 3. First let us notice, that the additive group R of real numbers is isomorphic with the group $(R \times R; +)$, where $(a, b) + (c, d) := (a + c, b + d)$. It is a consequences of the fact, that both the vector spaces $R(Q)$ and $R^2(Q)$, over the field Q of the rational numbers, have the same dimension equal to c. Let us denote this isomorphism from R onto $R \times R$ by $I = (I_1, I_2)$ and put $E :=$

$:= (R^+ \times R) - \{(0, b) \in R \times R : b > 0\}$, $H := I^{-1}(E)$, $Y = \langle 1, \infty)$ and $\mu(t, y, u) :=$ $:= y \,.\, 2^{I_1(u)}$. We can easy acertain, that the set E is such a subsemigroup of the group $R \times R$, that $E \cup (-E) = R \times R$ and $E \cap (-E) = \{(0, 0)\}$. Hence H is a subsemigroup of R such, that $H \cup (-H) = R$ and $H \cap (-H) = \{0\}$. Moreover, (Y, R, H, μ) is a pseudo-process. For every u from the set $I^{-1}(\{(0, b) \in R \times R: b \leqq 0\})$ we have $I_1(u) = 0$, thus $\mu(t, y, u) = y$ for every t from R. Consequently $\mu(t, ., u)$ is a bijection of the set Y for every u from $I^{-1}(\{(0, b) \in R \times R: b \leqq 0\})$ and t from R. However, it is not true, that for every (t, u) from $R \times H$, $\mu(t, ., u)$ is a bijection of the set Y, because for such a u, that $I_1(u) > 0$, we have $\mu(t, y, u) \leqq 2^{I_1(u)} > 1$. Hence, on the ground of Theorem 4 from the paper [2], the pseudo-process (Y, R, H, μ) is not extendable.

A. Mach in the paper [3] proves the theorem, which in terminology of dynamical systems has the form:

– If in the pseudo-dynamical system (X, H, π) the set H is a subsemigroup with zero of the abelian group $(G; +)$ and B is the subset of the set H with the property:

$$\bigwedge_{u \in H} \left(\bigvee_{\substack{r \in N \\ u_1, u_2, \ldots, u_r \in B \\ n_1, n_2, \ldots, n_r \in N}} \right) (-u + n_1 u_1 + n_2 u_2 + \ldots + n_r u_r \in H) \tag{3}$$

then, from

$$\bigwedge_{u \in B} \pi(u, .) \quad \text{is the bijection of the set } X$$

it follows

$$\bigwedge_{u \in H} \pi(u, .) \quad \text{is the bijection of the set } X.$$

The subset $B \subset H$ having the property (3) always exists because it is sufficient to assume that $B = H$. If H also satisfies the conditions $H \cup (-H) = G$ and $H \cap$ $\cap (-H) = \{0\}$ then:

– the subset B possessing the property (3) and being such the least in this sense that every its subset $B_1 \subsetneqq B$ does not possess this property (3), either exists and is one-element set or it does not exist.

The above theorem and the theorem 4 from the paper [2], on the ground of Proposition 1 make following

Theorem 2. *The pseudo-process (Y, G, H, μ), where H is subsemigroup of the abelian group G satisfying the condition $H \cup (-H) = G$, can be extended onto $G \times G$ if and only if:*

– *for every $(t, u) \in G \times B$, the mapping $\mu(t, ., u)$ is the bijection of the set Y, where B is any subset H possessing the property* (3). *If $H \cap (-H) = \{0\}$ additionally, then the least subset B possessing the property* (3), *when it exists, then it is the one-element set. If, moreover, H is a subsemigroup of non-negative elements of the linear-ordered abelian group G, then B possessing the property* (3) *is such a subset that the least convex subgroup converting the set B is G* (see [3], remark 3.5).

3. Now let us give *an example of the pseudo-dynamical system, which cannot be extended from the subgroup on the group and it can be extended if it is considered as the pseudo-process.* Let G be the multiplicative group of positive real numbers $R^+ - \{0\}$ and H be the subgroup of positive rational numbers $Q^+ - \{0\}$. Moreover, put $Y = H = Q^+ - \{0\}$ and $\pi(u, y) = u \,.\, y$ for u from H and y from Y. Then G with account of simple inequality is Archimedean group, and (Y, H, π) is the pseudo-dynamical system. It is not extendable to the system $(Y, G, \bar{\pi})$, because the necessary condition of the extendability, which was formulated in the paper [5], is not fulfilled. In this case the condition of the extendability has the form:

– it exists such subgroup G^* of the group G, that

$$\{1\} = H \cap G^* \qquad \text{and} \qquad G = H \,.\, G^*.$$

If such subgroup G^* would exist, then $\sqrt{2} = w \,.\, a$ for certains: w from H and a from G^*. Hence $a \neq 1$ and $1 \neq a^2 = \dfrac{2}{w^2} \in H \cap G^*$, what is contradictory with $\{1\} = H \cap G^*$.

Putting $\mu(t, y, u) := \pi(u, y) = u \,.\, y$, we obtain the pseudo-process (Y, G, H, μ). It can be extended to the pseudo-process $(Y, G, G, \bar{\mu})$, what we prove on the ground of the theorem 3 from the paper [2]:

– the pseudo-process (Y, G, H, μ), where $(G; +)$ is an abelian group, can be extended on the set $G \times Y \times G$ if and only if there exists a decomposition R of the set $Y \times G$ such that for every $s \in G$, the set of the cosets $h_s^{-1}(\{\bar{y}\})$ for $\bar{y}$ from Y of every one of the functions

$$h_s \colon Y \times \{t \in G \colon \bigvee_{u \in H} t + u = s\} \xrightarrow[\text{onto}]{} Y, \qquad \text{for which}$$

$$\mu(t, y, u) = \} h_{t+u}[h_t^{-1}(\{y\})] \{ \qquad \text{for } (t, y, u) \in G \times Y \times H,$$

is a selective decomposition for the decomposition R, i.e.

$$\bigwedge_{E \in R} \bigwedge_{s \in G} \bigvee_{\bar{y} \in Y}^{1} h_s^{-1}(\{\bar{y}\}) \subset E.$$

We have $\mu(t, y, u) = \} h_{t.u}(h_t^{-1}(\{y\})) \{$, where $h_s(y, t) = \dfrac{ys}{t}$ for s from G, y from Y and t from $G_s = \{t \in G \colon \bigvee_{u \in H} t \,.\, u = s\}$. The set $h_s^{-1}(\{\bar{y}\})$, for $\bar{y}$ from Y, is contained in the pseudo-ray $\left\{(y, t) \in Y \times G_s \colon t = \dfrac{s}{\bar{y}} y\right\}$. A decomposition R of the set $Y \times G$ such that the set of the cosets of the function h_s for every $s \in G$ is a selective decomposition for R, can be defined as follows. Let us denote by S any selector of the quotient group G/H. For every w from H, a component $S(w)$ of the decomposition R will be the sum $\bigcup_{m \in S.w} P_m(w)$ of the pseudo-rays $P_m(w) = \{(y, t) \in Y \times G\colon$

$t = m \,.\, y\}$ for $m \in S \,.\, w$. The unique set from family of the cosets of the function h_s, which is contained in the component $S(w)$, will be the set

$$h_s^{-1}\left(\left\{\frac{s}{\bar{s} \,.\, w}\right\}\right) = \left\{(y, t) \in Y \times G_s ; \frac{s \,.\, y}{t} = \frac{s}{\bar{s} \,.\, w}\right\} =$$
$$= \{(y, t) \in Y \times G_s : t = \bar{s} \,.\, w \,.\, y\},$$

where $\bar{s}$ is the unique element with S, such that $\frac{s}{\bar{s}} \in H$.

If $H \cup (-H) = G$, then the system (Y, H, π) is extendable iff the system is extendable as pseudo-process, it is because an existing of any one of the extendabilities is equivalent to bijectivity of $\pi(u, .)$ on Y for all u from H ([2] and [4]).

4. M. C. Zdun in [10] has prooved the theorem, which in the terminology of the theory of dynamical systems, has the following form:

- If in the pseudo-dynamical system (X, R^+, π) the set X is a compact metric space, in the additive semigroup of non-negative real numbers R^+ there is the simple topology, $\pi(., y) : R^+ \to X$ is the measurable function for every y from X, that is means, that a contrimage of any open set of X by the function $\pi(., y)$ is measurable by Lebesgue's in R^+, $\pi(u, .) \colon X \to X$ is the continuous function for every u from R^+, then $\pi \colon R^+ \times X \to X$ is the continuous function, that it means that (X, R^+, π) is the dynamical system.

The above theorem does not directly lead to the pseudo-process, which can be proved by

Example 4. Let us assume $(G; +) = (R, +)$ and $H = R^+$ with simple topologies, and put $Y = \{0, 1\}$ with a simple metric and

$$\mu(t, y, u) := \begin{cases} y, & \text{when } t, t + u \in Q \text{ or } t, t + u \in R - Q, \\ 1 - y, & \text{when } (t \in Q \text{ and } t + u \in R - Q) \text{ or } (t \in R - Q \text{ and } t + u \in Q), \end{cases}$$

where Q is the set of rational numbers.

(Y, R, R^+, μ) is the pseudo-process which can be prooved directly or on the ground of theorem 1 from the paper [2] because

$$\mu(t, y, u) = \} \, h_{t+u}(h_t^{-1}(\{y\})) \, \{ \qquad \text{for}$$

$$h_t(\tau, x) = \begin{cases} x, & \text{when } t \in Q, \\ 1 - x, & \text{when } t \in R - Q. \end{cases}$$

For every $u \in R^+$ we obtain, when $t \in R$:

$$h_t^{-1}(\{y\}) = (-\infty, t) \times \{y\} \subset (-\infty, t + u) \times \{y\} = \begin{cases} h_{t+u}^{-1}(\{1 - y\}), & \text{when } u \in R - Q, \\ h_{t+u}^{-1}(\{y\}), & \text{when } u \in Q \end{cases}$$

and when $t \in R - Q$:

$$h_t^{-1}(\{y\}) = (-\infty, t)\times\{1-y\} \subset (-\infty, t+u)\times\{1-y\} =$$
$$= \begin{cases} h_{t+u}^{-1}(\{y\}), & \text{when } t+u \in R - Q, \\ h_{t+u}^{-1}(\{1-y\}), & \text{when } t+u \in Q, \end{cases}$$

so the assumptions of the theorem 1 from the paper [2] are satisfied. Of course, the function $\mu(t, ., u)$ is continuous for every (t, u) from $R\times R^+$.

Because for $u \in R^+ \cap Q$ and $y \in \{0, 1\}$

$$\{t \in R\colon \mu(t, y, u) = y\} = R \quad \text{and} \quad \{t \in R\colon \mu(t, y, u) = 1 - y\} = \emptyset$$

and for $u \in R^+ - Q$ and $y \in \{0, 1\}$

$$\{t \in R\colon \mu(t, y, u) = y\} = (R-Q)-(-u + Q) \text{ and } \{t \in R\colon \mu(t, y, u) = 1 - y\} =$$
$$= Q \cup (-u + Q),$$

so $\mu(., y, u)$ is the measurable function on every $(y, u) \in \{0, 1\}\times R^+$. It is similar for $t \in Q$ and $y \in \{0, 1\}$ we obtain:

$$\{u \in R^+\colon \mu(t, y, u) = y\} = Q \cap R^+ \quad \text{and} \quad \{u \in R^+\colon \mu(t, y, u) = 1 - y\} =$$
$$= (R-Q) \cap R^+$$

and for $t \in R-Q$ and $y \in \{0, 1\}$.

$$\{u \in R^+\colon \mu(t, y, u) = y\} = [(R-Q)-(-t + Q)] \cap R^+ \quad \text{and}$$
$$\{u \in R^+\colon \mu(t, y, u) = 1 - y\} = [Q \cup (-t + Q)] \cap R^+$$

and each of these sets is measurable, so $\mu(t, y, .)$ is the measurable function for $(t, y) \in R\times\{0, 1\}$.

Moreover:

$$E_1 := \{(t, y) \in R\times R^+\colon \mu(t, y, u) = y\} =$$
$$= R\times(Q \cap R^+) \cup \{[(R-Q)\times(R-Q)] - \bigcup_{w\in Q} \{(t, w - t)\colon t \in R-Q\}\} \cap (R\times R^+).$$

$$E_2 := \{(t, y) \in R\times R^+\colon \mu(t, y, u) = 1 - y\} =$$
$$= [Q\times(R-Q) \cup \bigcup_{w\in Q} \{(t, w - t)\colon t \in R-Q\}] \cap (R\times R^+).$$

For $w \in Q$ the sets $\{(t, w - t)\colon t \in R-Q\}$ are adequatly contained in the straight lines $y = -x + w$, so their double-dimensional Lebesgue's measure equals zero, thus the sum of the countable numbers of zeromeasured sets, is the set of zero-measure. Thereby the sets E_2 are zero-measured and the sets E_1 are measurable as their well-matched complements. It prooves that the function $\mu(., y, .)$ is measurable. The function μ is neither continuous in relation to the first variable nor to the third one, because when $t_n \to \sqrt{2}$, $u_n \to \sqrt{2}$, and when $t_n, u_n \in Q$ then

$$\mu(t_n, y, \sqrt{2}) = 1 - y \to 1 - y \neq y = \mu(\sqrt{2}, y, \sqrt{2})$$

and

$$\mu(-\sqrt{2}, y, u_n) = y \to y + 1 - y = \mu(-\sqrt{2}, y, \sqrt{2}).$$

The considered pseudo-process is even the process, that it means that for every $u \in R^+$, a family $\{\mu(t, ., u)\}$ $t \in R$ is the equicontinuous family, that it means:

$$\bigwedge_{u \in R^+} \bigwedge_{y_1 \in Y} \bigwedge_{\varepsilon > 0} \bigvee_{\delta > 0} \bigwedge_{y_2 \in Y} \bigwedge_{t \in R} [|y_1 - y_2| < \delta \Rightarrow |\mu(t, y_1, u) - \mu(t, y_2, u)| < \varepsilon].$$

It is sufficient to accept $\delta = 1$ in the above condition. Of course, this process is not continuous.

By Proposition 1 we can obtain from Zdun's theorem the following

Theorem 3. *If in the pseudo-process (Y, G, H, μ) the set H is a semigroup R^+ of non-negative numbers with ordinary addition and topology, G is the oversemigroup of the semigroup R^+ endowed with a compact metric, that every translation of an element from H will be continuous and every translation by an element from $G-H$ converted to H will be measurable and Y is the compact metric space, $\mu(., ., u)$: $G \times Y \to Y$, with Tychonoff's topology on $G \times Y$, is the continuous function for every u from R^+, $\mu(t, y, .)$: $R^+ \to Y$ is the measurable function, then*

a) *μ is the continuous function,*

b) *(Y, G, H, μ) is the uniformly continuous process.*

Proof. For a proof of thesis a) it is sufficient to notice that $G \times Y$ is compact with Tychonoff's topology, so it is separable, so every open set is the sum of the at most countable sum of the sets, being the products of the open set in G and of the open set in Y. So it follows from our assumptions, that the function $\pi(., (t, y))$ defined in Proposition 1 is measurable. Because it follows from the above assumption that $\pi(u, (., .))$ is continuous, so we obtain from Zdun's theorem, that π is the continuous function, which make μ be continuous.

The function $\mu(., ., u)$ is uniformly continuous because it is continuous on the compact set $G \times Y$, which prooves the thesis b).

5. For the pseudo-process (Y, G, H, μ) the system $(G \times H; .)$ in which the operation "." is defined like this:

$$(t, u) . (s, v) := (t, u + v) \Leftrightarrow s = t + u,$$

is the category. The function $F(y, (t, u)) := \mu(t, y, u)$ satisfies the following equations:

$$F(y, (t, u) . (s, v)) = F(F(y, (t, u)), (s, v)),$$

$$F(y, (t, 0)) = y,$$

known as the translation equation and the identity condition. So the generalization of the pseudo-process will be a triple (Y, K, F), where $(K; .)$ is any category, Y is any set, and $F : Y \times K \to Y$ satisfies the following two conditions:

a) the translation equation

$$F(y, a\,.\,b) = F(F(y, a), b)$$

and

b) the identity condition

$$F(y, e) = y,$$

for every y from Y and for every unity e from the category K.

There are many papers which deal with a problem of the translation equations. One can see them in different branches of mathematics (see [6], [7], and the literature in this papers).

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