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REGULAR AND NORMAL QUANTALES

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Abstract. Quantales can be viewed as a framework for a non–commutative topology. Basic properties of quantales are given. There are considered regular and normal quantales. The main. result asserts that regular quantales are frames.

Key words. Residuated lattice, multiplicative lattice, quantale, regular quantale, normal quantale, Gelfand ring.

The idea of considering complete lattices equipped with an additional binary operation . goes back to Ward and Dilworth [10]. Their motivating examples were lattices of ring ideals. In 1983, C. J. Mulvey suggested to consider multiplicative lattices as a framework for a non-commutative topology and proposed to call them quantales. His approach was followed by Borceux [3], [4] who found some basic topological properties of quantales. The present paper contributes to this direction. The main result is that regular quantales are frames (i.e. the multiplication . coincides with the meet $\wedge$; concerning frames see Johnstone [5]). The consequence is that cogroups in the category of idempotent quantales are localic, i.e. they are frames. Further results generalize some topological properties of frames to quantales. As an application, there are considered Gelfand rings (in the sense of [5]). It is connected with results of B. Banaschewski and R. Harting communicated in 1984. The author would like to express his thanks to J. Rosický for his valuable assistence in this work.

§ 1. m-semilattices

1.1. Definition. An *m-semilattice* is a $\vee$-semilattice S with the top element 1 and the bottom element 0 equipped with an associative binary operation . so that

$$x \,.\, \Big(\bigvee_{i=1}^{n} x_i\Big) = \bigvee_{i=1}^{n} (x \,.\, x_i),$$

$$\Big(\bigvee_{i=1}^{n} x_i\Big) \,.\, x = \bigvee_{i=1}^{n} (x_i \,.\, x),$$

$$1 \,.\, x = x$$

for all $x, x_1, \dots, x_n \in S$.

A morphism $f: S_1 \to S_2$ of m-semilattices is a mapping f such that

(1) f preserves finite joins,

(2) $f(1) = 1$,

(3) $f(a) \,.\, f(b) \leqq f(a \,.\, b)$ for each $a, b \in S_1$.

The category of m-semilattices will be denoted by $\mathscr{M}$.

An ideal of an m-semilattice S will be just an ideal of a $\vee$-semilattice. An ideal I is called *m-prime* if $x \,.\, y \in I$ implies $x \in I$ or $y \in I$ for all $x, y \in S$.

An element a of an m-semilattice S is called

(1) *2-sided* if $a \,.\, 1 \leqq a$,

(2) *idempotent* if $a \,.\, a = a$.

An m-semilattice is called *2-sided* (*idempotent*) if any its element is 2-sided (idempotent). An m-semilattice S is called a q-semilattice if $a \,.\, b = 1$ implies $a = 1 = b$ for all $a, b \in S$.

1.2. Definition. Let S be an m-semilattice, $a, b \in S$. We say that b is *well inside a* if there exists $c \in S$ with $c \,.\, b = 0$, $c \vee a = 1$. We write $b \lhd a$ ([5] uses the notation $b \gtreqqless a$).

1.3. Lemma. *Let S be an m-semilattice, $a, b, c, d \in S$. Then*

(i) *$a \leqq b \lhd c \leqq d$ implies $a \lhd d$,*

(ii) *$a \lhd b$, $c \lhd d$ implies $a \,.\, c \vee c \,.\, a \lhd b \,.\, d$ and $u \lhd b \,.\, d$ for all $u \in S$, $u \leqq a$, $u \leqq c$.*

(iii) *If S is idempotent or a q-semilattice then $a \lhd b$, $c \lhd b$ implies $a \vee c \lhd b$.*

Proof. (i) It is trivial.

(ii) Let $e \,.\, a = 0 = f \,.\, c$, $e \vee b = 1 = f \vee d$. We put $z = (b \,.\, f) \vee (e \,.\, d) \vee (e \,.\, f)$. Then $z \,.\, u = 0$, $z \vee (b \,.\, d) = 1, 0 = z \,.\, (a \,.\, c \vee c \,.\, a)$.

(iii) Let $e \,.\, a = 0 = f \,.\, c$, $e \vee b = 1 = f \vee b$. We define $z = e \,.\, f$. Then $z \,.\, (a \vee c) = 0$, $z \vee b = (e \,.\, 1) \vee b \geqq (e \,.\, e) \vee b$. If S is idempotent we are ready. If S is a q-semilattice then $e \vee b = 1$ implies $(e \,.\, 1) \vee b = 1$. Namely, $1 = (e \vee b) \,.\, 1 = (b \,.\, 1) \vee (e \,.\, 1 \,.\, 1) = (e \,.\, 1 \vee b) \,.\, 1$ and from the definition of a q-semilattice it follows $(e \,.\, 1) \vee b = 1$.

An element $p \neq 1$ of an m-semilattice S is called *prime* i if $a \,.\, b \leqq p$ implies $a \leqq p$ or $b \leqq p$ for all $a, b \in S$.

1.4. Lemma. *Let S be an idempotent m-semilattice or a q-semilattice, $p \in S$ a dual atom. Then p is prime.*

Proof. If $a \,.\, b \leqq p$, $b \nleqq p$ then $b \vee p = 1$. Therefore $a \,.\, a \leqq a \,.\, 1 = a \,.\, (b \vee p) = a \,.\, b \vee a \,.\, p \leqq p$. If S is idempotent we are ready. Let S be a q-semilattice. Then $a \vee p = 1$ implies $a \,.\, 1 \vee p = 1$ and this is a contradiction.

1.5. Definition. An m-semilattice S is said to be *normal* if, given $a, b \in S$ with $a \vee b = 1$, we can find $d, c \in S$ with $d \,.\, c = 0$, $d \vee a = 1 = b \vee c$.

The next proposition generalizes the well known result (see [5] II.3.7) that in a normal distributive lattice every prime ideal is contained in a unique maximal ideal.

1.6. Proposition. *Let S be an normal m-semilattice. Then every prime element in S is contained in at most one dual atom.*

Proof. Let $p \in S$ be prime, $a \neq b$ dual atoms, $p \leqq a, b$. Clearly $a \vee b = 1$, i.e. there exist $c, d \in S$ so that $d \,.\, c = 0$, $d \vee a = 1 = c \vee b$. Then $d \leqq p$ or $c \leqq p$. Hence $p \vee a = 1$ or $b \vee p = 1$, which is a contradiction.

The next result generalizes [5] IV.1.6 and it is due to Banaschewski and Harting. The proof follows [5].

1.7. Proposition. *In a normal m-semilattice S $a \triangleleft b$ implies that there exists $c \in S$ so that $a \triangleleft c \triangleleft b$. If moreover $a \leqq b$ then there exists $d \in S$ so that $a \leqq d \leqq b$, $a \triangleleft d \triangleleft b$.*

§ 2. Regularity in quantales

2.1. Definition. A *quantale* K is a complete m-semilattice in which . distributes over arbitrary joins.

A *morphism* $f: K \to L$ of quantales is a morphism of m-semilattices which preserves arbitrary joins. The category of quantales will be denoted by $\mathscr{K}$.

2.2. Definition. Let K be a quantale. We say that an element $a \in K$ is *re gular* i there exists $D \subseteqq I_a$, where $I_a = \{b \in K;\ b \triangleleft a\}$, so that $a = \bigvee D$. Let RK be the set of all regular elements of K.

2.3. Lemma. *Let K be a quantale. Then RK is an idempotent quantale.*

Proof. Let $a \in RK$ i.e. $a = \bigvee J$ for $J \subseteqq I_a$. If $b \in J$ then there exists $c \in K$ so that $c \,.\, b = 0$, $c \vee a = 1$. Thus $b = 1 \,.\, b = (c \vee a) \,.\, b = a \,.\, b$ and therefore $a = \bigvee J = a \,.\, \bigvee J = a \,.\, a$.

2.4. Definition. We say that a quantale K is *regular* if $K = RK$.

2.5. Theorem. *Any regular quantale is a frame.*

Proof. From 2.3 we know that K is an idempotent quantale. Let $a \in K$, $a = \bigvee J$, $J \subseteqq I_a$. If $b \in J$ then there exists $c \in K$ so that $c \,.\, b = 0$, $c \vee a = 1$. Following [4], $c \,.\, b = 0$ implies $b \,.\, c = 0$. Hence $b \,.\, 1 = b \,.\, (a \vee c) = b \,.\, a$ i.e. $a \,.\, 1 = a \,.\, a = a$. We proved that K is 2-sided and idempotent. Consequently, K is a frame (see [3]).

It is easy to check that RK is a frame if K is an idempotent quantale.

For a quantale K put $R^0(K) = K$, $R^{\alpha+1}(K) = R(R^\alpha(K))$ for any ordinal α, $R^\alpha(K) = \bigcap_{\gamma<\alpha} R^\gamma(K)$ for any limit ordinal α and $Reg(K) = \bigcap_{\alpha \in \mathcal{O}_n} R^\alpha(K)$. $Reg(K)$ is a regular quantale (i.e. a regular frame) and *Reg* is the coreflection functor of quantales into regular frames.

Borceux (see [4]) proved that the category of idempotent quantales has arbitrary sums. If K, L are idempotent quantales then their sum $K + L$ is the $\bigvee$-subsemilattice of the tensor product $K \otimes L$ of $\bigvee$-semilattices (see [6]) generated by elements $a \otimes b$ such that at least one of elements a, b is 2-sided.

2.6. Definition. A *quantic group L* (briefly *K-group*) is a cogroup in the category of idempotent quantales, i.e. a K-group L is equipped with morphisms

$$\mu : L \to L + L \text{ (multiplication)}$$
$$\iota : L \to L \text{ (inversion)}$$
$$\varepsilon : L \to 2 \text{ (unit)}$$

satisfying

$$(\mu + 1) \circ \mu = (1 + \mu) \circ \mu,$$
$$(\varepsilon + 1) \circ \mu = 1_{2+L},$$
$$(1 + \varepsilon) \circ \mu = 1_{L+2}$$

and

$$\nabla \circ (\iota + 1) \circ \mu = \sigma \circ \varepsilon = \nabla \circ (1 + \iota) \circ \mu.$$

Here $\nabla : L + L \to L$ is a codiagonal i.e. $\nabla(x \otimes y) = x \,.\, y$ and $\sigma : 2 \to L$ is a unique morphism from the initial quantale 2.

We define $\nu = (\iota + 1) \circ \mu$ and for $u \in L$, $\varepsilon(u) = 1$ we put $B(u) = \{x \,.\, y;\, x \otimes y \in \in L + L,\, x \otimes y \leqq \nu(u)\}$, $a_{B(u)} = \bigvee\{z \in B(u);\, z \,.\, a \neq 0\}$.

The next result extends the reasoning of [8] and shows that any K-group is a localic group.

2.7. Theorem. *Let L be a K-group. Then L is a regular frame.*

Proof. It first, we will prove

(*) If $u, x, y \in L$, $\varepsilon(y) = 1$ and at least one of the elements x, y is 2-sided then $x \otimes y \leqq \mu(u)$ implies $x_{B(y)} \leqq u$.

Namely, let $p \,.\, q \in B(y)$. Then $p \otimes p \,.\, q \leqq p \otimes q \leqq \nu(y)$ i.e. $\iota(p) \otimes p \,.\, q \leqq \mu(y)$. Hence $x \otimes \iota(p) \otimes p \,.\, q \leqq x \otimes \mu(y) = (1 + \mu) \circ (x \otimes y) \leqq (1 + \mu) \circ \mu(u) = = (\mu + 1) \circ \mu(u)$ and $(x \,.\, p) \otimes p \,.\, q \leqq (\nabla(1 + \iota) + 1) \circ (\mu + 1) \circ \mu(u) = (\sigma \circ \varepsilon + 1) \circ \circ \mu(u) = 1 \otimes u$. Therefore $x \,.\, p \neq 0$ implies $p \,.\, q \leqq u$. If $p \,.\, q \,.\, x \neq 0$ then $q \,.\, p \,.\, x \neq 0$ by [4], i.e. $p \,.\, x \neq 0$. Hence $x \,.\, p \neq 0$ and we have $x_{B(y)} \leqq u$.

Let $a \in L$. Then $\mu(a) = \bigvee_j a'_j \otimes a''_j$. Since $a \otimes 1 = (1 + \varepsilon) \circ \mu(a)$, it holds $a = \bigvee\{a'_j;\, \varepsilon(a''_j) = 1\}$. Hence $(a'_j)_{B(a''_j)} \leqq a$. We put $c = \bigvee\{z \in B(a''_j);\, z \,.\, a'_j = 0\}$. Then $c \,.\, a'_j = 0$, $c \vee a \geqq c \vee (a'_j)_{B(a''_j)} = 1$, i.e. $a'_j \lhd a$, $a'_j \leqq a$ and a is a regular element. Since every $a \in L$ is regular L is a regular frame.

At the end of this chapter we will characterize regular elements in the quantale *Lid*(A) of all left ideals of a ring A with a unit. Recall (cf. [5]) that a 2-sided ideal is called neat if for all $a \in I$ there exists an element $e \in I$ so that $a = e . a$. Let $Nid(A)$ denote the set of all neat ideals.

2.8. Proposition. *Let A be a ring, $J \in Lid(A)$. Then the following conditions are equivalent:*

(i) *J is regular.*

(ii) *For all $a \in J$ there exists an element $e \in J$ so that $b = e . b$ for all $b \in \langle a \rangle$, where $\langle a \rangle$ is a left ideal generated by an element a.*

Proof. (i) $\Rightarrow$ (ii) Let $a \in J$, $J = \bigvee D, D \subseteq I_J$. Then $a = a_1 + \dots + a_n$ for some $a_i \in L_i$, $L_i \in D$ i.e. there exist $K_i \in Lid(A)$ so that $K_i . L_i = \{0\}$, $K_i \vee J = A$. Then for some $k_i \in K_i$ and $d_i \in J$ it holds $k_i + d_i = 1$ and $x \in \langle a_i \rangle$ implies $k_i . x = 0$. We put $k = k_1 . \dots . k_n$. Then for all $y \in \langle a \rangle$ it is $k . y = 0$. Further $1 = d_1 + k_1 . \{d_2 + k_2 . [d_3 + k_3 . (\dots)]\} = d_1 + k_1 . d_2 + k_1 . k_2 . d_3 + \dots + k_1 . \dots . k_{n-1} . d_n + k$ i.e. $1 = k + d$, $d \in J$. Hence $y = 1 . y = (k + d) . y = d . y$.

(ii) $\Rightarrow$ (i) Clearly $J = \bigvee_{a \in J} \langle a \rangle$. We will show that $\langle a \rangle \triangleleft J$ for all $a \in J$. We consider a left ideal $\langle (1 - e) \rangle$ such that $e \in J$ and $x = e . x$ for all $x \in \langle a \rangle$. Then $\langle (1 - e) \rangle . \langle a \rangle = \{0\}$, $\langle (1 - e) \rangle \vee J = A$.

The following result is due to Banaschewski and Harting [1].

2.9. Corollary. *Let A be a commutative ring, $J \in Id(A)$, where $Id(A)$ is the quantale of all ideals of A. Then the following conditions are equivalent:*

(i) *J is regular.*

(ii) *J is neat.*

§ 3. Normal quantales

Recall that a lattice is compact if 1 is compact.

3.1. Proposition. *Let K be a compact normal quantale. Then every prime element in K is contained in a unique dual atom.*

Proof. It follows from 1.5 and from the fact that in a compact complete lattice any element is contained in a dual atom.

Let K be a quantale, $a \in K$. We define

$$h(a) = \bigvee \{x \leqq a; x \in RK\}$$

i.e. $h(a)$ is the greatest regular element lying under a;

$$r(a) = \bigvee \{x \leqq a; x \triangleleft a\}.$$

It is trivial to check that $h(a) \leqq r(a)$. The next proposition is due to Banaschewski and Harting (see [1]).

3.2. Proposition. *Let K be a normal quantale, $a \in K$. Then $r(a) = h(a)$ and moreover $R^2K = RK$ i.e. $Reg(K) = RK$.*

Proof. If $x \leqq a$, $x \triangleleft a$ then there exists $z \in K$ so that $x \leqq z \leqq a$, $x \triangleleft z \triangleleft a$. Hence $z \leqq r(a)$. Clearly $x \triangleleft r(a)$, $x \leqq a$. Then $r(a) \in RK$, $r(a) \leqq a$ i.e. $r(a) \leqq h(a)$.

To prove $RK = R^2K$ we shall need to check that

$$a = \bigvee\{r(x);\ x \leqq a,\ x \triangleleft a\}$$

for any $a \in RK$. If $u \triangleleft a$, $u \leqq a$ then there exists $w \in K$ so that $u \leqq w \leqq a$, $u \triangleleft w \triangleleft a$ i.e. $u \leqq r(w)$.

Let K be a quantale. We say that K is a q-quantale if K is a q-semilattice.

3.3. Lemma. *Let K be a normal q-quantale. Given two elements $a, b \in K$, then $a \vee b = 1$ implies*

$$r(a) \vee b = 1 \qquad \text{i.e.} \qquad h(a) \vee b = 1.$$

Proof. Let $a, b \in K$, $a \vee b = 1$. Then there exist $c, d \in K$ so that $d \,.\, c = 0$, $d \vee a = 1 = c \vee b$ i.e. $c \triangleleft a$, $c \vee b = 1$. Obviously $1 = c \,.\, 1 \vee b = c \,.\, d \vee c \,.\, a \vee b = c \,.\, d \,.\, 1 \vee c \,.\, a \vee b = c \,.\, d \,.\, c \vee c \,.\, d \,.\, b \vee c \,.\, a \vee b = c \,.\, a \vee b$ and $c \,.\, a \leqq a$, $c \,.\, a \triangleleft a$. Then $r(a) \vee b = 1$.

3.4. Theorem. *Let K be a compact q-quantale. Then the following conditions are equivalent:*

(i) *K is normal.*

(ii) *$h(a) \vee h(b) = h(a \vee b)$ for all $a, b \in K$.*

Proof. (i) $\Rightarrow$ (ii) Clearly $h(a) \vee h(b) \leqq h(a \vee b)$. Conversely, let $h(x) \leqq h(a \vee b)$, $h(x) \triangleleft h(a \vee b)$. Then there exists $e \in K$ so that $e \,.\, h(x) = 0$, $e \vee h(a \vee b) = 1$. Hence $e \vee a \vee b = 1$. Applying 3.3 we become $h(e) \vee h(a) \vee h(b) = 1$. Then $h(a \vee b) = h(a \vee b) \,.\, 1 = \bigvee\{h(x);\ h(x) \leqq h(a \vee b),\ h(x) \triangleleft h(a \vee b)\} \,.\, 1 = \bigvee\{h(x) \,.\, (h(a) \vee h(b) \vee h(e))\} = h(a \vee b) \,.\, (h(a) \vee h(b)) \leqq h(a) \vee h(b)$.

(ii) $\Rightarrow$ (i) Conversely, let $a, b \in K$, $a \vee b = 1$. Then $h(a) \vee h(b) = 1$ i.e. $r(a) \vee r(b) = 1$ and by compactness there exist $x, y \in K$ so that $1 = x \vee y$, $x \leqq a$, $x \triangleleft a$, $y \leqq b$, $y \triangleleft b$. Hence there exists $e \in K$ so that $e \,.\, x = 0$, $e \vee a = 1 = x \vee b$.

We recall that a ring A is called *Gelfand* if for all $I, J \in Lid(A)$ it holds $(I \vee J)^c = I^c \vee J^c$; here I^c is the greatest neat ideal contained in I (see [5]). Equivalently, a ring A is Gelfand if for any distinct maximal left ideals M, N of A there exist elements $a \notin M$, $b \notin N$ of the ring A for which $a \,.\, A \,.\, b = \{0\}$ (see [2]).

It is well known that $Nid(A)$ is a compact regular frame for a Gelfand ring A (see [2]). The following result was proved by Banaschewski and Harting for commutative rings.

3.5. Theorem. *Let A be a ring. The following conditions are equivalent:*

(i) *$Lid(A)$ is a normal quantale.*

(ii) *A is a Gelfand ring.*

Proof. (i) $\Rightarrow$ (ii) Let $M \neq N$ be maximal left ideals. Then there exist $D, C \in Lid(A)$ so that $D \,.\, C = \{0\}$, $D \vee M = A = C \vee N$ i.e. there exist $a \in D$, $b \in C$,

$m \in M$, $n \in N$ such that $a \,.\, A \,.\, b = \{0\}$, $a + m = 1 = b + n$ i.e. $a \,.\, A \,.\, b = \{0\}$, $a \notin M$, $b \notin N$.

(ii) $\Rightarrow$ (i) Let $I \vee J = A$ then $I^c \vee J^c = A$ and from the normality of $Nid(A)$ there exist $K, L \in Nid(A)$ so that $K \,.\, L = \{0\}$, $K \vee I^c = A = L \vee J^c$.

§ 4. Coherent quantales

Let $Id(S)$ be the quantale of all ideals of an m-semilattice S ($I \,.\, J$ being generated by $\{x \,.\, y;\ x \in I,\ y \in J\}$).

4.1. Definition. A quantale K is called *coherent* if it is a compact algebraic quantale and $x \,.\, y \in c(K)$ for all $x, y \in c(K)$; here $c(K)$ is the set of all compact elements of K.

It is well known (cf. [5], II.3.2) that coherent frames are just frames of ideals of a distributive lattice. The following result is due to Keimel (see [7]).

4.2. Proposition. *For a quantale K the following conditions are equivalent:*

(i) *K is coherent quantale.*

(ii) *K is isomorphic to the quantale of all ideals of an m-semilattice.*

4.3. Proposition. The functor $Id : M \to K$ is left adjoint to the forgetful functor $Q : K \to M$.

Proof. Let $\eta : M \to QId(M)$ be defined by the prescription $\eta(a) = \downarrow(a)$. Then η is a morphism of m-semilattices and the same holds for $\eta^0 : QId(M) \to M$, $\eta^0(J) = \bigvee J$. Having a morphism $f : M \to Q(K)$, then $f \,.\, \eta^0$ is a unique factorization of f through η.

4.4. Proposition. *Let S be an m-semilattice. Then the prime elements of $Id(S)$ are precisely the m-prime ideals of S.*

Proof. Analogous to [5], II.3.4.

The following result is well known for distributive lattices (see [5]).

4.5. Theorem. *Let S be an m-semilattice. Then the following conditions are equivalent:*

(i) *S is normal.*

(ii) *$Id(S)$ is normal.*

Proof. (i) $\Rightarrow$ (ii) Let S be normal, $A, B \in Id(S)$, $A \vee B = S$. Then there exist $x \in A$, $y \in B$ so that $x \vee y = 1$ and from normality it follows that there exist $c, d \in S$ such that $d \,.\, c = 0$, $d \vee x = 1 = c \vee y$. Hence $\downarrow(d) \,.\, \downarrow(c) = \{0\}$, $\downarrow(d) \vee A = S = \downarrow(c) \vee B$.

(ii) $\Rightarrow$ (i) Conversely, let $Id(S)$ be normal, $a, b \in S$, $a \vee b = 1$. Then $\downarrow(a) \vee \downarrow(b) = S$ i.e. there exist $C, D \in Id(S)$ so that $D \,.\, C = \{0\}$, $D \vee \downarrow(a) = S = C \vee \downarrow(b)$. Hence there exist $c, d \in S$ so that $d \,.\, c = 0$, $d \vee a = 1$, $c \vee b = 1$.

4.6. Corollary. *Let S be a normal m-semilattice. Then every m-prime ideal in S is contained in a unique maximal ideal.*

4.7. Corollary. *Let A be a ring. Then the following conditions are equivalent:*
(i) *$c(Lid(A))$ is a normal m-semilattice.*
(ii) *A is a Gelfand ring.*

Proof. Follows immediately from 3.5 and 4.5.

REFERENCES

[1] B. Banaschewski and R. Harting: *V-rings and locales*, communication at the Category Theory Conference in Murten (1984).
[2] F. Borceux, H. Simmons, G. Van den Bossche: *A sheaf representation for modules with application to Gelfand rings*, preprint.
[3] F. Borceaux: *About quantales and quantic spaces*, Sydney Category Seminar Reports 1984.
[4] F. Borceux, G. Van den Bossche: *Quantales and their sheaves*, Rapp. Sém. Math. Pure No. 63, Université Louvain (1985).
[5] P. T. Johnstone: *Stone Spaces*, Cambridge University Press, Cambridge 1982.
[6] A. Joyal, M. Tierney: *A generalization of the descent theory of Grothendick*, Memoirs of the A. M. S. (1984).
[7] K. Keimel: *A unified theory of minimal prime ideals*, Acta Math. Acad. Sci. Hung. 23 (1972), 51–69.
[8] I. Kříž, A. Pultr, J. Rosický: *Localic groups*, manuscript (1985).
[9] C. J. Mulvey: *A Generalization of Gelfand duality*, J. Algebra 56 (1979), 499–505.
[10] M. Ward, R. P. Dilworth: *Residuated lattices*, Trans. Amer. Math. Soc., 45 (1939), 335–354.

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