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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION

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Abstract. The linear differential equation $y^{(n)} + p(t)\, y^{(k)} + q(t)\, y = 0$ is concerned in this paper. Under the conditions that ratios of certain powers of the coefficients and some their derivatives of this equation are small, the asymptotic behaviour as $t \to \infty$ of the fundamental set of solutions are given.

Key words. Linear differential equation of n-th order, asymptotic behaviour, asymptotic formula, fundamental system.

MS Classification. 34064

1. Introduction

Asymptotic behaviour of the n-order linear differential equations under the conditions $\int^{\infty} t^q \mid p_k(t) \mid \mathrm{d}t < \infty$ or $p_k(t) \to 0$, where $p_k(t)$ are coefficients of this equation, were studied in many papers and they can be found in the monographs E. A. Coddington and N. Levinson [1], P. Hartman [2]. The asymptotic behaviour of this equation under the weaker assumptions that $\int^{\infty} t^q p_k(t)\, \mathrm{d}t < \infty$ we can find in [10, 11]. In 1947 A. Wintner [12] derived asymptotic formulae for the differential equation $y'' + q(t)\, y = 0$ (see them in the corollaries of this paper) which have wide application in quantum mechanics under conditions that ratios of certain powers $q(t)$ and $q'(t)$ are small (inproper integrals on $[a, \infty)$ exist). The similar conditions have been used in other papers [3, 4, 5, 8, 9] for differential equations of the second, the third, the fourth and binomial of the n-th order.

Since some results of the n-order linear differential equations with two coefficients have been lately published [6, 7, 13], the aim of this paper will be investigation of asymptotic properties of the differential equation

$$y^{(n)} + p(t)\, y^{(k)} - (-1)^m q(t)\, y = 0. \tag{1}$$

The results of this paper generalize the results for the third and fourth order and give new results for the second and the n-th order linear differential equation generally.

2. Preliminary results

Let us consider the equation (1), where n, k, m are integers, $n > 1$, $1 \leqq k < n$, $m = 1, 2$, $p(t)$, $q(t) > 0$ are continuous functions including the derivatives that stand in theorems.

A vector-matrix form of the equation (1) is

$$z' = A(t)\, z, \tag{2}$$

where $A(t) = (a_{ij}(t))$ is $n\,.\,n$ matrix defined as follows

$$a_{ij}(t) = \begin{cases} 1 & \text{if } j = i + 1 \\ (-1)^m q(t) & \text{if } i = n \text{ and } j = 1 \\ -p(t) & \text{if } i = n \text{ and } j = k + 1 \\ 0 & \text{otherwise} \end{cases}$$

and $z = [y, y', \ldots, y^{(n-1)}]^T$. If we make a linear transformation $w = Tz$ with continuously differentiable nonsingular matrix

$$T(t) = dia\left[q(t)^{1-\frac{1}{n}}, q(t)^{1-\frac{2}{n}}, \ldots, q(t)^{\frac{1}{n}}, 1\right],$$

we get the equation

$$w' = \left[A_0 q(t)^{\frac{1}{n}} + A_1 p(t)\, q(t)^{\frac{k+1}{n}-1} + A_2 q'(t)\, q(t)^{-1}\right] w, \tag{3}$$

where $A_0 = (a_{ij}^0)$, $A_1 = (a_{ij}^1)$, A_2 are $n\,.\,n$ constant matrix defined as follows

$$a_{ij}^0 = \begin{cases} 1 & \text{if } j = i + 1 \\ (-1)^m & \text{if } i = n \text{ and } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$a_{ij}^1 = \begin{cases} -1 & \text{if } i = n \text{ and } j = k + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$A_2 = dia\left[1 - \frac{1}{n}, 1 - \frac{2}{n}, \ldots, \frac{1}{n}, 0\right].$$

Suppose $\int_a^\infty q(t)^{\frac{1}{n}}\, dt = \infty$. By putting the substitution $t = \alpha(s)$ into (3), where $\alpha(s)$ is the inverse function with $\omega(t) = \int_a^t q(s)^{\frac{1}{n}}\, ds$, the equation (3) reduces to

$$x' = [A_0 + A_1 f(s) + A_2 q(s)]\, x, \tag{4}$$

where $x(s) = (s)$,

$$f(s) = \frac{p[\alpha(s)]}{q[\alpha(s)]^{1-\frac{k}{n}}}, \qquad g(s) = \frac{q'[\alpha(s)]}{q[\alpha(s)]^{1+\frac{1}{n}}}.$$

An asymptotic behaviour of solutions of the equation

$$x' = [A + V(s) + R(s)]\, x$$

was proved in Theorem 8.1 in [1]. Now we proceed to apply this theorem for the equation (4) in two ways. In the first case we put $V(s) = A_1 f(s) + A_2 g(s)$ and $R(s) = 0$, in the second one we put $V(s) = A_1 f(s)$ and $R(s) = A_2 g(s)$.

Throughout by $L[a, \infty)$ we denote the Banach space of all complex valued functions which are Lebesque integrable on $[a, \infty)$. The next Lemma will be needed.

Lemma. (D. B. Hinton [5].) *Let* $h(t) > 0$ *on* $[a, \infty)$ *and* $h''(t) \,.\, h(t)^{-1-\frac{1}{n}} \in L[a, \infty)$. *Then*

(i) $h(t)^{\frac{1}{n}} \notin L[a, \infty)$,

(ii) $\left[h'(t) \,.\, h(t)^{-1-\frac{1}{n}}\right]' \in L[a, \infty)$,

(iii) $\left[h'(t) \,.\, h(t)^{-1-\frac{1}{2n}}\right]^2 \in L[a, \infty)$.

3. Main results

Theorem 1. *Let the functions* $p'(t)$ *and* $q''(t)$ *be continuous on* $[a, \infty)$. *Let*

$$\frac{q''(t)}{q(t)^{1+\frac{1}{n}}}, \quad \frac{p'(t)}{q(t)^{1-\frac{k}{n}}} \quad \text{and} \quad \frac{p(t)^2}{q(t)^{2-\frac{1+2k}{n}}} \tag{5}$$

be in $L[a, \infty)$. *Then there exists a fundamental system* $z_i(t)$ *of the equation* (2) *such that*

$$Tz_i q(t)^{\frac{1-n}{2n}} \exp\left\{-\lambda_i \int_a^t \left[q(\tau)^{\frac{1}{n}} - (-1)^m \frac{\lambda_i^k}{n} \frac{p(\tau)}{q(\tau)^{1-\frac{k+1}{n}}}\right] \mathrm{d}\tau\right\} \to p_i,$$

where λ_i *are the roots of the equation* $\lambda^n - (-1)^m = 0$ *and* $p_i = [1, \lambda_i, \lambda_i, \ldots, \lambda_i^{n-1}]^T$.

Proof. To apply Theorem 8.1 of [1] denote $A_0 = A$ and $V(s) = A_1 f(s) + A_2 g(s)$. Since $\det[\lambda E - A_0] = \lambda^n - (-1)^m$, the characteristic roots λ_i of A_0 are distinct and $p_i = [1, \lambda_i, \ldots, \lambda^{n-1}]^T$ are characteristic vectors of A_0 corresponding to λ_i.

By change of variable $t = \alpha(s)$ we obtain

$$\int_0^\infty |f'(s)|\, \mathrm{d}s = \int_0^\infty \left| \left[\frac{p[\alpha(s)]}{q[\alpha(s)]^{1-\frac{k}{n}}}\right]' \right| \mathrm{d}s \leqq$$

$$\leqq \int_a^\infty \left| \frac{p'(t)}{q(t)^{1-\frac{k}{n}}} \right| \mathrm{d}t + \left(1 - \frac{k}{n}\right) \int_a^\infty \left| \frac{p(t)\, q'(t)}{q(t)^{2-\frac{k}{n}}} \right| \mathrm{d}t \leqq$$

$$= \int_a^\infty \left| \frac{p'(t)}{q(t)^{1-\frac{k}{n}}} \right| dt + \left(1 - \frac{k}{n}\right) \left[\int_a^\infty \left| \frac{p^2(t)}{q(t)^{2-\frac{1+2k}{n}}} \right| dt \right]^{1/2} \cdot \left[\int_a^\infty \left| \frac{q'(t)}{q(t)^{1+\frac{1}{2n}}} \right| dt \right]^{1/2}.$$

From the conditions of the theorem and from the Lemma we get $f'(s) \in L[0, \infty)$. Similarly we deduce

$$\int_0^\infty | g'(s) | ds \leqq \int_a^\infty \left| \frac{q''(t)}{q(t)^{1+\frac{1}{n}}} \right| dt + \left(1 + \frac{1}{n}\right) \int_a^\infty \left[\frac{q'(t)}{q(t)^{1+\frac{1}{2n}}} \right]^2 dt < \infty,$$

$$\int_0^\infty f(s)^2 ds \leqq \int_a^\infty \frac{p(t)^2}{q(t)^{2-\frac{1+2k}{n}}} dt < \infty,$$

$$\int_0^\infty g(s)^2 ds \leqq \int_a^\infty \left[\frac{q'(t)}{q(t)^{1+\frac{1}{2n}}} \right]^2 dt < \infty.$$

So we obtained $\int_0^\infty | V'(s) | ds < \infty$ and $V(s) \to 0$ as $s \to \infty$.

Now we investigate the characteristic roots $\lambda(s)$ of the matrix $A_0 + V(s)$. The characteristic equation has the form

$$(6) \quad P[\lambda(s)] = -(-1)^m + f(s) \prod_{i=1}^{k} \left[\lambda - \frac{n-i}{n} g(s) \right] + \prod_{i=1}^{n} \left[\lambda - \frac{n-i}{n} g(s) \right] = 0.$$

It is evident that $P[\lambda(s)] \to \lambda^n - (-1)^m$, because $f(s) \to 0$ and $g(s) \to 0$ as $s \to \infty$. In the notation of Theorem 8.1 of [1] all j, $1 \leqq j \leqq n$ for a given i are supposed to fall into one of two classes I_1 and I_2, where

$$j \in I_1, \quad \text{if} \quad \int_0^s D_{ij}(s)\, ds \to \infty \quad \text{and} \quad \int_{s_1}^{s_2} D_{ij}(s)\, ds > -K,$$

$$j \in I_2, \quad \text{if} \quad \int_{s_1}^{s_2} D_{ij}(s)\, ds < K, \quad (s_2 \geqq s_1 \geqq 0),$$

where $K > 0$ is a constant and $D_{ij}(s) = \operatorname{Re}[\lambda_i(s) - \lambda_j(s)]$.

To proove this fact we express $\lambda(s)$ in the form

$$\lambda(s) = \lambda + \beta(s) + \gamma(s),$$

where $\beta(s) \to 0$, $\gamma(s) \to 0$ as $s \to \infty$ and $\gamma(s) \in L[0, \infty)$. For this aim we look for numbers c_1, c_2 such that

$$\beta(s) = c_1 f(s) + c_2 g(s)$$

and $P[\lambda + \beta(s)] \in L[0, \infty)$. From (6) it follows

$$(7) \qquad P[\lambda + \beta(s)] = -(-1)^m + f(s) \prod_{i=1}^{k} \left[\lambda + \beta(s) - \frac{n-i}{n} g(s) \right] +$$

$$+\prod_{i=1}^{n}\left[\lambda+\beta(s)-\frac{n-i}{n}g(s)\right].$$

All terms of the first product in (7), except $[\lambda+\beta(s)]^k f(s)$, contain $f^2(s)$, $g^2(s)$ or $f(s)\,g(s)$ and so they are in $L[0,\infty)$. Since we choose $\beta(s)=c_1 f(s)+c_2 g(s)$, all terms of $[\lambda+\beta(s)]^k f(s)$ are in $L[0,\infty)$. Therefore, if we put

$$-(-1)^m+f(s)\,\lambda^k+\lambda^n-\lambda^{n-1}g(s)\frac{n-1}{2}+n\beta(s)\,\lambda^{n-1}=0,$$

i.e.

$$\beta(s)=-\frac{1}{n}\frac{\lambda^k}{\lambda^{n-1}}f(s)+\frac{1}{2}\frac{n-1}{n}g(s), \tag{8}$$

we obtain that each term of $P[\lambda+\beta(s)]$ contains $f^2(s)$, $g^2(s)$ or $f(s)\,g(s)$ and hence $P[\lambda+\beta(s)]\in L[0,\infty)$. Evidently $\beta(s)\to 0$ as $s\to\infty$.

Now we are to proove $\gamma(s)\to 0$ as $s\to\infty$ and $\gamma(s)\in L[0,\infty)$. Since $\lambda(s)=\lambda+$ $+\beta(s)+\gamma(s)$ is the characteristic root of $A_0+V(s)$, hence

$$P[\lambda+\beta(s)+\gamma(s)]=A(s)\,\gamma(s)+P[\lambda+\beta(s)]=0$$

and so

$$|\,A(s)\,\gamma(s)\,|=|\,P[\lambda+\beta(s)]\,|. \tag{9}$$

By the same way as (7) we see that

$$P[\lambda+\beta(s)+\gamma(s)]=-(-1)^m+f(s)\prod_{i=1}^{k}\left[\lambda+\beta(s)+\gamma(s)-\frac{n-1}{n}g(s)\right]+$$
$$+\prod_{i=1}^{n}\left[\lambda+\beta(s)+\gamma(s)-\frac{n-1}{n}g(s)\right]. \tag{10}$$

From (10) it follows that $A(s)$ consists of the terms which tend to zero except $n\lambda^{n-1}$, i.e.

$$\lim_{s\to\infty}A(s)=n\lambda^{n-1}.$$

Then

$$|\,|A(s)|-|n\lambda^{n-1}|\,|<\frac{1}{2}$$

and

$$|\,A(s)\,|>n\lambda^{n-1}-\frac{1}{2}=n-\frac{1}{2}\geqq\frac{1}{2}$$

for sufficiently large s. Then (9) gives

$$|\,\gamma(s)\,|\leqq|\,2P[\lambda+\beta(s)]\,|$$

and hence $\gamma(s)\in L[0,\infty)$, $\gamma(s)\to 0$ as $s\to\infty$. Consequently we obtain that the characteristic roots $\lambda_i(s)$ of $A_0+V(s)$ may be written as

$$\lambda_i(s) = \lambda_i - \frac{1}{n}\frac{\lambda_i^k}{\lambda_i^{n-1}} f(s) + \frac{1}{2}\frac{n-1}{n} g(s) + \gamma_i(s), \tag{11}$$

where λ_i are the roots of $\lambda^n - (-1)^m = 0$, $\gamma_i(s) \in L[0, \infty)$ and $\gamma_i(s) \to 0$ as $s \to \infty$.

From (11) it follows that $D_{ij}(s)$ for all $i, j = 1, 2, \ldots, n$ may have the following forms

a) $D_{ij}(s) = G(s)$,
b) $D_{ij}(s) = c + F(s) + G(s)$,
d) $D_{ij}(s) = -c + F(s) + G(s)$,

where $c > 0$ is a number, $F(s)$, $G(s)$ are continuous functions on $[0, \infty)$, $F(s) \to 0$, $G(s) \to 0$ as $s \to \infty$ and $G(s) \in L[0, \infty)$.

a) If $G(s) \in L[0, \infty)$, then there exists a number $K > 0$ such that

$$\int_{s_1}^{s_2} D_{ij}(s)\, \mathrm{d}s < K \qquad (s_2 \geqq s_1 \geqq 0)$$

and hence $j \in I_2$.

b) If $F(s) \to 0$ as $s \to \infty$, then there exists a number $s' \in [0, \infty)$ such that $c + F(s) + G(s) \geqq \frac{c}{2} + G(s)$ for all $s > s'$. Then

$$\int_0^\infty D_{ij}(s)\, \mathrm{d}s = \int_0^\infty [c + F(s) + G(s)]\, \mathrm{d}s = \infty$$

and

$$\int_{s_1}^{s_2} D_{ij}(s)\, \mathrm{d}s > -K \qquad (s_2 \geqq s_1 \geqq 0),\ K > 0$$

because of $c + F(s) + G(s) \to c$ as $s \to \infty$. Hence $j \in I_1$.

c) From the condition $F(s) \to 0$ as $s \to \infty$ it yields that there exists a number $s'' > 0$ such that

$$-c + F(s) + G(s) < -\frac{c}{2} + G(s)$$

for all $s > s''$ and hence

$$\int_{s_1}^{s_2} D_{ij}(s)\, \mathrm{d}s < K \qquad (s_2 \geqq s_1 \geqq 0),\ K > 0.$$

So $j \in I_2$.

Thus all assumptions of Theorem 8.1 of [1] are fulfiled. Therefore there exist n linearly independent solutions $x_i(s)$, $i = 1, 2, \ldots, n$ of (4) such that

$$x_i(s) \exp\left[-\int_{s_0}^{s} \lambda_i(\xi)\, \mathrm{d}\xi\right] \to p_i,$$

i.e.

$$(12)\qquad x_i(s)\exp\left(-\int_{s_0}^{s}\left[\lambda_i-\frac{1}{n}\frac{\lambda_i^k}{\lambda_i^{n-1}}f(\xi)+\frac{1}{2}\frac{n-1}{n}g(\xi)+\gamma_i(\xi)\right]d\xi\right)\to p_i.$$

By substituting $\xi=\omega(\tau)$ in (12) and putting

$$L_i=q[\alpha(s_0)]^{\frac{1}{2}\frac{n-1}{n}}\exp\left[-\int_{s_0}^{\infty}\gamma_i(\tau)\,d\tau\right]$$

we have

$$L_iw_i(t)\,q(t)^{\frac{1-n}{2n}}\exp\left(-\lambda_i\int_{t_0}^{t}\left[q(\tau)^{\frac{1}{n}}-(-1)^m\frac{\lambda_i^k}{n}\frac{p(\tau)}{q(\tau)^{1-\frac{k+1}{n}}}\right]d\tau\right)\to p_i.$$

Since $w_i=Tz_i$ and the equation (2) is linear, we have the assertion of Theorem 1.

Theorem 2. *Let $p(t)$ and $q''(t)$ be continuous functions on $[a,\infty)$. Let*

$$(13)\qquad \frac{q''(t)}{q(t)^{1+\frac{1}{n}}}\quad\text{and}\quad\frac{p(t)}{q(t)^{1-\frac{k+1}{n}}}$$

be in $L(a,\infty)$. Then there exists a fundamental system $z_i(t)$ of the equation (2) *such that*

$$(14)\qquad Tz_iq(t)^{\frac{1-n}{2n}}\exp\left[-\lambda_i\int_{t_0}^{t}q(\tau)^{\frac{1}{n}}\,d\tau\right]\to p_i,$$

where λ_i are roots of $\lambda^n-(-1)^m=0$ and $p_i=[1,\lambda_i,\dots,\lambda_i^{n-1}]^T$.

Proof. In the notation of Theorem 8.1 in [1] we denote $V(s)=A_2g(s)$ and $R(s)=A_1f(s)$ in (4). Then

$$\int_0^\infty|g'(s)|\,ds<\infty\quad\text{and}\quad\int_0^\infty g^2(s)\,ds<\infty$$

by the same arguments used in the proof of Theorem 1. So $\int_0^\infty|V'(s)|\,ds<\infty$ and $V(s)\to0$ as $s\to\infty$. Since

$$\int_0^\infty|f(s)|\,ds=\int_a^\infty\left|\frac{p(t)}{q(t)^{1-\frac{k+1}{n}}}\right|dt<\infty$$

it holds that $\int_0^\infty|R(s)|\,ds<\infty$.

The characteristic equation of $A_0+V(s)$ is

$$(15)\qquad P[\lambda(s)]=-(-1)^m+\prod_{i=1}^{n}\left[\lambda-\frac{n-1}{n}g(s)\right]=0.$$

Similarly as in the proof of Theorem 1 we get that the characteristic roots of (15) may be expressed in the form

$$\lambda_i(s) = \lambda_i + \frac{1}{2}\frac{n-1}{n} g(s) + \gamma_i(s),$$

where $\gamma_i(s) \in L[0, \infty)$ and $\gamma_i(s) \to 0$ as $s \to \infty$. All the other assumptions of Theorem 8.1 in [1] are fulfiled, therefore there exists a fundamental system $x_i(s)$ of (4) such that

$$x_i(s) \exp\left[-\int_{s_0}^{s} \left[\lambda_i + \frac{1}{2}\frac{n-1}{n} g(\xi) + \gamma_i(\xi) \right] d\xi \right] \to p_i .$$

If we put $\xi = \omega(\tau)$ and consider $\gamma_i(s) \in L[0, \infty)$ we get the assertion (14).

4. Corollaries

Corollary 1. *Suppose the assumptions of Theorem 1 are fulfiled. Then the equation* (1) *has a fundamental system* $y_i(t)$, $i = 1, 2, \ldots, n$ *such that*

$$y_i^{(j)}(t) = \lambda_i^j q(t)^{\frac{2j+1-n}{2n}} \times$$
$$\times \exp\left(\lambda_i \int_{t_0}^{t} \left[q(t)^{\frac{1}{n}} - (-1)^m \frac{\lambda_i^k}{n} \frac{p(\tau)}{q(\tau)^{1-\frac{k+1}{n}}} \right] d\tau \right) . (1 + o(1)),$$

where $j = 0, 1, \ldots, n-1$ *and* λ_i *are the roots of* $\lambda^n - (-1)^m = 0$.

Corollary 2. *Suppose the assumptions of Theorem 2 are fulfiled. Then the equation* (1) *has a fundamental system* $y_i(t)$, $i = 1, 2, \ldots, n$ *such that*

(17) $$y_i^{(j)}(t) = \lambda_i^j q(t)^{\frac{2j+1-n}{2n}} \exp\left(\lambda_i \int_{t_0}^{t} q(\tau)^{\frac{1}{n}} d\tau \right) . (1 + o(1)).$$

Proof of Corollary 2. If we put the matrix T into (14) we have

$$dia\left[q(t)^{\frac{n-1}{2n}}, q(t)^{\frac{n-3}{2n}}, \ldots, q(t)^{\frac{n-(2n-1)}{2n}} \right] \times$$
$$\times \exp\left(-\lambda_i \int_{t_0}^{t} q(\tau)\, d\tau\right) . [y_i, y_i', \ldots, y_i^{(n-1)}]^T \to [1, \lambda_i, \ldots, \lambda_i^{n-1}]^T .$$

From this equality evidently follows (17).

If in the Corollary 2 we put $n = 2$ and $p(t) = 0$ we obtain

Corollary 3. *Let* $q(t) > 0$ *and* $q''(t)$ *be continuous on* $[a, \infty)$. *Let*

(18) $$q''(t)\, q(t)^{-3/2} \in L[0, \infty).$$

Then the equation

$$y'' + q(t)\, y = 0$$

has the general solution

(19) $$y(t) \doteq q(t)^{-1/4}[\cos(\int_{t_0}^{t} q(\tau)^{1/2}\,d\tau)(c_1 + o(1)) + \sin(\int_{t_0}^{t} q(\tau)^{1/2}\,d\tau)(c_2 + o(1))]$$

and for $y'(t)$ it yields

(20) $$y'(t) = q(t)^{1/4}[-\sin(\int_{t_0}^{t} q(\tau)^{1/2}\,d\tau)(c_1 + o(1)) + \cos(\int_{t_0}^{t} q(\tau)^{1/2}\,d\tau)(c_2 + o(1))].$$

A. Wintner [12] proved the assertions (19) and (20) under the conditions

(21) $$\int_0^\infty q(t)^{1/2}\,dt = \infty \quad \text{and} \quad \int_0^\infty \left| \frac{5q'(t)^2}{16q(t)^3} - \frac{q''(t)}{4q(t)^2} \right| q(t)^{1/2}\,dt < \infty.$$

To compare the assumptions (21) and (18) we easily verify that (18) implies (21), so Wintner theorem is a little general. However the Theorems 1 and 2 give other asymptotic formulae for differential equations of the second order.

If in Theorems 1 and 2 we put $n = 3$, resp. $n = 4$ and $k = 1$ we obtain the results of the papers [8] and [9].

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