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OSCILLATORY BEHAVIOR OF ITERATIVE LINEAR ORDINARY DIFFERENTIAL EQUATIONS DEPENDS ON THEIR ORDER

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Abstract. Each iterative ordinary linear homogeneous differential equation of an odd order has nonvanishing solutions regardless to the oscillatory behavior of the iterated second order equation, whereas each solution of any iterative equation of an even order oscillates if the iterated second order equation is oscillatory.

Key words. Ordinary linear homogeneous iterative differential equations, zeros, disconjugacy, oscillation.

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1. Introduction

It is known that each iterative linear differential equation of the third order

$$y''' + 4p(x)\,y' + 2p'(x)\,y = 0 \qquad \text{on } I \subset \boldsymbol{R},$$

obtained by iterating the second order linear differential equation

$$u'' + p(x)\,u = 0, \qquad p \in C^1(I),$$

has always a nonvanishing solution on the whole interval I, see, e.g. [2].

This fact plays an important rôle in investigation of certain monotonic properties of solutions of the second order linear differential equations, [6].

The aim of this paper is to show in a very simple way that each iterative equation of an odd order has nonvanishing solutions regardless to the oscillatory behavior of the iterated second order differential equation, whereas each solution of any iterative equation of an even order oscillates if the iterated second order equation is oscillatory. For similar results see also V. Šeda [5].

2. Notation and basic facts

Let $n \geqq 2$ be an integer. Consider a second order linear ordinary differential equation of the form

$$(p) \qquad u'' + p(x)\,u = 0, \qquad p \in C^{n-2}(I).$$

Let u_1 and u_2 be two of its linearly independent solutions. Evidently $u_1 \in C^n(I)$, $u_2 \in C^n(I)$, and their Wronskian $W[u_1, u_2](x)$ is a nonzero constant k. Define

$$(1) \qquad y_i(x) := u_1^{n-i}(x) \,.\, u_2^{i-1}(x), \qquad i = 1, \dots, n, x \in I.$$

Since $y_i \in C^n(I)$ for each $i = 1, \dots, n$, and the Wronskian of the n functions, $W[y_1, \dots, y_n](x)$ is nonvanishing everywhere on I (in fact, it is a nonzero constant), there exists a linear homogeneous differential equation

$$(2) \qquad y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \dots + p_0(x)\,y = 0 \qquad \text{on } I,$$

having all the y_i's as its solutions. This equation depends on the original equation (p) but due to (1), it does not depend on a particular choice of solutions u_1 and u_2 of (p).

The equation (2) is called the iterative equation obtained by iterating the iterated equation (p) and shortly denoted by $It_n[p] = 0$. It can be shown that

$$It_3[p] = y''' + 4p(x)\,y' + 2p'(x)\,y = 0,$$
$$It_4[p] = y^{\mathrm{IV}} + 10p(x)\,y'' + 10p'(x)\,y' + (3p''(x) + 9p^2(x))\,y = 0,$$
$$\dots,$$
$$It_n[p] = y^{(n)} + \binom{n+1}{3} p(x)\,y^{(n-2)} + 2\binom{n+1}{4} p'(x)\,y^{(n-3)} + \dots = 0;$$

for p_{n-i}, $i = 2, 3, 4, 5$ and 6 see e.g. [3]. However, we shall not need the explicit form of the coefficients for our considerations.

Let us recall the following results.

Lemma 1 (see O. Borůvka [1]).

Each solution of the second order equation (p) is of the form

$$(3) \qquad k_1 \mid \alpha'(x) \mid^{-1/2} \sin(\alpha(x) + k_2), \qquad x \in I,$$

where k_1 and k_2 are real constants, and α is a function of the class $C^{n+1}(I)$ with $\mathrm{d}\alpha(x)/\mathrm{d}x \neq 0$ on I.

The equation (p) is one-side oscillatory (on I) if and only if $\alpha(I)$ is an interval of the form (a, ∞) or $(-\infty, b)$, a and b finite. The equation is both-side oscillatory (on I) if and only if $\alpha(I) = (-\infty, \infty)$.

The function α in (3) is called the (first) phase of the equation (p) in Borůvka's terminology. Every phase α of (p) satisfies

$$\alpha(x) = \varepsilon \int_{x_0}^{x} (u_1^2(\sigma) + u_2^2(\sigma))^{-1}\,\mathrm{d}\sigma, \qquad \varepsilon = 1 \text{ or } -1,$$

for $x_0 \in I$, where u_1 and u_2 are linearly independent solutions of (p).

Lemma 2 (see [4]).

Consider a linear homogeneous ordinary differential equation of the n-th order defined on an interval I and its n-tuple of linearly independent solutions $y_1, \dots, y_n$ forming coordinates of a curve $\boldsymbol{y}: I \to R^n$ in n-dimensional vector space V_n, $n \geqq 2$.

To each solution $y_c(x) = \sum_{i=1}^{n} c_i y_i(x)$, $c \neq 0$, of the differential equation we assing the hyperplane $H(c) = \{(\eta_1, \dots, \eta_n) \in V_n;\ \sum_{i=1}^{n} c_i \eta_i = 0\}$ in V_n passing the origin $0 \in V_n$. In this correspondence, to each zero of a solution y_c there corresponds the parameter of an intersection of the curve $\boldsymbol{y}$ with the hyperplane $H(\boldsymbol{c})$ including multiplicity, and conversely.

For each C^n-diffeomorphism h of an open interval J onto I and nonvanishing function $f \in C^n(J)$ the coordinates z_i of the curve

$$z(t) = f(t)\, y(h(t)),$$

belong to $C^n(J)$ and their Wronskian is nonvanishing on I.

If $y_c(x)$ again denotes a solution $\sum_{i=1}^{n} c_i y_i(x)$ for $c \neq 0$, and $z_c(t) = \sum_{i=1}^{n} c_i z_i(t)$, $t \in J$, then to each zero x_0 of a solution y_c, i.e., $y_c(x_0) = 0$, there corresponds the zero $t_0 = h^{-1}(x_0)$ of the function (solution) z_c including multiplicity, and conversely.

3. Main result

Theorem. *Let $n \geqq 2$ be an integer and $It_n[p] = 0$ be the iterative linear homogeneous differential equation of the n-th order iterated from the second order equation* (p), *where $p \in C^{n-2}(I)$.*

If n is odd and α denotes any (first) phase of the equation (p), *then*

$$|\alpha'|^{(1-n)/2},$$

$' = \mathrm{d}/\mathrm{d}x$, is a positive solution of $It_n[p] = 0$ on I. Moreover, there are n linearly independent nonvanishing solutions of $It_n[p] = 0$ on I.

If n is even then the oscillatory behavior of solutions of (p) *is the same as the oscillatory behavior of every solution of $It_n[p] = 0$ on I, i.e., if* (p) *is not oscillatory then no solution of $It_n[p] = 0$ is oscillatory on I, if* (p) *is one-side oscillatory on I then each solution of $It_n[p] = 0$ is one-side oscillatory on I and if* (p) *is both-side oscillatory then each nontrivial solution of $It_n[p] = 0$ is both-side oscillatory on I.*

Corollary. *Each iterative equation $It_n[p] = 0$ of and odd order has a nonvanishing solution, every solution of an iterative equation $It_n[p] = 0$ of an even order oscillates on I if the equation* (p) *is oscillatory on I, see V. Šeda* [5].

Proof. Due to the formula (3) in Lemma 1 and the definition (1), an n-tuple of linearly independent solutions y_i of an iterative equation $It_n[p] = 0$ can be written in the form

$$(4) \qquad y_i(x) = |\alpha'(x)|^{(1-n)/2}(\sin\alpha(x))^{n-i}(\cos\alpha(x))^{i-1}, \qquad i = 1, \dots, n,$$

where α is a (first) phase of the equation (p); $\alpha'(x) \neq 0$ on I, $\alpha \in C^{n+1}(I)$.

If n is odd then

$$|\alpha'(x)|^{(1-n)/2} = |\alpha'(x)|^{(1-n)/2}(\sin^2\alpha(x) + \cos^2\alpha(x))^{(n-1)/2} =$$
$$= |\alpha'(x)|^{(1-n)/2}\left[\sin^{n-1}\alpha(x) + \binom{(n-1)/2}{1}\sin^{n-3}\alpha(x)\cos^2\alpha(x) + \right.$$
$$\left. + \dots + \binom{(n-1)/2}{(n-3)/2}\sin^2\alpha(x)\cos^{n-3}\alpha(x) + \cos^{n-1}\alpha(x)\right] =$$
$$= y_1(x) + \binom{(n-1)/2}{1}y_3(x) + \dots + \binom{(n-1)/2}{(n-3)/2}y_{n-2} + y_n(x)$$

is a solution of $It_n[p] = 0$. Evidently, it does not vanish on I.

For $h := \alpha$, $f := |\alpha'|^{-1/2}$ and y_i given by the formula (4), Lemma 2 states that parameters of the intersections of the hyperplane $H(c)$ passing the origin with curve z formed by the coordinates

$$z_i(t) = \sin^{n-i} t \cos^{i-1} t, \qquad i = 1, \dots, n,$$

where $t \in \alpha(I)$, establish zeros of the solution y_c of $It_n[p] = 0$.

The set of all points $\{z(t);\ t \in \alpha(I)\}$ is (generally) a subset of a compact (closed and bounded) set U in V_n (with the euclidean norm) of all points $u(t)$ with i-th coordinate $u_i(t) = \sin^{n-i} t \cos^{i-1} t$, $t \in [0, 2\pi]$ (closed interval). For odd n there holds

$$u_1(t) + \binom{(n-1)/2}{1}u_3(t) + \dots + \binom{(n-1)/2}{(n-3)/2}u_{n-2}(t) + u_n(t) =$$
$$= (\sin^2 t + \cos^2 t)^{(n-1)/2} = 1 > 0,$$

hence the hyperplane

$$H\left(1, 0, \binom{(n-1)/2}{1}, 0, \dots, 0, \binom{(n-1)/2}{(n-3)/2}, 0, 1\right) = 0$$

has no common point with the set U. In fact, the compact set U lies in the hyperplane

$$\eta_1 + \binom{(n-1)/2}{1}\eta_3 + \dots + \binom{(n-1)/2}{(n-3)/2}\eta_{n-2} + \eta_n = 1,$$

not passing the origin. Hence there exist n hyperplanes $H(c_i) = 0$ passing the origin with independent vectors c_i, $i = 1, \dots, n$, that have no common point with U. Thereby no hyperplane $H(c_i) = 0$ can intersect the curve $z(t) \in U$ for

$t \in \alpha(I)$. Due to Lemma 2, this fact guarantees the existence of n linearly independent nonvanishing solutions of $It_n[p] = 0$ for each odd n.

Now, let n be even. For each point $u(t_0)$ of the set U, $t_0 \in [0, 2\pi]$, consider the point $u(t_1)$, where $t_1 \in [0, 2\pi]$ and $|t_1 - t_0| = \pi$. The points $u(t_0)$ and $u(t_1)$, are opposite in V_n, i.e.

$$u(t_0) = -u(t_1).$$

Each hyperplane $H(c) = 0$ passing the origin in V_n has a common point, say $u(t_c)$, $t_c \in [0, 2\pi)$, with the set U. In fact, either $u(t_0)$ (and then also $u(t_1)$) lie in the hyperplane $H(c) = 0$, or neither of them is in the hyperplane. In the first case, t_0 or t_1 can be taken for t_c. In the second case the points $u(t_0)$ and $u(t_1)$ lie in the opposite half-spaces of V_n with the boundary hyperplane $H(c) = 0$. Due to continuity of $u(t)$ on the closed interval with end-points t_0 and t_1, the existence of t_c for which $u(t_c)$ belongs to $H(c) = 0$ is established.

Keeping n even consider a hyperplane $H(c) = 0$. There always exists at least one point $u \in U$ lying in the hyperplane. On the other side, we have at most finite number of points of the set U that belong to the hyperplane $H(c) = 0$, otherwise the Wronskian of $u_i(t)$, $i = 1, \ldots, n$, is vanishing on $[0, 2\pi]$ that contradicts to Lemma 2. Now, if the equation (p) is not oscillatory then, according to Lemma 1, the interval $\alpha(I)$ is finite and there exists at most finite number of parameters $t_c \in \alpha(I)$ for which $z(t_c)$ belongs to the hyperplane $H(c) = 0$ for each fixed $c \neq 0$. Thus, due to Lemma 2, each solution of $It_n[p] = 0$ has only finite number of zeros in I. If (p) is one-side oscillatory, then $\alpha(I)$ is of the form (a, ∞) or $(-\infty, b)$, a and b being finite. In each hyperplane $H(c) = 0$ there exists a finite number of points $u(t_c) \in U$, $t_c \in [0, 2\pi)$. Since $z(t) = -z(t + \pi) = z(t + 2\pi)$ for all $t \in \alpha(I)$, and the interval $\alpha(I)$ is one-side unbounded, the solution y_c, corresponding to the hyperplane $H(c) = 0$, is one-side oscillatory due to Lemma 2. Finally, when the equation (p) is both-side oscillatory then $\alpha(I) = (-\infty, \infty)$ and each hyperplane $H(c) = 0$ intersects U in finite number of points but each of the points, say $u(t^*)$, coincides with $z(t^* + 2k\pi)$ for all $k \in Z$, because $t^* + 2k\pi \in \alpha(I)$. Hence, due to Lemma 2, each solution of $It_n[p] = 0$ is both-side oscillatory. Q.E.D.

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