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MATRICES REPRESENTABLE BY DIRECTED GRAPHS

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Abstract. Let $G = (V, E)$ be a directed graph such that $V = \{v_1, v_2, \dots, v_n\}$, $\{d_1^+, d_2^+, \dots, d_k^+\}$ is the set of all outer-degrees and $\{d_1^-, d_2^-, \dots, d_l^-\}$ is the set of all inner-degrees of vertices of G. To the graph G we can assigne a pair of matrices $M_G^+ = [a_{ij}]_{l\times n}$ and $M_G^- = [b_{jm}]_{n\times k}$ as follows: the element a_{ij} is the number of vertices $v \in V$ having the inner-degree d_i^- such that $(v_j, v) \in E$, the element b_{jm} is the number of vertices $v \in V$ having the outer-degree d_m^+ such that $(v, v_j) \in E$. Matrices M_G^+ and M_G^- will be called the out-distribution matrix of G and the in-distribution matrix of G respectively. In this paper we give criteria for graphicity of a pair (M^+, M^-) matrices i.e. necessary and sufficient conditions under which there exists a digraph G such that $M^+ = M_G^+$ and $M^- = M_G^-$. We give also a procedure for constructing a graph realizing a graphic pair (M^+, M^-) and we characterize the set of all graphs on the fixed vertex-set and with the same out-distribution and in-distribution matrices.

Key words. Out-distribution (in-distribution) matrix of the graph, realization of the pair of matrices, graphic pair of matrices, demi-bipartite graph (d. b-graph), switching, pair of sequences realizable by d. b-graph, alternate anti-cycle (a, a-cycle), a, a-cyclic partition of the graph.

Introduction. Let $G = (V, E)$ be a simple graph where $V = \{v_1, v_2, \dots, v_n\}$ and let $\{d_1, d_2, \dots, d_k\}$ be a set of non-negative integers. A matrix $M_G = [a_{ij}]$ where $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n$ will be called the distribution matrix of G iff a_{ij} is the number of vertices of the degree d_i adjacent with the vertex v_j (see [6]). The distribution matrix of a graph G contains informations not only on degrees of vertices but also about degrees of the neighbours of any vertex.

In [6] the necessary and sufficient conditions for a matrix M of non-negative integers were formulated under which M was the distribution matrix of a simple graph. It was also characterized the set of all graphs on the fixed vertex set having the same distribution matrix.

In this paper we solve the same problems for directed graphs, namely:

Let $G = (V, E)$ be a directed graph where $V = \{v_1, v_2, \dots, v_n\}$. Let $D^+(G) = \{d_1^+, d_2^+, \dots, d_k^+\}$ be the set of all outer-degrees of vertices of G and $D^-(G) = \{d_1^-, d_2^-, \dots, d_l^-\}$ be the set of all inner-degrees of vertices of G. We denote by a_{ij} ($i = 1, 2, \dots, l$ and $j = 1, 2, \dots, n$) the number of vertices $v \in V$ having the inner-degree d_i^- such that $(v_j, v) \in E$. Similarily we denote by b_{mj} ($m = 1, 2, \dots, k$) the number of vertices $v \in V$ having the outer-degree d_m^+ such that $(v, v_j) \in E$.

The matrix $M_G^+ = [a_{ij}]$ will be called the out-distribution matrix of G and the matrix $M_G^- = [b_{mj}]$ will be called the in-distribution matrix of G. So to any directed graph G we can assign the pair (M_G^+, M_G^-) of the distribution matrices.

In Section 1 we define precisely the matrices M_G^+ and M_G^-.

A pair (M^+, M^-) of matrices of non-negative integers will be called graphic iff there exists a directed graph G such that $M^+ = M_G^+$ and $M^- = M_G^-$.

In Section 2 we reduce the problem of graphicity of a pair of matrices to the problem of graphicity of a pair of sequences.

In Section 3 we formulate criteria of graphicity of a pair of sequences. We also give an algorithm for constructing a graph realizing a pair of sequences.

In Section 4 we give criteria of graphicity of a pair of matrices.

In Section 5 we present a procedure for constructing a graph realizing a pair (M^+, M^-).

In Section 6 we characterize the set of all realizations of a graphic pair of matrices.

1. The distribution matrices of a directed graph

Let $G = (V, E)$ be a finite directed graph, i.e. V is a non-empty set, $E \subseteq V \times V$. For $v \in V$ we denote:

$$\Gamma_G^+(v) = \{u \in V; (v, u) \in E\}, \qquad \Gamma_G^-(v) = \{u \in V; (u, v) \in E\},$$
$$\deg_G^+(v) = |\Gamma_G^+(v)|, \qquad \deg_G^-(v) = |\Gamma_G^-(v)|,$$
$$D^+(G) = \{\deg_G^+(v); v \in V\}, \qquad D^-(G) = \{\deg_G^-(v); v \in V\}.$$

Assume that for the graph G we have:

$$D^+(G) = \{d_1^+, d_2^+, \ldots, d_k^+\}, \qquad D^-(G) = \{d_1^-, d_2^-, \ldots, d_l^-\}.$$

For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$ we define:

$$V_i^+ = \{v \in V; \deg_G^+(v) = d_i^+\}, \qquad V_j^- = \{v \in V; \deg_G^-(v) = d_j^-\}, \tag{1}$$

$$E_{ij} = \{(u, v) \in E; u \in V_i^+, v \in V_j^-\}, \tag{2}$$

$$t^{+j}(v) = |V_j^- \cap \Gamma_G^+(v)|, \qquad t^{-i}(v) = |V_i^+ \cap \Gamma_G^-(v)|. \tag{3}$$

For a graph G we define two functions, namely out-distribution t_G^+ and in-distribution t_G^- as follows:

$$\begin{aligned} &t_G^+ : V \to N^l, \qquad t_G^- : V \to N^k,\\ &t_G^+(v) = \big(t^{+1}(v), t^{+2}(v), \ldots, t^{+l}(v)\big)^T,\\ &t_G^-(v) = \big(t^{-1}(v), t^{-2}(v), \ldots, t^{-k}(v)\big)^T, \end{aligned} \tag{4}$$

(N is the set of all non-negative integers).

Let $V = \{v_1, v_2, \ldots, v_n\}$. Then we get a pair (M_G^+, M_G^-) of matrices where

$$\begin{aligned} M_G^+ &= [t_G^+(v_1), t_G^+(v_2), \dots, t_G^+(v_n)], \\ M_G^- &= [t_G^-(v_1), t_G^-(v_2), \dots, t_G^-(v_n)]. \end{aligned} \tag{5}$$

M_G^+ is a $(l \times n)$-matrix called the out-distribution matrix of G and M_G^- is a $(k \times n)$-matrix called the in-distribution matrix of G. Then the graph G is called a realization of the pair (M_G^+, M_G^-).

Example 1. Let G be a graph in Fig. 1.

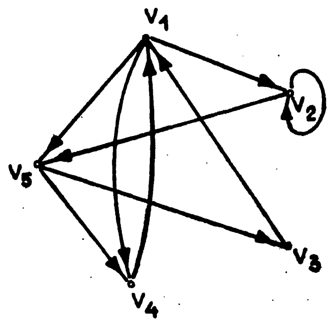

Fig. 1

For the graph G we have:

$$\begin{aligned} &D^+(G) = \{3, 2, 1\}, && D^-(G) = \{2, 1\}, \\ &V_1^+ = \{v_1\}, && V_1^- = \{v_1, v_2, v_4, v_5\}, \\ &V_2^+ = \{v_2, v_5\}, && V_2^- = \ v_3\}, \\ &V_3^+ = \{v_3, v_4\}, && \end{aligned}$$

$$M_G^+ = \begin{bmatrix} 3 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad M_G^- = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. Graphic pair of matrices

Let $M = [\alpha_1, \alpha_2, \dots, \alpha_n]$ be a matrix of non-negative integers such that a column α_i is of the form $\alpha_i = (a_i^1, a_i^2, \dots, a_i^r)^T$ for $i = 1, 2, \dots, n$. Let $M^{(*)}$ denote an $(r \times n)$ matrix with the same columns as in M but ordered as follows:

$$\alpha_i \text{ precedes } \alpha_j \text{ iff } \Big(\sum_{s=1}^{r} a_i^s > \sum_{s=1}^{r} a_j^s\Big) \quad \text{or} \quad \Big(\sum_{s=1}^{r} a_i^s = \sum_{s=1}^{r} a_j^s \text{ and } i < j\Big).$$

For the matrices from Example 1 we have:

$$M_G^{+(*)} = \begin{bmatrix} 3 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad M_G^{-(*)} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let for some matrices M^+, M^- the pair $(M^{+(*)}, M^{-(*)})$ have the form:

$$M^{+(*)} = \begin{bmatrix} t_{11}^{+1} & \dots & t_{1s_1}^{+1} & \dots & t_{i1}^{+1} & \dots & t_{is_i}^{+1} & \dots & t_{k1}^{+1} & \dots & t_{ks_k}^{+1} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ t_{11}^{+l} & \dots & t_{1s_1}^{+l} & \dots & t_{i1}^{+l} & \dots & t_{is_i}^{+l} & \dots & t_{k1}^{+l} & \dots & t_{ks_k}^{+l} \end{bmatrix},$$

(6)

$$M^{-(*)} = \begin{bmatrix} t_{11}^{-1} & \dots & t_{1r_1}^{-1} & \dots & t_{j1}^{-1} & \dots & t_{jr_j}^{-1} & \dots & t_{l1}^{-1} & \dots & t_{lr_l}^{-1} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ t_{11}^{-k} & \dots & t_{1r_1}^{-k} & \dots & t_{j1}^{-k} & \dots & t_{jr_j}^{-k} & \dots & t_{l1}^{-k} & \dots & t_{lr_l}^{-k} \end{bmatrix},$$

where $\sum_{i=1}^{k} s_i = \sum_{j=1}^{l} r_j = n$ and for any $i = 1, 2, \dots, k, j = 1, 2, \dots, l, p = 1, 2, \dots, s_i$, $q = 1, 2, \dots, r_j$ we have:

$$t_{ip}^{+1} + \dots + t_{ip}^{+l} = d_i^+, \qquad d_1^+ > \dots > d_k^+,$$
$$t_{jq}^{-1} + \dots + t_{jq}^{-k} = d_j^-, \qquad d_1^- > \dots > d_l^-.$$

The aim of this paper is to give necessary and sufficient conditions for the pair (M^+, M^-) to be graphic. This problem reduces itself to finding criteria of the pair $(M^{+(*)}, M^{-(*)})$.

First we need some auxiliary notions.

Considere a triple $D = (X, Y, E)$, where X, Y are non-empty sets and $E \subseteq X \times Y$. The triple $D = (X, Y, E)$ we shall call a demi-bipartite graph or briefly a d . b-graph.

From any graph $G = (V, E)$ we can form a d . b-graph (X, Y, E) as follows: For any $v \in V$: $\deg_G^+(v) \neq 0 \Rightarrow v \in X$, $\deg_G^-(v) \neq 0 \Rightarrow v \in Y$,

$$\deg_G^+(v) = \deg_G^-(v) = 0 \Rightarrow v \in X \quad \text{or } v \in Y.$$

Note that the edge-set in G and its d . b-graph is the same, so if G has no isolated vertices then d . b-graph obtained in this way is unique. However to any graph $G = (V, E)$ there corresponds a trivial d . b-graph (V, V, E).

Let $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, b_2, \dots, b_m)$ be two sequences of non-negative integers. A d.b-graph $D = (X, Y, E)$ where $X = \{u_1, u_2, \dots, u_n\}$, $Y = \{v_1, v_2, \dots, v_m\}$ we shall call a realization of the pair of sequences (a, b) iff for any $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ the following conditions are satisfied:

(7)
$$\deg_D^+(u_i) = a_i, \qquad \deg_D^-(v_j) = b_j,$$
$$\deg_D^-(u_i) = 0 \quad \text{for } u_i \notin X \cap Y, \deg_D^+(v_j) = 0 \quad \text{for } v_j \notin X \cap Y.$$

Then the pair (a, b) will be called the pair of demi-degree sequences of D and we denote it by $(a, b)_D$.

The pair (a, b) of sequences of non-negative integers will be called graphic iff there exists a d . b-graph D such that $(a, b) = (a, b)_D$.

Let $G = (V, E)$ be a directed graph,

$$D^+(G) = \{d_1^+, d_2^+, \ldots, d_k^+\}, \qquad D^-(G) = \{d_1^-, d_2^-, \ldots, d_l^-\}.$$

For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$ we define the following subgraphs of G:

$$G_{ij} = (V_i^+ \cup V_j^-, E_{ij}). \tag{8}$$

By (1) and (2) the triple (V_i^+, V_j^-, E_{ij}) is a d.b-graph. In the sequel we shall treat notation (8) and notation $G_{ij} = (V_i^+, V_j^-, E_{ij})$ as equivalent.

It is easy to see that

$$G \bigcup_{\substack{i=1,\ldots,k\\ j=1,\ldots,l}} G_{ij} = \Big(\bigcup_{\substack{i=1,\ldots,k\\ j=1,\ldots,l}} V_i^+ \cup V_j^-, \bigcup_{\substack{i=1,\ldots,k\\ j=1,\ldots,l}} E_{ij}\Big). \tag{9}$$

Lemma 1. *If a graph $G = (V, E)$ is a realization of a pair of matrices $(M^{+(*)}, M^{-(*)})$ of the form (6), then for any $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$ the d.b-graph G_{ij} is a realization of a pair of sequences (t_i^{+j}, t_j^{-i}), where $t_i^{+j} = (t_{i1}^{+j}, \ldots, t_{is_i}^{+j})$, $t_j^{-i} = (t_{j1}^{-i}, \ldots, t_{jr_j}^{-i})$.*

Proof. Let $G_{ij} = (V_i^+, V_j^-, E_{ij})$. Put

$$M_i^{+(*)} = \begin{bmatrix} t_{i1}^{+1} & \ldots & t_{is_i}^{+1} \\ \vdots & & \vdots \\ t_{i1}^{+l} & \ldots & t_{is_i}^{+l} \end{bmatrix}, \qquad M_j^{-(*)} = \begin{bmatrix} t_{j1}^{-1} & \ldots & t_{jr_j}^{-1} \\ \vdots & & \vdots \\ t_{j1}^{-k} & \ldots & t_{jr_j}^{-k} \end{bmatrix}. \tag{10}$$

The columns of $M_i^{+(*)}$ are out-distributions of vertices of V_i^+; the columns of $M_j^{-(*)}$ are in-distributions of vertices of V_j^-.

Let

$$V_i^+ = \{v_{i1}, \ldots, v_{is_i}\}, \qquad V_j^- = \{w_{j1}, \ldots, w_{jr_j}\}$$

and

$$M_i^{+(*)} = [t_G^+(v_{i1}), \ldots, t_G^+(v_{is_i})], \qquad M_j^{-(*)} = [t_G^-(w_{j1}), \ldots, t_G^-(w_{jr_j})].$$

By (3), (4) and (7) we infer that the graph G_{ij} is a realization of the pair of sequences

$$((t_{i1}^{+j}, \ldots, t_{is_i}^{+j}), (t_{j1}^{-i}, \ldots, t_{jr_j}^{-i})).$$

Lemma 2. *Let $(M^{+(*)}, M^{-(*)})$ be a pair of matrices of the form (6). Let*

$$V = \{v_{11}, \ldots, v_{1s_1}, \ldots, v_{i1}, \ldots, v_{is_i}, \ldots, v_{k1}, \ldots, v_{ks_k}\}$$

and $X_i = \{v_{i1}, \ldots, v_{is_i}\}$ for *$i = 1, 2, \ldots, k$. Let for $j = 1, 2, \ldots, l$ Y_j be a set such that $\bigcup_{j=1,\ldots,l} Y_j = V$, $Y_{j_1} \cap Y_{j_2} = \emptyset$ for $j_1 \neq j_2$, $j_1, j_2 \in \{1, 2, \ldots, l\}$. Further let $H_{ij} = (X_i, Y_j, F_{ij})$ be a d.b-graph realizing a pair (t_i^{+j}, t_j^{-i}) for $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$. Then the graph $H = \bigcup_{\substack{i=1,\ldots,k\\ j=1,\ldots,l}} H_{ij}$ is a realization of a pair of matrices $(M^{(*)}, M^{-(*)})$.*

Proof. Let for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$ $H_{ij} = (X_i, Y_j, F_{ij})$, $X_i = \{v_{i1}, \ldots, v_{is_i}\}$, $Y_j = \{w_{j1}, \ldots, w_{jr_j}\}$, $Y_j \subseteq V$, $Y_{j_1} \cap Y_{j_2} = \emptyset$ for $j_1 \neq j_2$.

Then

$$\bigcup_{i=1,\dots,k} X_i = \bigcup_{j=1,\dots,l} Y_j = V.$$

Let us fix $v \in V$. Let $v \in X_{i_0}$ and $v \in Y_{j_0}$. So $v = v_{i_0 p} = w_{j_0 q}$ for some $p \in \{1, 2, \dots, s_{i_0}\}$ and $q \in \{1, 2, \dots, r_{j_0}\}$. Observe that v is a vertex of all graphs $H_{i_0 j}$ for $j = 1, 2, \dots, l$, is a vertex of all graphs $H_{i j_0}$ for $i = 1, 2, \dots, k$ and v does not belong to others.

By assumptions we have:

$$\deg^+_{H_{i_0 j}}(v_{i_0 p}) = t^{+j}_{i_0 p}, \quad \deg^-_{H_{i j_0}}(w_{j_0 q}) = t^{-i}_{j_0 q}.$$

Hence and by the fact that the sets F_{ij} are mutually disjoint, we get:

$$t^+_H(v_{i_0 p}) = (t^{+1}_{i_0 p}, \dots, t^{+l}_{i_0 p})^T,$$
$$t^-_H(w_{j_0 q}) = (t^{-1}_{j_0 q}, \dots, t^{-k}_{j_0 q})^T.$$

Since the sequences $(t^{+1}_{i_0 p}, \dots, t^{+l}_{i_0 p})$ and $(t^{-1}_{j_0 q}, \dots, t^{-k}_{j_0 q})$ are columns of $M_i^{+(*)}$ and $M_j^{-(*)}$ of the form (10), the proof is finished.

3. Criteria of graphicity of a pair of sequences

From Lemmas 1 and 2 it follows that the problem of graphicity of a pair of matrices $(M^{+(*)}, M^{-(*)})$ reduces itself to the problem of graphicity of pairs of sequences. We start to solve this problem.

We define some operations on d . b-graphs.

Let $D = (X, Y, E)$ be a d . b-graph realizing of a pair of sequences $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, b_2, \dots, b_m)$ and let $X = \{u_1, u_2, \dots, u_n\}$, $Y = \{v_1, v_2, \dots, v_m\}$.

a) Assume that for some $i \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, m\}$ we have $u_i, v_j \notin X \cap Y$.

Put

$$D_{J(u_i, v_j)} = (X, Y', E') \quad \text{where } Y' = (Y \setminus \{v_j\}) \cup \{u_i\}$$

and E' is obtained from E by substituting any arc (u, v_j) by (u, u_i) for every $u \in \Gamma^-_D(v_j)$.

Then we say that the graph $D_{J(u_i, v_j)}$ arises from the graph D by joining vertices u_i and v_j.

b) Assume that for some $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, m\}$ we have $u_i \in X \cap Y$, $\deg^+(u_i) = a_i$, $\deg^-(u_i) = b_j$, $v_j \notin Y$.

Put

$$D_{u_i/v_j} = (X, Y'', E'') \quad \text{where } Y'' = (Y \setminus \{u_i\}) \cup \{v_j\}$$

and E'' is obtained from E by substituting any arc (u, u_i) by (u, v_j) for every $u \in \Gamma^-_D(u_i)$.

Then we say that the graph D_{u_t/v_j} arises from the graph D by splitting the vertex u_t into vertices u_i and v_j.

c) Assume that (u_i, v_j, u_k, v_l) is a sequence of vertices of the graph D such that:

1° $u_i, u_k \in X, v_j, v_l \in Y, u_i \neq u_k$ and $v_j \neq v_l$,

2° $(u_i, v_j), (u_k, v_l) \in E$ and $(u_i, v_l), (u_k, v_j) \notin E$.

Put

$$D_{(u_i, v_j, u_k, v_l)} = (X, Y, E'''),$$

where $E''' = (E \setminus \{(u_i, v_j), (u_k, v_l)\}) \cup \{(u_i, v_l), (u_k, v_j)\}$.

Then we say briefly that the graph $D_{(u_i, v_j, u_k, v_l)}$ is a $(\overrightarrow{*})$-switching of D.

Note that the graphs $D_{J(u_i, v_j)}$, D_{u_i/v_j}, $D_{(u_i, v_j, u_k, v_l)}$ are d . b-graphs realizing the pair (a, b).

Lemma 3. *Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m)$ be sequences of non-negative integers and $s = \min\{n, m\}$. If the pair (a, b) is graphic, then there exists a sequence $D_0, D_1, \ldots, D_s$ of d . b-graphs realizing the pair (a, b) and such that for $r \in \{0, 1, \ldots, s\}$ $|V(D_r)| = m + n - r$, where $V(D_r)$ denotes the set of all vertices of D_r.*

Proof. Let $D = (X, Y, E)$ be a realization of the pair (a, b). The sequence $D_0, D_1, \ldots, D_s$ we form as follows.

If $X \cap Y = \emptyset$, then $D_0 = D$.

If $X \cap Y \neq \emptyset$, then we obtain the graph D_0 by splitting any vertex of $X \cap Y$ into two.

Put

$$D_0 = (X_0, Y_0, E_0),$$

$$X_0 = \{u_1, u_2, \ldots, u_n\}, \qquad Y_0 = \{v_1, v_2, \ldots, v_m\}.$$

Note that for D_0 we have: $X_0 \cap Y_0 = \emptyset$, $\deg^+(u_i) = a_i$, $\deg^-(u_i) = 0$ for $i = 1, 2, \ldots, n$, $\deg^+(v_j) = 0$, $\deg^-(v_j) = b_j$ for $j = 1, 2, \ldots, m$.

Let for some $r \in \{1, 2, \ldots, s\}$ $D_r = (X_r, Y_r, E_r)$ be a d . b-graph obtained from D_0 by joining exactly $2r$ vertices. Then the sequence $D_0, D_1, \ldots, D_s$ satisfies the required conditions.

Obviously for $n, m > 1$ there can exist many d . b-graphs D_r since in D_0 we can join arbitrary r vertices of X_0 with arbitrary r vertices of Y_0.

By the method used in the proof of the Lemma 3 we have

Corollary 1. Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m)$ be a pair of sequences of non-negative integers. The pair (a, b) is graphic iff the sequence

$$((0, b_1), \ldots, (0, b_m), (a_1, 0), \ldots, (a_n, 0))$$

of pairs is realizable by a directed graph.

Lemma 4. *Let* $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m)$ *and* $b_1 \geqq b_2 \geqq \ldots \geqq b_m$. *Then the pair* (a, b) *is realizable by a d. b-graph iff the following conditions* (11) *and* (12) *are satisfied:*

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j, \tag{11}$$

$$\sum_{j=1}^{k} b_j \leqq \sum_{i=1}^{n} \min\{k, a_i\}, \quad \text{where} \quad k = 1, 2, \ldots, m. \tag{12}$$

Proof. This result follows from Corollary 1 in this paper and Theorem 1 Chapter 6 in [1].

The recursive Havel—Hakimi's criterium for a sequence to be graphic was proved for simple graphs ([4], [2]). However in this paper we need such criterium for realizability of a pair of sequences by demi-bipartite graphs. Now we start to determine such criterium, using a similar idea as that of Havel—Hakimi's (see [3]).

Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m)$ be non-increasing sequences of non-negative integers such that $a_1 \leqq m$.

Denote

$$\operatorname{red}(a, b) = ((0, a_2, \ldots, a_n), (b_1 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, \ldots, b_m)).$$

Lemma 5. *A pair* (a, b) *is realizable by a d. b-graph iff the pair* red (a, b) *is.*

Proof. Let $G' = (X', Y', E')$ where $X' = \{u_1, u_2, \ldots, u_n\}$, $Y' = \{v_1, v_2, \ldots, v_m\}$ realizes the pair red (a, b). We form a d. b-graph $G = (X, Y, E)$ where $X = X'$, $Y = Y'$, $E = E' \cup \{(u_1, v_k)\}_{k \in \{1, 2, \ldots, a_1\}}$. Obviously G realizes the pair (a, b).

Let $G = (X, Y, E)$ realizes the pair (a, b) and $X = \{u_1, u_2, \ldots, u_n\}$, $Y = \{v_1, v_2, \ldots, v_m\}$. We show that from G we can obtain a d. b-graph $H = (X, Y, F)$ realizing the pair (a, b) and such that $\{(u_1, v_k)\}_{k \in \{1, 2, \ldots, a_1\}} \subset F$. Let $G \neq H$. Then there exists $r \leqq a_1$ such that $(u_1, v_r) \notin E$. So there exist $s > a_1$ such that $(u_1, v_s) \in E$. Since $\deg_G^-(v_r) \geqq \deg_G^-(v_s)$, so there exists $t > 1$ such that $(u_t, v_r) \in E$ and $(u_t, v_s) \notin E$. Thus we can apply the following $(\overset{\rightarrow}{*})$-switching:

$$G_1 = G_{(u_1, v_s, u_t, v_r)}.$$

The same procedure as for G we can repeat for G_1 and after a finite number of steps we get H.

Now we form a d. b-graph $G' = (X', Y', F')$ such that $X' = X$, $Y' = Y$, $F' = F \setminus \{(u_1, v_k)\}_{k \in \{1, 2, \ldots, a_1\}}$. The d. b-graph G' realizes the pair red (a, b).

By Lemma 5 the following Algorithms 1 and 2 are correct.

Algorithm 1 (for graphicity of a pair of sequences)

Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m)$ be non-increasing sequences of non-negative integers.

```
IF a_1 ≦ m THEN
  REPEAT
    (a^1, b^1): = red (a, b);
    a: = sort a^1; {sort a^1 decreasing}
    b: = sort b^1;
  UNTIL (a^1 is the zero-sequence OR b^1 is the zero-sequence OR there is a negative
        element in b^1)
  IF a and b are zero-sequences THEN (a, b) is graphic;
  ELSE (a, b) is not graphic;
ELSE (a, b) is not graphic;
```

Algorithm 2 (for constructing d. b-graph realizing a pair of sequences)

Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m)$ be non-increasing sequences of non-negative integers and let the pair (a, b) be graphic.

The required realization of the pair (a, b) is a triple $G = (X, Y, E)$ such that $X = \{u_1, u_2, \ldots, u_n\}$, $Y = \{v_1, v_2, \ldots, v_m\}$, $E = \bigcup\limits_{k=1,\ldots,n} E_k$, where $E_k = \{(u_k, v_j)\}_{j \in T_k}$. We give the procedure for finding the set T_k.

```
FOR k: = 1 TO n DO
  BEGIN
    l: = 0 {the number of found positive integers in b}
    j: = 1 {index of element in b}
    T_k: = ∅
  WHILE l < a_k DO
    BEGIN
      IF b_j >0 THEN
      BEGIN
        T_k: = T_k ∪ {j};
        b_j: = b_j − 1;
        l: = l + 1
      END
    j: = j + 1
  END
END
```

4. Criteria for graphicity of a pair of matrices (M^+, M^-)

Let $c = (c_1, c_2, \ldots, c_k)$ be a sequence of non-negative integers. Denote $\underline{c} = (\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_k)$, where $(\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_k)$ is a permutation of c such that $\underline{c}_1 \geqq \underline{c}_2 \geqq \ldots \geqq \underline{c}_k$.

Theorem 1. *Let (M^+, M^-) be a pair of matrices of non-negative integers such* *that the pair* $(M^{+(*)}, M^{-(*)})$ *is of the form* (6). *Then the pair* (M^+, M^-) *is graphic iff for any* $i = 1, 2, \ldots, k$ *and* $j = 1, 2, \ldots, l$ *the following conditions hold:*

(i) $$\sum_{r=1}^{s_i} t_{ir}^{+j} = \sum_{p=1}^{r_j} t_{jp}^{-i},$$

(ii) $$\sum_{r=1}^{m} t_{jr}^{-i} \leqq \sum_{r=1}^{s_i} \min\{m, t_{ir}^{+j}\} \qquad \text{for} \qquad m = 1, 2, \ldots, r_j.$$

Proof. ⇒ Assume that G realizes the pair (M^+, M^-), so the pair $(M^{+(*)}, M^{-(*)})$ too. By Lemma 1 the subgraphs G_{ij} realize the pairs of sequences (t_i^{+j}, t_j^{-i}) for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$. From Lemma 4 it follows that the conditions (i) and (ii) are satisfied.

⇐ By Lemma 4 for any $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$ the pair (t_i^{+j}, t_j^{-i}) is realizable by a d. b-graph. In view of Lemma 3 as realizations of (t_i^{+j}, t_j^{-i}) we can take the d. b-graphs H_{ij} from Lemma 2. Thus the d. b-graph $H = \bigcup_{\substack{i=1,\ldots,k \\ j=1,\ldots,l}} H_{ij}$ is a realization of the pair $(M^{+(*)}, M^{-(*)})$, so of the pair (M^+, M^-) as well.

Theorem 2. *A pair of matrices* $(M^{+(*)}, M^{-(*)})$ *of the form* (6) *is graphic iff for any* $i = 1, 2, \ldots, k$ *and* $j = 1, 2, \ldots, l$ *the pair* $\operatorname{red}(t_i^{+j}, t_j^{-i})$ *is realizable by a d. b-graph.*

The proof follows from Lemmas 1, 2, 3, 5 and is similar to that of Theorem 1.

5. Construction of a graph realizing a pair (M^+, M^-)

Let us consider a graphic pair (M^+, M^-) of matrices of non-negative integers where M^+ is an $(l \times n)$-matrix and M^- is a $(k \times n)$-matrix.

Let $G = (V, E)$ be a graph we look for.

A procedure of finding G consists of the following 5 steps.

1. We denote

$$V = \{v_1, v_2, \ldots, v_n\},$$
$$M^+ = [t_G^+(v_1), t_G^+(v_2), \ldots, t_G^+(v_n)],$$
$$M^- = [t_G^-(v_1), t_G^-(v_2), \ldots, t_G^-(v_n)].$$

2. We form matrices $M^{+(*)}$ and $M^{-(*)}$ from (6).
3. We write down the sets V_i^+, V_j^- for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$.
4. We construct the d. b-graphs $G_{ij} = (V_i^+, V_j^-, E_{ij})$ realizing pairs (t_i^{+j}, t_j^{-i}) for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$.
5. We form the graph $G = \bigcup_{\substack{i=1,2,\ldots,k \\ j=1,2,\ldots,l}} G_{ij}$.

Observe that the d. b-graphs G_{ij} and consequently the graph G need not be uniquely determined.

Example 2. Let (M^+, M^-) be a pair of matrices from Example 1. We shall construct a realization $G = (V, E)$ of (M^+, M^-).

1. Let $V = \{v_1, v_2, v_3, v_4, v_5\}$,

$$M^+ = [t_G^+(v_1), \dots, t_G^+(v_5)],$$
$$M^- = [t_G^-(v_1), \dots, t_G^-(v_5)].$$

2. $$M^{+(*)} = \left[\begin{array}{c|cc|cc} 3 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] = [t_G^+(v_1), t_G^+(v_2), t_G^+(v_5), t_G^+(v_3), t_G^+(v_4)],$$

$$M^{-(*)} = \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{array}\right] = [t_G^-(v_1), t_G^-(v_2), t_G^-(v_4), t_G^-(v_5), t_G^-(v_3)].$$

3. $V_1^+ = \{v_1\}$, $\quad V_1^- = \{v_1, v_2, v_4, v_5\}$,
$V_2^+ = \{v_2, v_5\}$, $\quad V_2^- = \{v_3\}$.
$V_3^+ = \{v_3, v_4\}$.

4. $t_1^{+1} = (3)$, $\quad t_1^{+2} = (0)$, $\quad t_2^{+1} = (2, 1)$,
$t_1^{-1} = (0, 1.1, 1)$, $\quad t_2^{-1} = (0)$, $\quad t_1^{-2} = (0, 1, 1, 1)$,
$t_2^{+2} = (0, 1)$, $\quad t_3^{+1} = (1, 1)$, $\quad t_3^{+2} = (0, 0)$,
$t_2^{-2} = (1)$, $\quad t_1^{-3} = (2, 0, 0, 0)$, $\quad t_2^{-3} = (0)$.

The d. b-graphs G_{ij} are in Fig. 2.

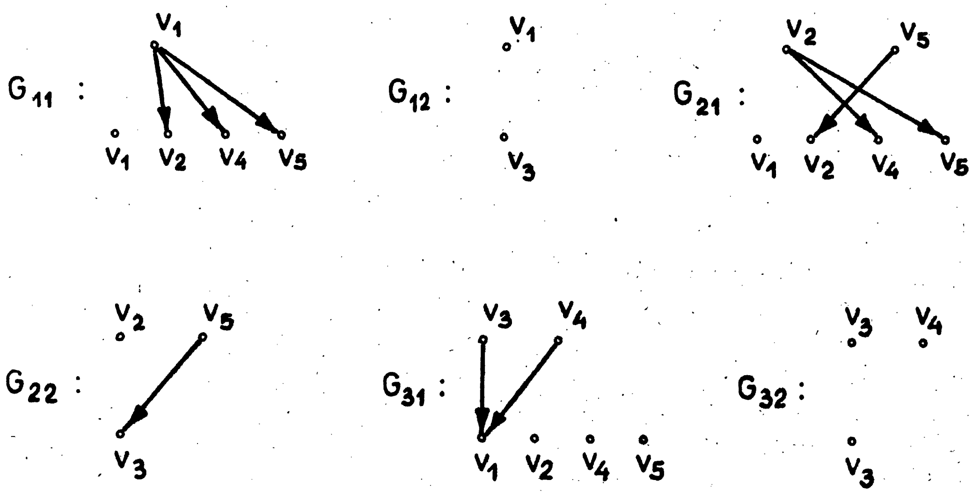

Fig. 2

5. $G = \bigcup_{\substack{i=1,2,3\\ j=1,2}} G_{ij}$. The graph G is in Fig. 3.

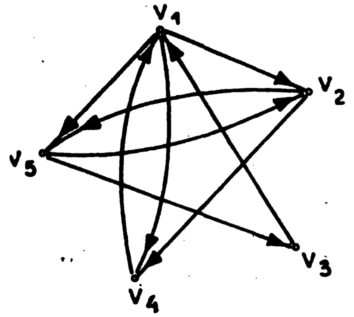

Fig. 3

6. The set of all realizations of a pair of matrices $(M^{+(*)}, M^{-(*)})$

Let $(M^{+(*)}, M^{-(*)})$ be a pair of matrices of the form (6). Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set. Denote by $R_V(M^{+(*)}, M^{-(*)})$ the set of all labeled directed graphs $G = (V, E)$ realizing the pair $(M^{+(*)}, M^{-(*)})$.

For the set $R_V(M^{+(*)}, M^{-(*)})$ we have similar properties as for the set $R_V(M^*)$ (see [6]).

Any of the operations $\overrightarrow{(*)}$-switching in a d. b-graph G_{ij} for $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, l\}$ will be called a $\overrightarrow{(*)}$-switching in G.

Let $G = (V, E)$, $H = (V, F)$ be directed graphs.

Denote $G \dot{-} H = (V, E \dot{-} F)$ where $\dot{-}$ is the symmetric difference.

Let c be a sequence of vertices of the graph $G \dot{-} H$ having the following form:

$$c = (u_1 w_1 u_2 w_2, \ldots, u_m w_m u_{m+1}), \tag{13}$$

where for $s, t \in \{1, 2, \ldots, m\}$ $u_{m+1} = u_1$, $(u_s, w_s) \in E$, $(u_{s+1}, w_s) \in F$ and for $s \neq t$: $(u_s, w_s) \neq (u_t, w_t)$, $(u_{s+1}, w_s) \neq (u_{t+1}, w_t)$.

The sequence c of the form (13) will be called an alternate anti-cycle or briefly a. a-cycle.

Lemma 6. *Let $G, H \in R_V(M^{+(*)}, M^{-(*)})$, $G = (V, E)$ and $H = (V, F)$. Then for $i = 1, 2, \ldots, k, j = 1, 2, \ldots, l$ any non-trivial component of the graph $G_{ij} \dot{-} H_{ij} = (V_i^+ \cup V_j^-, E_{ij} \dot{-} F_{ij})$ can be considered as an a. a-cycle of the form* (13) *where $u_s \in V_i^+$, $w_s \in V_j^-$, $(u_s, w_s) \in E_{ij} \backslash F_{ij}$ and $(u_{s+1}, w_s) \in F_{ij} \backslash E_{ij}$ for $s = 1, 2, \ldots, m$.*

Proof. For $u \in V_i^+$ denote $\Gamma_{ij}^+(u, G) = \{w \in V_j^- : (u, w) \in E_{ij}\}$, $\Gamma_{ij}^+(u, H) = \{w \in V_j^- : (u, w) \in F_{ij}\}$ where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$. Since $t_G^+(u) = t_H^+(u)$, so there exists an arc $x' \in E_{ij} \backslash F_{ij}$ with initial point in u iff there exists an arc $x'' \in F_{ij} \backslash E_{ij}$ with initial point in u. So $|\Gamma_{ij}^+(u, G)| = |\Gamma_{ji}^+(u, H)|$. Analogously $|\Gamma_{ij}^-(w, G)| = |\Gamma_{ij}^-(w, H)|$ for $w \in V_j^-$. Hence it follows that if $\Gamma_{ij}^+(u, G) \neq \Gamma_{ij}^+(u, H)$ then u belongs to a cycle of the form (13). The same holds for the vertex w.

Let $G, H \in R_V(M^{+(*)}, M^{-(*)})$, $G = (V, E)$, $H = (V, F)$ and for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$ $G_{ij} = (V_i^+ \cup V_j^-, E_{ij})$, $H_{ij} = (V_i^+ \cup V_j^-, F_{ij})$. Denote by $\mathscr{C}_{ij}$ a set of all alternate anticycles formed from the arcs of the graph $G_{ij} \dot{-} H_{ij}$ in this way that every arc of $G_{ij} \dot{-} H_{ij}$ belongs exactly to one cycle.

The set $\mathscr{C}_{ij}$ will be called an a . a-cyclic partition of the graph $G_{ij} \dot{-} H_{ij}$.

The number

$$\delta(G_{ij}, H_{ij}, \mathscr{C}_{ij}) = \frac{1}{2} |E(G_{ij} \dot{-} H_{ij})| - |\mathscr{C}_{ij}|$$

will be called the distance of the graphs G_{ij} and H_{ij} with respect to the set $\mathscr{C}_{ij}$.

Lemma 7. *Let $G, H \in R_V(M^{+(*)}, M^{-(*)})$, $G = (V, E)$, $H = (V, F)$ and for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$ we have: $G_{ij} = (V_i^+ \cup V_j^-, E_{ij})$, $H_{ij} = (V_i^+ \cup V_j^-, F_{ij})$ and $\mathscr{C}_{ij}$ is an a.a-cyclic partition of the graph $G_{ij} \dot{-} H_{ij}$. Further let $\delta(G_{ij}, H_{ij}, \mathscr{C}_{ij}) = p$ and $p > 0$. Then there exists a sequence $G_{ij} = G_{ij}^0, G_{ij}^1, \ldots, G_{ij}^m = H_{ij}$ of graphs and a sequence $\mathscr{C}_{ij}^1, \ldots, \mathscr{C}_{ij}^m$ of a.a-cyclic partitions of the graphs $G_{ij}^1 \dot{-} H_{ij}, \ldots, G_{ij}^m \dot{-} H_{ij}$ respectively – such that $m \leq p$ and for any $r = 1, 2, \ldots, m$ the following two conditions are satisfied:*

$$G_{ij}^r \text{ is a } \overrightarrow{(*)} \text{ switching of } G_{ij}^{r-1},$$

$$\delta(G_{ij}^r, H_{ij}, \mathscr{C}_{ij}^r) < \delta(G_{ij}^{r-1}, H_{ij}, \mathscr{C}_{ij}^{r-1}).$$

Proof. The proof of the Lemma 7 is analogous to the proof of the Lemma 4 from [6]. Here we list the differences:

In the Case 1 instead of $u_1 \neq w_2$ and $\{u_1, w_2\} \notin E_{ij}$ we consider the case: $(u_1, w_2) \notin E_{ij}$.

In the Case 2 instead of $u_1 = w_2$ or $\{u_1, w_2\} \in E_{ij}$ we consider the case: $(u_1, w_2) \in E_{ij}$.

In the Case 2.1 we have here: $(u_1, w_{s-1}) \in E_{ij}$.

We omit the Case 2.2.

Theorem 3. *Let $G = (V, E)$ and $H = (V, F)$ be two different realizations of a pair $(M^{+(*)}, M^{-(*)})$ of matrices of the form (6). Then there exists a sequence $G = G^0, G^1, \ldots, G^r = H$ of graphs belonging to $R_V(M^{+(*)}, M^{-(*)})$ such that G^{s+1} is a $\overrightarrow{(*)}$-switching of G^s for $s \in \{0, 1, \ldots, r-1\}$.*

Proof. We decompose the graph $G \dot{-} H$ into the subgraphs $G_{ij} \dot{-} H_{ij}$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$. Next we apply the Lemma 7 and we get the sequence $G_{ij} = G_{ij}^0, G_{ij}^1, \ldots, G_{ij}^{m(i,j)} = H_{ij}$ for any graph $G_{ij} \dot{-} H_{ij}$. Then we order all graphs as follows.

$$G^u_{ij} \prec G^w_{mn} \Leftrightarrow (i, j, u) \prec_l (m, n, w),$$

where $\prec_l$ denotes the lexicographic order.

Remark 1. The statement of Theorem 3 can be also derived from [5, Th. 1] but it is not a simple consequence.

Corollary 2. The set $R_V(M^{+(*)}, M^{-(*)})$ can be generated by a single graph $G \in R_V(M^{+(*)}, M^{-(*)})$ using $\overrightarrow{(*)}$-switching operations finitely many times.

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