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Josef Kalas

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# SOME RESULTS ON THE ASYMPTOTIC BEHAVIOUR OF THE EQUATION $\dot{z} = f(t, z)$ WITH A COMPLEX-VALUED FUNCTION $f$

JOSEF KALAS, Brno

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**Abstract.** Asymptotic properties of the solutions of an equation $\dot{z} = f(t, z)$ with a complex-valued function $f$ are studied. The technique of the proofs of results is based on the modified Ljapunov function method. The applicability of results is illustrated by an example.

**Key words.** Asymptotic behaviour, singular point, Ljapunov function.

Consider a differential equation

$$\dot{z} = G(t, z)\left[h(z) + g(t, z)\right], \tag{1}$$

in which $G$ is a real-valued function and $h, g$ are complex-valued functions, $t$ and $z$ being a real and a complex variable, respectively. In [3] we investigated the asymptotic nature of the solutions of (1) under the assumptions that $h$ is holomorphic in a simply connected region $\Omega$, $h(z) = 0 \Leftrightarrow z = 0$, $h^{(j)}(0) = 0$ for $j = 1, \dots, n - 1$, $h^{(n)}(0) \neq 0$, where $n \geqq 2$ is an integer. The purpose of the present paper is to give some further results on the asymptotic behaviour of the equation (1) under the above mentioned assumptions. In the whole paper we use the notation from [2] and [3]. Assume $G \in C(I \times (\Omega - \{0\}))$, $g \in \tilde{C}(I \times (\Omega - \{0\}))$.

**Theorem 1.** *Let* $0 < \vartheta \leqq \lambda_+$. *Suppose that*

(i) *for any* $\tau \geqq t_0$, *the initial value problem* (1), $z(\tau) = 0$, *possesses the unique solution* $z \equiv 0$;

(ii) *there exists a function* $E(t) \in C[t_0, \infty)$ *such that*

$$\sup_{t_0 \leqq t < \infty} \int_{t_0}^{t} E(s)\, \mathrm{d}s = \varkappa < \infty \tag{2}$$

*and*

$$G(t, z) \operatorname{Re}\left\{k h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t) \tag{3}$$

*holds for* $t \geqq t_0$, $z \in K(0, \vartheta)$.

*If a solution* $z(t)$ (1) *satisfies*

$$z(t_1) \in Cl\ K(0, \gamma),$$

*where* $t_1 \geqq t_0$ *and*

$$0 < \beta = \gamma e^{\varkappa} \exp\left[-\int_{t_0}^{t_1} E(s)\,ds\right] < \vartheta,$$

*then*

$$z(t) \in Cl\ K(0, \beta)$$

*for* $t \geqq t_1$.

Proof. Put $\mathcal{M} = \{t \geqq t_1 : z(t) \in K(0, \vartheta)\}$. For $t \in \mathcal{M}$ we have

$$\dot{W}(z) = G(t, z)\,W(z)\,\mathrm{Re}\left\{k h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\}, \tag{4}$$

where $z = z(t)$. By virtue of (3) we get

$$\dot{W}(z(t)) \leqq E(t)\,W(z(t)) \qquad \text{for } t \in \mathcal{M}. \tag{5}$$

Suppose there is a $t^* > t_1$ such that $z(t^*) \in K(\beta, \vartheta)$ and $z(t) \in K(0, \vartheta)$ for $t \in [t_1, t^*]$. The inequality (5) is equivalent to

$$\frac{d}{dt}\left\{W(z(t)) \exp\left[-\int_{t_1}^{t} E(s)\,ds\right]\right\} \leqq 0, \qquad t \in \mathcal{M}.$$

Integrating over $[t_1, t^*]$ we obtain

$$W(z(t^*)) \exp\left[-\int_{t_1}^{t^*} E(s)\,ds\right] - W(z(t_1)) \leqq 0,$$

whence

$$W(z(t^*)) \leqq W(z(t_1)) \exp\left[\int_{t_1}^{t^*} E(s)\,ds\right] \leqq$$

$$\leqq \gamma \exp\left[\varkappa - \int_{t_0}^{t_1} E(s)\,ds\right] \leqq \beta < W(z(t^*)).$$

This contradiction proves $z(t) \in Cl\ K(0, \beta)$ for $t \geqq t_1$.

**Theorem 2.** *Suppose that the hypotheses of Theorem 1 are fulfilled and*

$$\int_{t_0}^{\infty} E(s)\,ds = -\infty. \tag{6}$$

*If a solution* $z(t)$ *of* (1) *satisfies*

$$z(t_1) \in K\left(0, \vartheta e^{-\varkappa} \exp\left[\int_{t_0}^{t_1} E(s)\,ds\right]\right) \cup \{0\}, \tag{7}$$

*where* $t_1 \geqq t_0$, *then to any* $\varepsilon$, $0 < \varepsilon < \lambda_+$, *there is a* $T = T(\varepsilon, t_1) > 0$ *independent of* $z(t)$, *such that*

$$z(t) \in K(0, \varepsilon) \cup \{0\}$$

*for* $t \geqq t_1 + T$.

Proof. Put $\mathcal{M} = \{t \geqq t_1 : z(t) \in K(0, \vartheta)\}$. For $t \in \mathcal{M}$ we get (4), where $z = z(t)$. From Theorem 1 it follows that $z(t) \in K(0, \vartheta) \cup \{0\}$ for $t \geqq t_1$.

Choose $\varepsilon$, $0 < \varepsilon < \lambda_+$. Without loss of generality it may be supposed $\varepsilon < \vartheta$. Pick $T = T(\varepsilon, t_1) > 0$ so that

$$\int_{t_1}^{t} E(s)\,\mathrm{d}s < \ln \frac{\varepsilon}{2\vartheta}$$

for $t \geqq t_1 + T$. We claim that $z(t) \in K(0, \varepsilon) \cup \{0\}$ for $t \geqq t_1 + T$. If it is not the case, there exists a $t^* \geqq t_1 + T$ for which

$$z(t^*) \notin K(0, \varepsilon) \cup \{0\}. \tag{8}$$

The inequality (5) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\{W(z(t)) \exp [-\int_{t_1}^{t} E(s)\,\mathrm{d}s]\} \leqq 0.$$

Since $z(t) \neq 0$ for $t \in [t_1, t^*]$, the integration of this inequality from $t_1$ to $t^*$ gives

$$W(z(t^*)) \exp [-\int_{t_1}^{t^*} E(s)\,\mathrm{d}s] - W(z(t_1)) \leqq 0.$$

Hence

$$W(z(t^*)) \leqq W(z(t_1)) \exp [\int_{t_1}^{t^*} E(s)\,\mathrm{d}s] \leqq \vartheta \frac{\varepsilon}{2\vartheta} = \frac{\varepsilon}{2} < \varepsilon,$$

which contradicts (8) and implies $z(t) \in K(0, \varepsilon) \cup \{0\}$ for $t \geqq t_1 + T$.

**Theorem 3.** *Let the assumptions of Theorem 2 be fulfilled except (6) is replaced by*

$$\int_{s}^{s+t} E(\xi)\,\mathrm{d}\xi \to -\infty \quad \text{as} \quad t \to \infty \tag{9}$$

*uniformly for* $s \in [t_0, \infty)$.

*If a solution* $z(t)$ *of* (1) *satisfies* (7), *where* $t_1 > t_0$, *then to any* $\varepsilon$, $0 < \varepsilon < \lambda_+$, *there is a* $T = T(\varepsilon) > 0$ *independent of* $t_1$ *and* $z(t)$ *such that*

$$z(t) \in K(0, \varepsilon) \cup \{0\}$$

*for* $t \geqq t_1 + T$.

Proof. Because of (9), there exists a $T = T(\varepsilon) > 0$ so that $t - t_1 \geqq T$ implies

$$\int_{t_1}^{t} E(\xi)\,\mathrm{d}\xi = \int_{t_1}^{t_1+(t-t_1)} E(\xi)\,\mathrm{d}\xi < \ln \frac{\varepsilon}{2\vartheta}.$$

The statement follows from the proof of Theorem 2.

Theorems 1–3 have their corresponding analogies (Theorems 1′–3′) for the case when we consider subsets of $K(\infty, \lambda_-) \cup \{0\}$ instead of those of $K(0, \lambda_+) \cup \{0\}$. We shall formulate here only the first of these results:

**Theorem 1′.** *Let $\lambda_- \leqq \vartheta < \infty$. Suppose that*

(i) *for any $\tau \geqq t_0$, the initial value problem* (1), $z(\tau) = 0$, *possesses the unique solution $z \equiv 0$;*

(ii) *there exists an $E(t) \in C[t_0, \infty)$ such that*

$$\sup_{t_0 \leqq t < \infty} \int_{t_0}^{t} E(s)\,ds = \varkappa < \infty \tag{2}$$

*and*

$$-G(t, z)\,\mathrm{Re}\left\{k h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t)$$

*holds for $t \geqq t_0$, $z \in K(\infty, \vartheta)$.*

*If a solution $z(t)$ of* (1) *satisfies*

$$z(t_1) \in Cl\ K(\infty, \gamma),$$

*where $t_1 \geqq t_0$ and*

$$\vartheta < \beta = \gamma e^{-\varkappa} \exp\left[\int_{t_0}^{t_1} E(s)\,ds\right] < \infty,$$

*then*

$$z(t) \in Cl\ K(\infty, \beta)$$

for $t \geqq t_1$.

**Example.** Consider an equation

$$\dot{z} = z^2 q(t, z), \tag{10}$$

where $q \in \tilde{C}(I \times C)$ satisfies locally a Lipschitz condition with respect to $z$. Putting $G(t, z) \equiv 1$, $h(z) = b(z - a)\,z^2$, $g(t, z) = [q(t, z) + b(a - z)]\,z^2$, where $a, b \in C$, $a \neq 0 \neq b$, we can write (10) in the form

$$\dot{z} = G(t, z)\,[h(z) + g(t, z)]. \tag{1}$$

From [2, Example 2] we have $h'(z) = b(3z - 2a)\,z$, $h''(z) = 2b(3z - a)$, $n = 2$, $W(z) = |a|\,|z|\,|z - a|^{-1} \exp\{\mathrm{Re}[-az^{-1}]\}$, $\lambda_+ = \lambda_- = |a|$, $k = a/2$. Supposing that there is an $H(t) \in C(I)$ such that $|q(t, z) + (a - z)\,b| \leqq H(t)\,|z - a|$ for $t \geqq t_0$, $z \in C$, we obtain

$$G(t, z)\,\mathrm{Re}\left\{k h^{(n)}(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq -\mathrm{Re}\,(a^2 b) + |a|^2 H(t).$$

By use of Theorem 1 and Theorem 2 we get the following assertion:

*If there exist $a, b \in C$, $H(t) \in C(I)$ such that $b \neq 0$,*

$$|q(t, z) + (a - z)\,b| \leqq H(t)\,|z - a| \quad \textit{for } t \geqq t_0, z \in C, \tag{11}$$

*and the function*

$$|a|^2 \int_{t_0}^{t} H(\xi)\,d\xi - \mathrm{Re}\,(a^2 b)\,t \quad \textit{is upper bounded on} \quad t_0 \leqq t < \infty, \tag{12}$$

*then every solution $z(t)$ of* (10) *satisfying*

$$|z(t_1)|\,|z(t_1) - a|^{-1} \exp\{\operatorname{Re}[-az^{-1}(t_1)]\} = \tag{13}$$
$$= \omega < e^{-\varkappa} \exp\Big[|a|^2 \int_{t_0}^{t_1} H(s)\,ds - \operatorname{Re}(a^2b)(t_1 - t_0)\Big],$$

*where $t_1 \geqq t_0$ and*

$$\varkappa = \sup_{t_0 \leqq t < \infty} \Big[|a|^2 \int_{t_0}^{t} H(\xi)\,d\xi - \operatorname{Re}(a^2b)(t - t_0)\Big], \tag{14}$$

*is defined for all $t \geqq t_1$, and*

$$|z(t)|\,|z(t) - a|^{-1} \exp\{\operatorname{Re}[-az^{-1}(t)]\} \leqq \omega e^{\varkappa} \exp\Big[-|a|^2 \int_{t_0}^{t_1} H(s)\,ds + \operatorname{Re}(a^2b)(t_1 - t_0)\Big]$$

*holds for $t \geqq t_1$. If, in addition,*

$$\lim_{t\to\infty} \Big[|a|^2 \int_{t_0}^{t} H(\xi)\,d\xi - \operatorname{Re}(a^2b)\,t\Big] = -\infty, \tag{15}$$

*then any solution $z(t)$ of* (10) *satisfying* (13), *where $t_1 \geqq t_0$ and $\varkappa$ is defined by* (14), *fulfils the condition*

$$\lim_{t\to\infty} z(t) = 0.$$

## REFERENCES


[1] Z. Tesařová: *The Riccati differential equation with complex-valued coefficients and application to the equation $x'' + P(t)x' + Q(t)x = 0$*, Arch. Math. (Brno) 18 (1982), 133–143.

[2] J. Kalas: *On a "Liapunov-like" function for an equation $\dot{z} = f(t, z)$ with a complex-valued function f*, Arch. Math. (Brno) 18 (1982), 65–76.

[3] J. Kalas: *Asymptotic nature of solutions of the equation $\dot{z} = f(t, z)$ with a complex-valued function f*, Arch. Math. (Brno) 20 (1984), 83—94.

[4] C. Kulig: *On a system of differential equations*, Zeszyty Naukowe Univ. Jagiellońskiego, Prace Mat., Zeszyt 9, 77 (1963), 37–48.

[5] M. Ráb: *Geometrical approach to the study of the Riccati differential equation with complex-valued coefficients*, J. Diff. Equations 25 (1977), 108–114.



*J. Kalas*
*Department of Mathematics, J. E. Purkyně University*
*662 95 Brno, Janáčkovo nám. 2a*
*Czechoslovakia*