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NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF APPROXIMATE PICARD'S ITERATES FOR NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. Nonlinear boundary value problem in R_n is considered and necessary and sufficient conditions for the convergence of approximate sequence of Picard's iterates to its unique solution are given. The generalized normed (vector norm) linear space and component-wise calculation are used.

Key words. Picard's iterates, approximate sequence, convergence to solution, component-wise, generalized norm space, boundary value problem, spectral radius, error criterion.

1. Introduction

This is in continuation to our work [1] on nonlinear boundary value problems

$$\frac{dx}{dt} = g(x, t) \tag{1.1}$$

$$f(x) = 0, \tag{1.2}$$

where x and $g(x, t)$ are n dimensional vectors and $f(x)$ is an operator from $C(I)$ into R^n, $C(I)$ is the space of all real n vector functions continuous on $I = [a, b]$.

In what follows, a particular equation say $(\alpha \,.\, \beta)$ of [1] will be refered as $(1 \,.\, \alpha, \beta)$.

Besides otherthings, theorem 1.4.1 ensures that the sequence $\{x_m(t)\}$ obtained from the iterative process

$$\begin{gathered} x_{m+1}(t) = H_1[g(x_m(t), t) - A(t)\, x_m(t)] + H_2[L[x_m] \pm f(x_m)], \\ x_0(t) = \bar{x}(t); \qquad m = 0, 1, \ldots \end{gathered} \tag{1.4.1}$$

converges to the unique solution $x^*(t)$ of (1.1), (1.2). However, in practical evaluation of these iterates, only an approximate sequence $\{y_m(t)\}$ is constructed which depends on approximating g and f by some simpler g^* and f^*. Therefore, the computed sequence $\{y_m(t)\}$ satisfies the iterative process

$$
(1.3)\quad \begin{aligned} y_{m+1}(t) &= H_1[g^*(y_m(t), t) - A(t)\, y_m(t)] + H_2[L[y_m] \pm f^*(y_m)], \\ y_0(t) &= x_0(t) = \bar{x}(t); \qquad m = 0, 1, \dots \end{aligned}
$$

In section 2, we shall approximate g and f by g^* and f^* following relative error and absolute error criterion, and present two corresponding results. The important feature of these results is the necessary and sufficient conditions for the convergence of the approximate sequence $\{y_m(t)\}$ to the solution $x^*(t)$ of (1.1), (1.2).

2. Convergence of the Approximate Iterates

As in [1−3], we shall consider the inequalities between two vectors in R^n component-wise whereas between two $n \times n$ matrices element-wise the generalized normed (vector norm) linear space B as $C(I)$, where $\| x \| = (\max_{t \in I} | x_1(t) |, \dots, \dots, \max_{t \in I} | x_n(t) |)$. In (1.1), (1.2) the function $g(x, t)$ is assumed to be continuously differentiable with respect to x in $R^n \times I$ and $g_x(x, t)$ represents the Jacobian matrix of $g(x, t)$ with respect to x; $f(x)$ is continuously differentiable in $C(I)$ and $f_x(x)$ denotes the linear operator mapping $C(I)$ into R^n.

Theorem 2.1. *With respect to* (1.1), (1.2) *we assume that there exists an approximate solution $\bar{x}(t)$ and conditions* (1.i)−(1.iii) *of theorem* 1.4.1 *are satisfied. Further, let for all $x(t) \in \bar{S}(\bar{x}, r)$, the following inequalities* (corresponding to the relative error in approximating g and f by g^* and f^*) *be satisfied*

$$
(2.1)\qquad \| g(x(t), t) - g^*(x(t), t) \| \leqq \Delta_1 \| g(x(t), t) \|,
$$

$$
(2.2)\qquad \| f(x) - f^*(x) \| \leqq \Delta_2 \| f(x) \|,
$$

where Δ_1 and Δ_2 are $n \times n$ nonnegative matrices with $\varrho(\Delta_1)$, $\varrho(\Delta_2) < 1$. We also assume that $\varrho(K_\Delta) < 1$, where

$$
(2.3)\qquad K_\Delta = M_1(E + \Delta_1) M_3 + M_2(E + \Delta_2) M_4 + M_1 \Delta_1 \| A(t) \| + M_2 \Delta_2 \| L \|
$$

and

$$
(2.4)\quad \begin{aligned} r_\Delta = (E - K_\Delta)^{-1} (M_1 \delta_1 + M_2 \delta_2 + M_1 \Delta_1 (E - \Delta_1)^{-1} \| g^*(\bar{x}(t), t) \| + \\ + M_2 \Delta_2 (E - \Delta_2)^{-1} \| f^*(\bar{x}) \|) \leqq r. \end{aligned}
$$

Then,

(1) *all the conclusions* (1)−(5) *of theorem* 1.4.1 *hold,*

(2) *the sequence $\{y_m(t)\}$ obtained from* (1.3) *remains in $\bar{S}(\bar{x}, r_\Delta)$,*

(3) *the sequence $\{y_m(t)\}$ converges to $x^*(t)$ the solution of* (1.1), (1.2) *if and only if*

$$
(1.3)\qquad \lim_{m \to \infty} \| y_{m+1}(t) - H_1[g(y_m(t), t) - A(t)\, y_m(t)] - H_2[L[y_m] \pm f(y_m)] \| = 0
$$

also,

$$\| x^* - y_{m+1} \| \leqq (E - K_0)^{-1} [M_1 \Delta_1 (E - \Delta_1)^{-1} \| g^*(y_m(t), t) \| + \\ + M_2 \Delta_2 (E - \Delta_2)^{-1} \| f^*(y_m) \| + K_0 \| y_{m+1} - y_m \|]. \tag{2.6}$$

Proof. Since $\varrho(K_\Delta) < 1$ implies $\varrho(K_0) < 1$ and obviously $r_\Delta \geqq (E - K_0)^{-1} \cdot (M_1 \delta_1 + M_2 \delta_2)$, the conditions of theorem 1.4.1 are satisfied and part (1) follows.

To prove part (2), we note that $\bar{x}(t) \in \bar{S}(\bar{x}, r_\Delta)$, and hence if $x(t) \in \bar{S}(\bar{x}, r_\Delta)$, it is sufficient to show that $Tx(t) \in \bar{S}(\bar{x}, r_\Delta)$, where

$$Tx(t) = H_1[g^*(x(t), t) - A(t)\, x(t)] + H_2[L[x] \pm f^*(x)]. \tag{2.7}$$

For this, from (1.4.2) and (2.7), we have successively

$$Tx(t) - \bar{x}(t) = H_1[g^*(x(t), t) - g(\bar{x}(t), t) - A(t)(x(t) - \bar{x}(t)) - \eta(t)] + \\ + H_2[L[x - \bar{x}] \pm (f^*(x) - f(\bar{x})) \pm e'] = \\ = H_1[g^*(x(t), y) - g(x(t), t) + \\ + \int_0^1 [g_x(x(t) + \Theta_1(\bar{x}(t) - x(t)), t) - A(t)](x(t) - \bar{x}(t))\, d\Theta_1 - \eta(t)] + \\ + H_2[\pm(f^*(x) - f(x)) \pm \int_0^1 [f_x(x + \Theta_2(\bar{x} - x)) \pm L](x - \bar{x})\, d\Theta_2 \pm e']$$

and hence

$$\| Tx - \bar{x} \| \leqq M_1[\Delta_1 \| g(x(t), t) \| + M_3 r_\Delta + \delta_1] + M_2[\Delta_2 \| f(x) \| + M_4 r_\Delta + \delta_2]. \tag{2.8}$$

Since $\varrho(\Delta_1)$, $\varrho(\Delta_2) < 1$, for all $x(t) \in \bar{S}(\bar{x}, r_\Delta)$ inequalities (2.1) and (2.2) provide

$$\| g(x(t), t) \| \leqq (E - \Delta_1)^{-1} \| g^*(x(t), t) \|, \tag{2.9}$$

$$\| f(x) \| \leqq (E - \Delta_2)^{-1} (E - \Delta_2)^{-1} \| f^*(x) \|. \tag{2.10}$$

Next, we have

$$\| g(x(t), t) \| \leqq \| g(x(t), t) - g(\bar{x}(t), t) - A(t)(x(t) - \bar{x}(t) \| + \\ + \| g(\bar{x}(t), t) \| + \| A(t) \| r_\Delta$$

and hence from (2.9), we get

$$\| g(x(t), t) \| \leqq M_3 r_\Delta + (E - \Delta_1)^{-1} \| g^*(\bar{x}(t), t) \| + \| A(t) \| r_\Delta. \tag{2.11}$$

Similarly, we find

$$\| f(x) \| \leqq M_4 r_\Delta + (E - \Delta_2)^{-1} \| f^*(\bar{x}) \| + \| L \| r_\Delta. \tag{2.12}$$

Using (2.11) and (2.12) in (2.8), we obtain

$$\| Tx - \bar{x} \| \leqq K_\Delta r_\Delta + \\ + (M_1 \delta_1 + M_2 \delta_2 + M_1 \Delta_1 (E - \Delta_1)^{-1} \| g^*(\bar{x}(t), t) \| + M_2 \Delta_2 (E - \Delta_2)^{-1} \| f^*(\bar{x}) \|) \leqq \\ \leqq K_\Delta r_\Delta + (E - K_\Delta) r_\Delta = r_\Delta.$$

This completes the proof of part (2).

Next, we shall prove part (3). From the definition of $x_{m+1}(t)$ and $y_{m+1}(t)$, we have

$$x_{m+1}(t) - y_{m+1}(t) = -y_{m+1}(t) + H_1[g(y_m(t), t) - A(t)\, y_m(t)] + H_2[L[y_m] \pm f(y_m)] + \\ + H_1[g(x_m(t), t) - g(y_m(t), t) - A(t)\,(x_m(t) - y_m(t))] + \\ + H_2[L[x_m - y_m] \pm (f(x_m) - f(y_m))]$$

and hence, as in part (2), we find

$$\| x_{m+1} - y_{m+1} \| \leqq \| y_{m+1}(t) - H_1[g(y_m(t), t) - A(t)\, y_m(t)] - H_2[L[y_m] \pm f(y_m)] \| + \\ + (M_1 M_3 + M_2 M_4) \| x_m - y_m \| . \tag{2.13}$$

Using the similar arguments for $x_m(t) - y_m(t)$ and substituting the obtained inequality in (2.13), we get

$$\| x_{m+1} - y_{m+1} \| \leqq \\ \leqq \sum_{i=m-1}^{m} K_0^{m-1} \| y_{i+1}(t) - H_1[g(y_i(t), t) - A(t)\, y_i(t)] - H_2[L_\llcorner y_i] \pm f(y_i)] \| + \\ + K_0^2 \| x_{m-1} - y_{m-1} \| .$$

Continuing in this way, we obtain

$$\| x_{m+1} - y_{m+1} \| \leqq \\ \leqq \sum_{i=0}^{m} K_0^{m-i} \| y_{i+1}(t) - H_{1\llcorner} g(y_i(t), t) - A(t)\, y_i(t)] - H_2[L[y_i] \pm f(y_i)] \| = A_m \text{ (say)}. \tag{1.10}$$

Using (2.14) in the triangle inequality, we find

$$\| x^* - y_{m+1} \| \leqq A_m + \| x_{m+1} - x^* \| . \tag{2.15}$$

In (2.15), theorem 1.4.1 ensures that $\lim_{m \to \infty} \| x_{m+1} - x^* \| = 0$. Thus, the condition (2.5) is necessary and sufficient for the convergence of the sequence $\{y_m(t)\}$ to $x^*(t)$ follows from the generalized Toeplitz lemma [4] „for square matrix $A \geqq 0$ with $\varrho(A) < 1$, let $s_m = \sum_{i=0}^{m} A^{m-i} d_i$; $m = 0, 1, \ldots$ Then, $\lim_{m \to \infty} s_m = 0$ if and only if $\lim_{m \to \infty} d_m = 0$."

Finally, we shall prove (2.6). For this, we have

$$x^*(t) - y_{m+1}(t) = H_1[g(x^*(t), t) - g^*(y_m(t), t) - A(t)\,(x^*(t) - y_m(t))] + \\ + H_2[L[x^* - y_m] \pm (f(x^*) - f(y_m))]$$

and as in part (1), we have

$$\| x^* - y_{m+1} \| \leqq M_1[M_3 \| x^* - y_m \| + \Delta_1(E - \Delta_1)^{-1} \| g^*(y_m(t), t) \|] + \\ + M_2[M_4 \| x^* - y_m \| + \Delta_2(E - \Delta_2)^{-1} \| f^*(y_m) \|] \leqq \\ \leqq K_0 \| x^* - y_{m+1} \| + [M_1 \Delta_1(E - \Delta_1)^{-1} \| g^*(y_m(t), t) \| + \\ + M_2 \Delta_2 (E - \Delta_2)^{-1} \| f^*(y_m) \| + K_0 \| y_{m+1} - y_m \|],$$

which is same as (2.6).

Theorem 2.2. *With respect to* (1.1), (1.2) *we assume that there exists an approximate solution* $\bar{x}(t)$ *and conditions* (1.i)–(1.iii) *of theorem* 1.4.1 *are satisfied. Further, let for all* $x(t) \in \bar{S}(\bar{x}, r)$, *the following inequalities be satisfied*

$$\| g(x(t), t) - g^*(x(t), t) \| \leqq \Delta_3, \tag{2.16}$$

$$\| f(x) - f^*(x) \| \leqq \Delta_4, \tag{2.17}$$

where Δ_3 *and* Δ_4 *are* $n \times n$ *nonnegative matrices. We also assume that* $\varrho(K_0) < 1$, *and*

$$r_\Delta^* = (E - K_0)^{-1} [M_1(\Delta_3 + \delta_1) + M_2(\Delta_4 + \delta_2)] \leqq r.$$

Then,

(1) *all the conclusions* (1)–(5) *of theorem* 1.4.1 *hold,*

(2) *the sequence* $\{y_m(t)\}$ *obtained from* (1.3) *remains in* $\bar{S}(\bar{x}, r_\Delta^*)$,

(3) *condition* (2.5) *is necessary and sufficient for the convergence of* $\{y_m(t)\}$ *to* $x^*(t)$ *the solution of* (1.1), (1.2) *and*

$$\| x^* - y_{m+1} \| \leqq (E - K_0)^{-1} [M_1 \Delta_3 + M_2 \Delta_4 + K_0 \| y_{m+1} - y_m \|].$$

Proof. The proof is contained in the proof of theorem 2.1.

Remark. Inequalities (2.16), (2.17) correspond to the absolute error in approximating g and f by g^* and f^*.

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