Ondřej Došlý

On transformations of self-adjoint linear differential systems

Archivum Mathematicum, Vol. 21 (1985), No. 3, 159--170

Persistent URL: http://dml.cz/dmlcz/107228

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 21, No. 3 (1985), 159–170

ON TRANSFORMATIONS OF SELF—ADJOINT LINEAR DIFFERENTIAL SYSTEMS

ONDŘEJ DOŠLÝ, Brno

(Received March 31, 1984)

Abstract. There are studied transformations of self-adjoint linear differential systems of the second order. Certain special form of these systems is suggested as the canonical representation of the class of mutually transformable differential systems. The obtained results are used for transformations of disconjugate systems.

Key words. Self-adjoint systems, trigonometric matrices, isotropic solution, equivalent systems, disconjugate systems.

1. Introduction

In the present paper we deal with linear differential systems of the second order in the matrix form

$$(F(x)\,Y')' + G(x)\,Y = 0, \tag{1.1}$$

where $F(x)$, $G(x)$ are symmetric continuous $n \times n$ matrices, $F(x)$ is positive definite and $Y(x)$ is an $n \times n$ matrix solution.

The following notation is used: $C^m(I)$ denotes the space of m-times differentiable real functions on an interval I. If $A(x)$ is an arbitrary matrix of functions we write $A(x) \in C^m(I)$ if each entry of $A(x)$ is of the class $C^m(I)$. A^T denotes the transpose of the matrix A, E and 0 denote the unit and the zero matrix of any dimension. System (1.1) is uniquely determined by a pair of matrices $F(x)$, $G(x)$, hence we shall denote such system $(F(x), G(x))$. I denotes an interval of arbitrary kind.

It is investigated the following transformation of systems (1.1).

Theorem A. *Let $H(x) \in C^1(I)$ be a nonsingular $n \times n$ matrix for which*

$$H^{T'}(x)\,F(x)\,H(x) - H^T(x)\,F(x)\,H'(x) = 0 \tag{1.2}$$

on I and let $Y(x) = H(x)\,U(x)$. Then

$$H^T(x)\,[(F(x)\,Y')' + G(x)\,Y] = (F_1(x)\,U')' + G_1(x)\,U, \tag{1.3}$$

where

$$F_1(x) = H^T(x)\, F(x)\, H(x) \tag{1.4}$$
$$G_1(x) = H^T(x)\, [(F(x)\, H'(x))' + G(x)\, H(x)].$$

Proof. $H^T[(FY')' + GY] = H^T(FHU' + FH'U)' + H^TGY = (H^TFH'U)' + (H^TFHU')' - H^{T'}(FH'U + FHU') + H^TGHU = H^{T'}FH'U + H^T(FH')'U + H^TFH'U' + (H^TFHU')' - H^{T'}FH'U - H^{T'}FHU' + H^TGHU = (H^TFHU')' + [H^T(FH')' + H^TGH]\, U = (F_1U')' + G_1U.$

Some aspects of this transformation have been studied in [1] and [4]. We shall show that by means of this transformation can be defined certain equivalence on the class of all differential systems of the form (1.1). We shall also suggest certain special forms of (1.1) to be the canonical representations of the class of decomposition determined by this equivalence. In the last section the obtained results are used for transformations of disconjugate differential systems.

2. Equivalent differential systems

In the beginning of this section we recall two theorems concerning transformations of self-adjoint differential systems of the second order. This theorems were proved in [5] and [6]. The first one concerns systems

$$Y'' + P(x)\, Y = 0, \tag{2.1}$$

where $P(x)$ is a symmetric continuous $n \times n$ matrix.

Theorem B. *There exists a nonsingular $n \times n$ matrix $R(x) \in C^1(I)$ for which*

$$R^{T'}(x)\, R(x) - R^T(x)\, R'(x) = 0 \tag{2.2}$$

such that the transformation $S(x) = R^{-1}(x)\, Y(x)$ transforms system (2.1) *in the system*

$$(Q^{-1}(x)\, S')' + Q(x)\, S = 0,$$

where $Q(x) = (R^T(x)\, R(x))^{-1}$ and $P(x) = -R''(x)\, R^{-1}(x) + (R(x)\, R^T(x))^{-2}$.

Proof. See [5, Theorem 1].

A linear differential system of the second order $(Q^{-1}(x)\, S')' + Q(x)\, S = 0$ can be rewritten as the system of the first order

$$\begin{aligned} S' &= Q(x)\, C, \\ C' &= -Q(x)\, S. \end{aligned} \tag{2.3}$$

Let $(S(x), C(x))$ be the solution of (2.3) for which $S(a) = 0$, $C(a) = E$, $a \in I$, then

$$\begin{aligned} &S^T(x)\, S(x) + C^T(x)\, C(x) = E, \qquad S(x)\, S^T(x) + C(x)\, C^T(x) = E, \\ &C^T(x)\, S(x) - S^T(x)\, C(x) = 0, \qquad S(x)\, C^T(x) - C(x)\, S^T(x) = 0 \end{aligned} \tag{2.4}$$

on I, see [9, p. 312]. In all points where the matrix $C(x)$ is nonsingular let us put $T(x) = C^{-1}(x)\, S(x)$. According to (2.4) $T(x)$ is symmetric. To emphasize that $(S(x), C(x))$ is a solution of (2.3) with a matrix $Q(x)$ we write $(S(x, Q), C(x, Q))$ and also $T(x, Q)$ instead of $T(x)$.

The following statement holds for transformations of systems (2.3).

Theorem C. *Let us consider two systems $(Q_i^{-1}(x), Q_i(x))$, where $Q_i(x)$ are positive definite, $i = 1, 2$. Then the following two statements are equivalent;*

i) *There exists a nonsingular $n \times n$ matrix $R(x) \in C^1(I)$ for which $R^T(x) \times Q_1^{-1}(x)\, R'(x) - R^{T'}(x)\, Q_1^{-1}(x)\, R(x) = 0$ such that the transformation $S_1(x) = R(x)\, S_2(x)$ gives*

$$R^T(x)\,[(Q_1^{-1}(x)\, S_1')' + Q_1(x)\, S_1] = (Q_2^{-1}(x)\, S_2')' + Q_2(x)\, S_2.$$

ii) *There exist constant $n \times n$ matrices K, L, M, N for which*

$$K^T L - L^T K = 0, \qquad M^T N - N^T M = 0, \qquad K^T N - L^T M = E \tag{2.5}$$

such that in all points where the following expression is defined we have

$$T(x, Q_2) = (T(x, Q_1)\, M + N)^{-1}(T(x, Q_1)\, K + L).$$

Proof. See [6, Theorem 1].

Now we define the following relation on the class of all self-adjoint differential systems of the second order.

Definition 1. Let $(F_i(x), G_i(x))$, $i = 1, 2$, be two differential systems of the form (1.1). We write $(F_1(x), G_1(x)) \leftrightarrow (F_2(x), G_2(x))$ if there exists a nonsingular $n \times n$ matrix $H(x) \in C^1(I)$ for which $H^{T'}(x)\, F_1(x)\, H(x) - H^T(x)\, F_1(x)\, H'(x) = 0$ and $F_2(x) = H^T(x)\, F_1(x)\, H(x)$, $G_2(x) = H^T(x)\,(F_1(x)\, H'(x))' + H^T(x)\, G_1(x)\, H(x)$. In this case we say that the matrix $H(x)$ transforms $(F_1(x), G_1(x))$ in $(F_2(x), G_2(x))$.

Theorem 1. *The relation $\leftrightarrow$ is an equivalence on the class of all self-adjoint linear differential systems of the form* (1.1).

Proof. Reflexivity of $\leftrightarrow$ is obvious by choosing $H(x) = E$. Let $H(x)$ transforms the system $(F_1(x), G_1(x))$ in the system $(F_2(x), G_2(x))$, i.e. $H^{T'}F_1H - H^TF_1H' = 0$, $F_2 = H^TF_1H$ and $G_2 = H^T((F_1H')' + G_1H)$. Then $H^{-1}(x)$ transforms $(F_2(x), G_2(x))$ in $(F_1(x), G_1(x))$. Indeed, $F_1 = H^{T-1}F_2H^{-1}$, $(H^{T-1})'\, F_2H^{-1} - H^{T-1}F_2 \times (H^{-1})' = -H^{T-1}H^{T'}H^{T-1}F_2H^{-1} + H^{T-1}F_2\, H^{-1}H'H^{-1} = -H^{T-1}H^{T'}F_1 + F_1H'H^{-1} = -H^{T-1}(H^{T'}F_1H - H^TF_1H')\, H^{-1} = 0$ and $G_1 = H^{T-1}G_2H^{-1} - (F_1H')'\, H^{-1} = H^{T-1}G_2H^{-1} - (H^{T-1}F_2H^{-1}H')'\, H^{-1} = H^{T-1}G_2H^{-1} - (H^{T-1}F_2H^{-1}H'H^{-1})' + H^{T-1}F_2H^{-1}H'(H^{-1})' = H^{T-1}G_2H^{-1} + H^{T-1} \times (F_2(H^{-1})')' + H^{T-1}H^{T'}F_1H'H^{-1} - F_1H'H^{-1}H'H^{-1} = H^{T-1}[(F_2(H^{-1})')' + G_2H^{-1}] + H^{T-1}(H^{T'}F_1H - H^TF_1H')\, H^{-1}H'H^{-1} = H^{T-1}[(F_2(H^{-1})')' + G_2H^{-1}]$. Now, let $H_1(x)$ transform $(F_1(x), G_1(x))$ in $(F_2(x), G_2(x))$ and $H_2(x)$

transforms $(F_2(x), G_2(x))$ in $(F_3(x), G_3(x))$. Then $F_3 = H_2^T F_2 H_2 = H_2^T H_1^T F_1 H_1 H_2 = (H_1H_2)^T F_1 (H_1 H_2)$, $(H_1H_2)^{T'} F_1 H_1 H_2 - (H_1H_2)^T F_1 (H_1H_2)' = H_2^T H_1^{T'} F_1 H_1 H_2 + H_2^{T'} H_1^T F_1 H_1 H_2 - H_2^T H_1^T F_1 H_1' H_2 - H_2^T H_1^T F_1 H_1 H_2' = 0$ and $G_3 = H_2^T (F_2 H_2')' + H_2^T G_2 H_2 = H_2^T (H_1^T F_1 H_1 H_2')' + H_2^T [H_1^T (F_1 H_1')' + H_1^T G_1 H_1]\, H_2 = H_2^T H_1^T \times (F_1 H_1 H_2')' + H_2^T H_1^{T'} F_1 H_1 H_2' + H_2^T H_1^T (F_1 H_1')' H_2 + (H_1H_2)^T G_1 H_1 H_2 = (H_1H_2)^T (F_1 (H_1H_2)')' - H_2^T H_1^T (F_1 H_1' H_2)' + H_2^T H_1^{T'} F_1 H_1 H_2' + H_2^T H_1^T (F_1 H_1')' \times H_2 + (H_1H_2)^T G_1 H_1 H_2 = (H_1H_2)^T (F_1 (H_1H_2)')' - H_2^T H_1^T (F_1 H_1')' H_2 + H_2^T \times H_1^{T'} F_1 H_1 H_2' - H_2^T H_1^T F_1 H_1' H_2' + H_2^T H_1^T (F_1 H_1')' H_2 + (H_1H_2)^T G_1 H_1 H_2 = (H_1H_2)^T (F_1 (H_1H_2)')' + (H_1H_2)^T G_1 H_1 H_2$. Thus $H_1(x) H_2(x)$ transforms $(F_1(x), G_1(x))$ in $(F_3(x), G_3(x))$ and the proof is complete.

Now, let $(F_1(x), G_1(x)) \leftrightarrow (F_2(x), G_2(x))$, i.e. there exists a nonsingular $n \times n$ matrix $H(x)$ which transforms $(F_1(x), G_1(x))$ in $(F_2(x), G_2(x))$. From Theorem A we see that $Y_1(x)$ is the solution of $(F_1(x), G_1(x))$ if and only if $Y_2(x) = H^{-1}(x) \times Y_1(x)$ is the solution of $(F_2(X), G_2(x))$. Hence the equivalence of systems (1.1) in the sence of the following definition is the same as the equivalence in the sence of Definition 1.

Definition 2. Two differential systems $(F_i(x), G_i(x))$, $i = 1, 2$, are said to be equivalent if there exists a nonsingular $n \times n$ matrix $H(x) \in C^1(I)$ such that $Y_1(x)$ is the solution of $(F_1(x), G_2(x))$ if and only if $Y_2(x) = H(x)\, Y_1(x)$ is the solution of $(F_2(x), G_2(x))$.

The basic result of this section is the following statement.

Theorem 2. *There exists a nonsingular $n \times n$ matrix $H(x) \in C^1(I)$ which transforms system* (1.1) *in the system $Y'' + P(x)\, Y = 0$, where*

$$P(x) = H^T(x)\, [(F(x)\, H'(x))' + G(x)\, H(x)], \tag{2.6}$$

i.e. there exists a nonsingular matrix $H(x)$ for which

$$\begin{aligned} H^{T'}(x)\, F(x)\, H(x) - H^T(x)\, F(x)\, H'(x) &= 0 \\ H^T(x)\, F(x)\, H(x) &= E. \end{aligned} \tag{2.7}$$

Proof. Let $D(x)$ be the symmetric positive definite $n \times n$ matrix for which $D^2(x) = F^{-1}(x)$ and let us set $K(x) = D'(x)\, D^{-1}(x) - D^{-1}(x)\, D'(x)$. Then $K^T(x) = -K(x)$ and the matrix solution of the differential system

$$G' = \frac{1}{2} K(x)\, G, \qquad G(a) = E \tag{2.8}$$

is orthonormal (i.e. $G^T(x) = G^{-1}(x)$) for all x. Let $H(x) = D(x)\, G(x)$. Then we have $H^T F H = G^T D F D G = G^T G = E$ and $H^{T'} F H - H^T F H' = (G^{T'} D + G^T D') \times FDG - G^T DF(D'G + DG') = -\frac{1}{2} G^T K D F D G + G^T D' F D G - G^T D F D' G -$

$-\frac{1}{2}G^TDFDKG = G^T\left(-\frac{1}{2}K + D'D^{-1} - D^{-1}D' - \frac{1}{2}K\right)' = 0$. The proof is complete since (2.6) follows from (1.4).

If the system $(F(x), G(x))$ is equivalent to the system $(E, P(x))$ then it is also equivalent to the system $(E, G_0^TP(x)\,G_0)$, where G_0 is a constant orthonormal $n \times n$ matrix. Indeed, if the initial condition in (2.8) is replaced by the condition $G(a) = G_0$, then the statement of Theorem 2 remains valid and $(F(x), G(x))$ is, according to (2.6), equivalent to $(E, G_0^TP(x)\,G_0)$. If we add the condition which forces $P(x)$ to be in the diagonal form for some fixed $a \in I$, then the transformation of $(F(x), G(x))$ in $(E, P(x))$ is unique. Hence differential system in the form (2.1) can be suggested as the canonical representation of the class of equivalent differential systems (1.1).

According to Theorem B and Theorem 2 the system $(F(x), G(x))$ can also be transformed in the system $(Q^{-1}(x), Q(x))$, where $Q(x)$ is a symmetric positive definite $n \times n$ matrix. Theorem C shows that this transformation is not unique. However, solutions of the system $(Q^{-1}(x), Q(x))$ have some nice properties (e.g. they are bounded, are oscillatory for $x \to \infty$ iff $\int^{\infty} Q(x)\,dx = \infty$, etc.). Thus if we are interesting in distribution of conjugate points, boundeness of solutions, etc. it is convenient to consider the canonical representation in the form $(Q^{-1}(x), Q(x))$, in spite of its above mentioned nonuniqueness. If we investigate invariants of the transformation under consideration and related topics, it is convenient to consider the canonical representation in the form $(E, P(x))$.

3. Transformations of disconjugate differential systems

In this section we use the results of the preceding section to obtain several statements concerning transformations of differential systems which are disconjugate on $R = (-\infty, \infty)$. We also give one oscillation criterion for (1.1).

First let us note that if the system (1.1) is disconjugate on some bounded interval $I = (a, b)$ then the transformation of independent variable $x = \frac{1}{2}(a + b) + \frac{1}{\pi}(b - a)\operatorname{arctg} t$ transforms this system in system which is disconjugate on R. If (1.1) is disconjugate on a half-bounded interval (b, ∞) then we use the transformation $x = b + e^t$. For some details concerning this transformation see [1].

Now we recall some definitions and auxiliary results. Let $U(x)$ be a solution of (1.1) then $U^{T'}(x)\,F(x)\,U(x) - U^T(x)\,F(x)\,U'(x) = K$, where K is a constant $n \times n$ matrix. If $K = 0$ then $U(x)$ is said to be isotropic solution. Two solutions $U(x)$, $V(x)$ of (1.1) are said to be independent if every solution $Z(x)$ of (1.1) can be expressed in the form $Z(x) = U(x)\,C_1 + V(x)\,C_2$, where C_1, C_2 are constant

$n\times n$ matrices. In [8] it was proved that isotropic solutions $U(x)$, $V(x)$ are independent if and only if the constant matrix $U^{T'}(x) F(x) V(x) - U^T(x) F(x) V'(x)$ is nonsingular. Two points x_1, x_2 are said to be conjugate according to system (1.1) if there exists a solution $U(x)$ of (1.1) for which $U(x_1) = 0$, $U'(x_1) = E$ and $U(x_2) c = 0$ for some constant n-vector $c \neq 0$. System (1.1) is said to be disconjugate on an interval I if there exist no conjugate points on I.

In the sequel, according to Theorem 2, we suppose that the given differential system is in the form (2.1).

Theorem 3. *Let the system $Y'' + P(x) Y = 0$ be disconjugate on R. Then there exist a nonsingular $n\times n$ matrix $R(x) \in C^1(R)$ and a symmetric positive definite $n\times n$ matrix $F(x) \in C^1(R)$ satisfying*

$$\begin{gathered} R^{T'}(x) R(x) - R^T(x) R'(x) = 0 \\ R^T(x) R(x) = F(x) \end{gathered} \tag{3.1}$$

such that the transformation $U(x) = R^{-1}(x) Y(x)$ transforms the system $Y'' + P(x) Y = 0$ in the system

$$(F(x) U')' = 0. \tag{3.2}$$

Proof. As the system (2.1) is disconjugate on R, there exists an isotropic solution $Y_0(x)$ of this system which is nonsingular on R, see [9, p. 337]. Let us set $R(x) = Y_0(x)$. Then $R(x)$ satisfies first relation of (3.1) and $R^T(x) [R''(x) + P(x) R(x)] = 0$.

The following theorem shows when two disconjugate differential systems are equivalent.

Theorem 4. *Two differential systems $(F_i(x), 0)$, $i = 1, 2$, are equivalent if and only if there exist $a, b \in R$ and constant $n\times n$ matrices K, L, M, N satisfying*

$$K^T L - L^T K = 0, \qquad M^T N - N^T M = 0, \qquad K^T N - L^T M = E \tag{3.3}$$

such that $(\int_a^b F_1^{-1}(t)\,dtK + L) = 0$, $(\int_a^x F_1^{-1}(t)\,dtM + N)$ is nonsingular on R and

$$\int_b^x F_2^{-1}(t)\,dt = (N + \int_a^x F_1^{-1}(t)\,dtM)^{-1} (L + \int_a^x F_1^{-1}(t)\,dtK). \tag{3.4}$$

Proof. Let $(F_1(x) U')' = 0$ and $(F_2(x) U')' = 0$ be equivalent. Then there exist nonsingular $n\times n$ matrices $R_i(x)$, $i = 1, 2$, satisfying

$$\begin{gathered} R_i^{T'}(x) F_i(x) R_i(x) - R_i^T(x) F_i(x) R_i'(x) = 0, \\ R_i^T(x) F_i(x) R_i(x) = E, \end{gathered}$$

such that $U_1(x) = R_1^{-1}(x) \int_a^x F_1^{-1}(t)\,dt$, $V_1(x) = R_1^{-1}(x)$ and $U_2(x) = R_2^{-1}(x)$

$\int_b^x F_2^{-1}(t)\,dt$, $V_2(x) = R_2^{-1}(x)$ are isotropic solutions of the same canonical equation $Y'' + P(x)\,Y = 0$ satisfying $U_i^{T'}(x)\,V_i(x) - U_i^T(x)\,V_i'(x) = E$, $i = 1, 2$. Hence there exist constant $n \times n$ matrices K, L, M, N such that

$$U_2(x) = U_1(x)\,K + V_1(x)\,L$$
$$V_2(x) = U_1(x)\,M + V_1(x)\,N.$$

Since $U_2(x)$ is isotropic $U_2^{T'}U_2 - U_2^TU_2' = (K^TU_1^{T'} + L^TV_1^{T'})(U_1K + V_1L) - (K^TU_1^T + L^TV_1^T)(U_1'K + V_1'L) = K^T(U_1^{T'}V_1 - U_1^TV_1')L - K^T(U_1^{T'}U_1 - U_1^TU_1') \times K + L^T(V_1^{T'}U_1 - V_1^TU_1')\,K + L^T(V_1^{T'}V_1 - V_1^TV_1')\,L = K^TL - L^TK = 0.$
Similarly we prove that $M^TN - N^TM = 0$ and $K^TN - L^TM = E$.
Using nonsingularity of the matrix $U_1(x)\,M + V_1(x)\,N$ we obtain

$$V_2^{-1}(x)\,U_2(x) = (V_1(x)\,N + U_1(x)\,M)^{-1}(V_1(x)\,L + U_1(x)\,K)$$

and hence

$$\int_b^x F_2^{-1}(t)\,dt = (\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}(\int_a^x F_1^{-1}(t)\,dtK + L).$$

Now, conversely, let (3.4) hold and K, L, M, N satisfy (3.3). Differentiation of both sides of (3.4) yields

$$F_2^{-1}(x) = -(\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}F_1^{-1}(x)\,M(\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}(\int_a^x F_1^{-1}(t)\,dtK + L) +$$
$$+ (\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}F_1^{-1}(x)\,K =$$
$$= (\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}F_1^{-1}(x)(-M(\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}(\int_a^x F_1^{-1}(t)\,dtK + L) + K) =$$
$$= (\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}F_1^{-1}(x)(-M(K^T\int_a^x F_1^{-1}(t)\,dt + L^T) \times$$
$$\times(\int_a^x F_1^{-1}(t)\,dtM + N)^{T-1} + K) = (\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}F_1^{-1}(x) \times$$
$$\times(-MK^T\int_a^x F_1^{-1}(t)\,dt - ML^T + KN^T + KM^T\int_a^x F_1^{-1}(t)\,dt)(\int_a^x F_1^{-1}(t)\,dtM + N)^{T-1},$$

where the symmetry of the matrix $(\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}(\int_a^x F_1^{-1}(t)\,dtK + L)$ have been used. From (3.3) and Lemma 1 of [6] it follows that matrices K, L, M, N satisfy relations

$$(3.6)\qquad KM^T - MK^T = 0, \qquad LN^T - NL^T = 0, \qquad KN^T - ML^T = E,$$

hence

$$F_2^{-1}(x) = (\int_a^x F_1^{-1}(t)\,dtM + N)^{-1}F_1^{-1}(x)(\int_a^x F_1^{-1}(t)\,dtM + N)^{T-1}$$

and thus

$$F_2(x) = (\int_a^x F_1^{-1}(t)\,\mathrm{d}tM + N)^T F_1(x) (\int_a^x F_1^{-1}(t)\,\mathrm{d}tM + N). \tag{3.7}$$

Let us set $R(x) = (\int_a^x F_1^{-1}(t)\,\mathrm{d}tM + N)$. Then $R^T(x)\,F_1(x)\,R'(x) - R^{T'}(x)\,F_1(x) \times \times R(x) = (M^T \int_a^x F_1^{-1}(t)\,\mathrm{d}t + N^T)\,F_1(x)\,F_1^{-1}(x)\,M - M^T F_1^{-1}(x)\;F_1(x)\;(\int_a^x F_1^{-1}(t) \times \times \mathrm{d}tM + N) = 0$ and $R^T(x)\;(F_1(x)\;R'(x))' = (M^T \int_a^x F_1(t)\;\mathrm{d}t + N^T)\;(F_1(x)\; F_1^{-1}(x)\,M)' = 0$. This, together with (3.7), completes the proof.

Now we give another statement concerning transformations of disconjugate differential systems.

Theorem 5. *Let $Y_1(x)$, $Y_2(x)$ be nonsingular isotropic solutions of* (2.1) *for which*

$$Y_1^{T'}(x)\,Y_2(x) - Y_1^T(x)\,Y_2'(x) = E. \tag{3.8}$$

Then there exist a nonsingular $n \times n$ matrix $R(x) \in C^1(R)$ and a symmetric positive definite $n \times n$ matrix $Q(x)$ for which

$$\begin{aligned} R^{T'}(x)\,R(x) - R^T(x)\,R'(x) &= 0 \\ (R^T(x)\,R(x))^{-1} &= Q(x) \end{aligned} \tag{3.9}$$

such that the transformation $U(x) = R^{-1}(x)\,Y(x)$ transforms the system $Y'' + P(x)\,Y = 0$ in the system

$$(Q^{-1}(x)\,U')' - Q(x)\,U = 0. \tag{3.10}$$

In the proof of this theorem we shall need the following auxiliary statements.

Lemma 1. *Let A be a symmetric $n \times n$ matrix and $c = (c_1, \ldots, c_n)^T$ be a vector for which $c^T A c < 0$. Then there exist n linearly independent vectors $d_1, \ldots, d_n$ such that $d_k^T A d_k < 0$, $k = 1, \ldots, n$.*

Proof. Let $x = (x_1, \ldots, x_n)^T$. There exists $\delta > 0$ such that for every x for which $\| x - c \|_e < \delta$, where $\| \,.\, \|_e$ is the Euclidean vector norm, we have $x^T A x < 0$. For $n = 1$ the statement of Lemma is obvious. Let us suppose that this statement is valid for $i = n - 1$, i.e. there exist linearly independent vectors $d_1, \ldots, d_{n-1}$ for which $\| d_k - c \|_e < \delta$, $k = 1, \ldots, n - 1$. These vectors form a hyperplane in the n-dimensional Euclidean space. From the geometrical point of view it is obvious that there exists a vector d_n which does not lay in this hyperplane and $\| d_n - c \|_e < \delta$. Hence $d_1, \ldots, d_n$ are linearly independent and $d_k^T A d_k < 0$, $k = 1, \ldots, n$.

Lemma 2. *Let $A(x)$ be a continuous symmetric positive definite $n \times n$ matrix and d_k be independent unit vectors for which $\lim_{x \to \infty} d_k^T \int_a^x A(s)\,\mathrm{d}s\, d_k = a_k < \infty$, $k = 1, \ldots, n$.*

Then for every unit vector b $\lim_{x\to\infty} b^T \int_a^x A(s)\,\mathrm{d}s b < \infty$.

As the precise proof of this statement is rather complicated and exeeds the scope of this paper, we omit it. The main ideas concerning this proof can be found in [3].

Proof of Theorem 5. Let $Y_1(x)$, $Y_2(x)$ be isotropic solutions of (2.1) for which (3.8) holds. Then

$$\begin{pmatrix} Y_2^{T'}(x) & -Y_2^T(x) \\ -Y_1^{T'}(x) & Y_1^T(x) \end{pmatrix} \begin{pmatrix} Y_1(x) & Y_2(x) \\ Y_1'(x) & Y_2'(x) \end{pmatrix} = \begin{pmatrix} - & 0 \\ 0 & -E \end{pmatrix}$$

hence

$$\begin{pmatrix} Y_1(x) & Y_2(x) \\ Y_1'(x) & Y_2'(x) \end{pmatrix} \begin{pmatrix} Y_2^{T'}(x) & -Y_2^T(x) \\ -Y_1^{T'}(x) & Y_1^T(x) \end{pmatrix} = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}$$

and thus

$$\begin{aligned} &Y_1(x)\,Y_2^{T'}(x) - Y_2(x)\,Y_1^{T'}(x) = -E \\ &Y_1(x)\,Y_2^T(x) - Y_2(x)\,Y_1^T(x) = 0 \\ &Y_1'(x)\,Y_2^{T'}(x) - Y_2'(x)\,Y_1^{T'}(x) = 0 \\ &Y_2'(x)\,Y_1^T(x) - Y_1'(x)\,Y_2^T(x) = -E. \end{aligned} \tag{3.11}$$

We see that the matrix $Y_1(x)\,Y_2^T(x)$ is symmetric and it follows that $Y_2^{-1}(x)\,Y_1(x)$ is also symmetric. Since $Y_1(x)$, $Y_2(x)$ are nonsingular, the number of positive eigenvalues of the matrix $Y_2^{-1}(x)\,Y_1(x)$ is constant on R and $(Y_2^{-1}Y_1)' =$ $= -Y_2^{-1}Y_2'Y_2^{-1}Y_1 + Y_2^{-1}Y_1' = Y_2^{-1}Y_2^{T-1}(-Y_2^{T'}Y_1 + Y_2^TY_1') = (Y_2^TY_2)^{-1}$, hence

$$Y_2^{-1}(x)\,Y_1(x) = \int_a^x (Y_2^T(t)\,Y_2(t))^{-1}\,\mathrm{d}t + Y_2^{-1}(a)\,Y_1(a), \qquad a \in R. \tag{3.12}$$

Let us suppose that there exists a unit vector c for which $c^T Y_2^{-1}(a)\,Y_1(a)\,c < 0$. Then by Lemma 1 there exist independent unit vectors $d_1, \ldots, d_k$ for which

$$d_k^T Y_2^{-1}(a)\,Y_1(a)\,d_k < 0, \qquad k = 1, \ldots, n. \tag{3.13}$$

For these vectors we have $\lim_{x\to\infty} d_k^T \int_a^x (Y_2^T(t)\,Y_2(t))^{-1}\,\mathrm{d}t d_k = a_k < \infty$. Indeed, if $\lim_{x\to\infty} d_j^T \int_a^x (Y_2^T(t)\,Y_2(t))^{-1}\,\mathrm{d}t d_j = \infty$ for some $j \in \{1, \ldots, n\}$ then, according to (3.12) and (3.13), there exists $x_0 > a$ for which $d_j Y_2^{-1}(x_0)\,Y_1(x_0)\,d_j = 0$. It is a contradiction, since $Y_1(x)$, $Y_2(x)$ are nonsingular and $Y_2^{-1}(x)\,Y_1(x)$ is symmetric.

Now, let b be arbitrary unit vector. According to Lemma 2 $\lim_{x\to\infty} b^T \int_a^x (Y_2^T(t)$ $Y_2(t))^{-1}\,\mathrm{d}t b < \infty$ and hence the matrix integral $\int_a^\infty (Y_2^T(t)\,Y_2(t))^{-1}\,\mathrm{d}t$ is convergent.

Let

$$(U(x), V(x)) = \begin{cases} (Y_1(x), Y_2(x)) & \text{if } Y_2^{-1}(a)\, Y_1(a) \quad \text{is positive definite} \\ (Y_2(x), Y_2(x) \int\limits_x^{\infty} (Y_2^T(t)\, Y_2(t))^{-1} \mathrm{d}t) & \text{in the opposite case} \end{cases}$$

Then $U^{T'}(x)\, V(x) - U^T(x)\, V'(x) = E$, $U(x)$, $V(x)$ are nonsingular isotropic solutions of (2.1) and $V(x)\, U^T(x)$ is symmetric and positive definite. Let $D(x)$ be the symmetric positive definite $n \times n$ matrix for which $D^2(x) = 2V(x)\, U^T(x)$ and let $K(x) = D'(x)\, D(x) - D(x)\, D'(x)$, $L(x) = K(x)\, D^{-2}(x) - D^{-2}(x)\, K(x)$. If $M(x)$ is the solution of the matrix equation

$$D^{-2}(x)\, M(x) + M(x)\, D^{-2}(x) = L(x)$$

then $M^T(x) = M(x)$, since $D^{-2}(x)$ and $L(x)$ are symmetric (see [3]). Let $T(x)$ be the solution of

$$T' = \frac{1}{2} D^{-2}(x) (K(x) + M(x))\, T, \qquad T(a) = E.$$

As $D^{-2}(K + M) + [D^{-2}(K + M)]^T = D^{-2}K + D^{-2}M - KD^{-2} + MD^{-2} = D^{-2}M + MD^{-2} - L = 0$, $T(x)$ is orthonormal on I. For

$$R(x) = D(x)\, T(x)$$

we have $R^{T'}R - R^T R' = (T^T D' + T^{T'} D)\, DT - T^T D(D'T + DT') = T^T D' DT - T^T DD'T + \frac{1}{2} T^T (M - K) D^{-2} D^2 T - \frac{1}{2} T^T D^2 D^{-2} (K + M)\, T = T^T (D'D - DD')\, T - \frac{1}{2} T^T KT - \frac{1}{2} T^T KT = T^T (D'D - DD')\, T - T^T (D'D - DD')\, T = 0$ and $RR^T = DTT^T D = D^2 = 2VU^T$.

From the first relation of (3.9) it follows that $RR^T R' R^T = RR^{T'} RR^T$. Simultaneously we have $RR^T V' U^T = 2VU^T V' U^T = 2V(U^{T'} V - E)\, U^T = (VU^{T'} - E)\, RR^T$ and $VU^{T'} RR^T = 2VU^{T'} VU^T = 2V(U^T V' + E)\, U^T = RR^T (V' U^T + E)$. Addition of the last equalities gives $RR^T (2V' U^T + E) = (2VU^{T'} - E)\, RR^T$. Let us denote $X_1 = R' R^T$, $X_2 = (2V' U^T + E)$, $Y_1 = RR^{T'}$, $Y_2 = (2VU^{T'} - E)$. It holds

$$RR^T X_i - Y_i RR^T = 0,$$

$$X_i + Y_i = (RR^T)', \qquad i = 1, 2,$$

hence

$$\begin{aligned} RR^T X_i + X_i RR^T &= (RR^T)' RR^T \\ RR^T Y_i + Y_i RR^T &= RR^T (RR^T)', \qquad i = 1, 2. \end{aligned} \tag{3.14}$$

Since the matrix RR^T is positive definite, both matrix equations (3.14) have unique solution and it follows $X_1 = X_2$, $Y_1 = Y_2$, i.e. $R' R^T = 2V' U^T + E$, $RR^{T'} = 2VU^{T'} - E$. Now, let us set $Q(x) = (R^T(x)\, R(x))^{-1}$. According to Theorem A it remains to verify that $R^T(x)\, R''(x) - R^T(x)\, P(x)\, R(x) = -Q(x)$.

$R^TR' + R^TPR = R^{-1}(RR^TR'R^T + RR^TPRR^T)\,R^{T-1} = R^{-1}(RR^T(R'R^T)' - RR^TR'R^{T'} + RR^TPRR^T)\,R^{T-1} = R^{-1}(RR^T(2V'U^T + E)' - RR^{T'}RR^{T'} + RR^TPRR^T)\,R^{T-1} = R^{-1}(-RR^TPRR^T + 4VU^TV'U^{T'} - (2VU^{T'} - E)^2)\,R^{T-1} = R^{-1}(4V(U^{T'}V - E)\,U^{T'} - 4VU^{T'}VU^{T'} + 4VU^{T'} - E)\,R^{T-1} = -R^{-1}R^{T-1} = -Q$. The proof is complete.

Remark. Theorem 3 and Theorem 5 can be comprehended as a generalization of some results of Krbila and Barvínek concerning other sorts of phase functions of the scalar differential equation

$$y'' + p(x)\,y = 0 \tag{3.15}$$

than is the usual phase function of (3.15) introduced by O. Borůvka, see [2]. Let $F(x)$ and $Q(x)$ be matrices from Theorem 3 and Theorem 5 respectively. For $n = 1$ (i.e. (2.1) is identical with (3.15)) the matrix $\int_a^x F^{-1}(t)\,dt$, $a \in R$, is identical with the parabolic phase function of (3.15) and $\int_a^x Q(t)\,dt$ is identical with the hyperbolic phase function of (3.15).

In the end of this section we use Theorem B and Theorem 1 to introduce an oscillation criterion for (1.1) which generalizes a similar criterion for systems (2.1) which was proved in [5].

Definition 3. System (1.1) is said to be oscillatory for large x if there exist an isotropic solution $U(x)$ of (1.1) and a sequence $x_n \to \infty$ such that $\det U(x_n) = 0$. In the opposite case system (1.1) is said to be nonoscillatory for large x. For equivalent definitions of this concept see [7] and [10].

Theorem 6. *If there exists an n-dimensional constant vector c such that for the matrix $F(x)$ in (1.1) $\lim_{x\to\infty} c^T \int_a^x F^{-1}(t)\,dtc = \infty$, $a \in R$, and all solutions of (1.1) are bounded on (a, ∞), then system (1.1) is oscillatory for large x.*

Proof. By Theorem B and Theorem 1 there exists a nonsingular $n \times n$ matrix $R(x) \in C^1((a, \infty))$ such that the transformation $S(x) = R^{-1}(x)\,Y(x)$ transforms system (1.1) in the system

$$(Q^{-1}(x)\,S')' + Q(x)\,S = 0 \tag{3.16}$$

and

$$Q(x) = R^{-1}(x)\,F^{-1}(x)\,R^{T-1}(x), \tag{3.17}$$

i.e. if $Y_1(x)$, $Y_2(x)$ are independent isotropic solutions of (1.1) then $S_1(x) = R^{-1}(x)\,Y_1(x)$, $S_2(x) = R^{-1}(x)\,Y_2(x)$ are independent isotropic solutions of (3.16). It is known, see [9, p. 352], that (3.16) is nonoscillatory for large x if and only if $\int^{\infty} \| Q(x) \|_e \mathrm{d}x < \infty$, where $\| Q(x) \|_e = \sup_{\| c \|_e = 1} c^T Q(x)\, c$ is the Euclidean matrix norm of $Q(x)$, and all solutions of (3.16) are bounded. Let us suppose that there exists a real constant k such that $\| R(x) \|_e \leqq k$. Then $\lim_{x \to \infty} \int_a^x \| Q(s) \|_e \mathrm{d}s \geqq$ $\geqq \lim_{x \to \infty} c^T \int_a^x Q(s) \mathrm{d}sc = \lim_{x \to \infty} c^T \int_a^x R^{-1}(s)\, F^{-1}(s)\, R^{T^{-1}}(s) \mathrm{d}sc \geqq \lim_{x \to \infty} k^{-2} c^T \int_a^x F^{-1}(s)\, dsc =$ $= \infty$, hence (3.16) is oscillatory for large x. Since the transformation $S(x) =$ $= R^{-1}(x)\, Y(x)$ preserves oscillatory character of equations, system (1.1) is oscillatory for large x.

REFERENCES

[1] C. D. Ahlbrand, D. B. Hinton, R. T. Lewis: *The effect of variable change on oscillation and disconjugate criteria with application to spectral theory,* J. Math. Anal. Appl. 81 (1981) 234–277.

[2] E. Barvínek: *Quadratic phases of differential equations* $y'' + q(t)\, y = 0$, Arch. Math. 8 (1972) 63–78.

[3] R. Bellman: *Introduction to matrix analysis,* McGraw–Hill New York 1971.

[4] D. F. St. Mary: *On transformation and oscillation of linear differential systems,* Canad. J. Math. 29 (1977) 392–399.

[5] O. Došlý: *A phase matrix of linear differential systems,* to appear in Časopis pro pěstování matematiky.

[6] O. Došlý: *The basic properties of phase matrices,* to appear in Arch. Math.

[7] V. A. Jakubovič: *Uslovia kalebatělnosti i nekalebatělnosti linejnych kanoničeskich system,* Dokl. Akad. Nauk SSSR 124 (1959) 994–997.

[8] J. Radon: *Zum Problem von Lagrange,* Abh. Math. Sem. Univ. Hamburg 6 (1928) 273–279.

[9] W. T. Reid: *Ordinary Differential Equations,* John Wiley, New York 1971.

[10] R. L. Sternberg: *Variational methods and non-oscillatory theorems for systems of differential equations,* Duke Math. J. 19 (1952) 311–322.

O. Došlý
Department of Mathematics
J. E. Purkyně University
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia