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# CHARACTERIZATIONS OF ELEMENTS OF BEST APPROXIMATION IN NON-ARCHIMEDEAN NORMED SPACES

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The problem of existence and uniqueness of best approximation in non-archimedean (n.a.) normed spaces has been discussed by Monna [2], [3], Ikada and Haifawi [1] and some others. In this note we shall give some characterizations of elements of best approximation in n.a. normed spaces.

Let $G$ be a subset of a n.a. normed space $X$ over some nontrivially valued field $F$ and $x \in X \setminus G$. An element $g_0 \in G$ is said to be a *best approximation* to $x$ if

$$\| x - g_0 \| \leqq \| x - g \|, \qquad g \in G.$$

We shall denote the set of all best approximations to $x$ in $G$ by $L_G(x)$ i.e.

$$L_G(x) = \{g_0 \in G : \| x - g_0 \| \leqq \| x - g \|, g \in G\}.$$

It can be easily seen that for a linear subspace $G$ of a n.a. normed space $X$, $g_0 \in L_G(x)$ if and only if $g_0 \in L_G[tx + (1 - t) g_0]$ for all scalars $t \in F$.

An element $x$ of a n.a. normed space $X$ is said to be *orthogonal* (cf. [6]) to an element $y \in X (x \perp y)$ if

$$\operatorname{dist}(x, [y]) = \| x \|,$$

i.e. if $\| x + \alpha y \| \geqq \| x \|$ for every scalar $\alpha \in F$.

$x$ is said to be orthogonal to a subset $G$ of $X$ if $x \perp y$ for all $y \in G$.

The following characterization of elements of best approximation was observed in [4]:

For a linear subspace $G$ of a n.a. normed space $X$, $g_0 \in L_G(x)$ if and only if $x - g_0 \perp G$.

A n.a. normed space $X$ is said to be *spherically complete* if every nest of closed spheres in $X$ has a non-empty intersection.

A n.a. normed space $X$ has the *extension property* if every bounded (continuous if the underlying field is nontrivially valued) linear transformation on any subspace $G$ of $X$ can be extended to whole of $X$ without increasing its norm.

It is well known (cf. [5]) that a n.a. normed space $X$ is spherically complete if and only if $X$ has the extension property.

The following theorem gives another characterization of elements of best approximation in spherically complete normed spaces.

**Theorem 1.** *Suppose $G$ is a linear subspace of a spherically complete n.a. normed linear space $X$, $x \in X \setminus G$ and $g_0 \in G$. Then $g_0 \in L_G(x)$ if and only if there exists $f \in X^*$ such that*

(i) $f(g) = 0$,

(ii) $|f(x - g_0)| = \| x - g_0 \|$

*and*

(iii) $|f(x - g)| \leqq \| x - g \|$

*for every $g \in G$.*

Proof. Let $g_0 \in L_G(x)$. Then for every $g \in G$,

$$\| x - g_0 \| \leqq \| x - g \|. \tag{1}$$

In particular, for $\alpha \neq 0$,

$$\| x - g_0 \| \leqq \left\| x - g_0 + \frac{g}{\alpha} \right\| \tag{2}$$

for every $g \in G$. Let

$$M = \{g + \alpha(x - g_0) : \alpha \in F\}.$$

Define $f_0$ on $M$ as

$$| f_0(g + \alpha(x - g_0)) | \; | = |\alpha| \; \| x - g_0 \|,$$

for each $g \in G$. Therefore $f_0(g) = 0$ and

$$| f_0(x - g_0) | = \| x - g_0 \|.$$

Now for $\alpha \neq 0$,

$$| f_0(g + \alpha(x - g_0)) | = |\alpha| \; \| x - g_0 \| \leqq$$
$$\leqq |\alpha| \left\| x - g_0 + \frac{g}{\alpha} \right\| \text{ by (2)} =$$
$$= \| g + \alpha(x - g_0) \|$$

for each $g \in G$. The inequality is trivial for $\alpha = 0$. Therefore for every $z \in M$,

$$| f_0(z) | \leqq \| z \|.$$

Since $X$ is spherically complete, it has extension property and so $f_0$ can be extended to a continuous linear functional $f$ on $X$ such that

$$| f(x) | \leqq \| x \|$$

for every $x \in X$ and

$$f(z) = f_0(z)$$

for every $z \in M$, whence $f(g) = 0$, $|f(x - g_0)| = \| x - g_0 \|$ and $|f(x - g)| \leqq \leqq \| x - g \|$ for every $g \in G$. Thus the relations (i), (ii) and (iii) are established.

Conversely, let the given conditions be satisfied. Then by (ii)

$$\begin{aligned} \| x - g_0 \| &= |f(x - g_0)| = \\ &= |f(x - g)| \leqq && \text{by (i)} \\ &\leqq \| x - g \| && \text{by (iii)} \end{aligned}$$

for every $g \in G$. Hence $g_0 \in L_G(x)$.

As a consequence of Theorem 1 we get the following.

**Theorem 2.** *Let $X$ be as in Theorem 1, $M$ a linear manifold in $X$, $x \in X \setminus M$ and $m_0 \in M$. Then $m_0 \in L_M(x)$ if and only if there exists $f \in X^*$ such that*

(iv) $f(m - m_0) = 0$,

(v) $|f(x - m_0)| = \| x - m_0 \|$

*and*

(vi) $|f(x - m)| \leqq \| x - m \|$

*for every $m \in M$.*

Proof. Since $M$ is a linear manifold in $X$ and $m_0 \in M$, $M - m_0$ is a linear subspace of $X$. Also $m_0 \in L_M(x)$ iff. $0 \in L_{M - m_0}(x - m_0)$. Hence, by Theorem 1, there exists $f \in X^*$ such that

$$f(m - m_0) = 0, \tag{3}$$

$$|f(x - m_0)| = \| x - m_0 \|, \tag{4}$$

$$|f(x - m_0 - m + m_0)| \leqq \| x - m_0 - m + m_0 \| \tag{5}$$

for every $m \in M$. These relations are (iv), (v) and (vi).

Conversely, let the conditions given in theorem be satisfied. Then by (v)

$$\begin{aligned} \| x - m_0 \| &= |f(x - m_0)| = \\ &= |f(x - m)| && \text{by (iv)} \\ &\leqq \| x - m \| && \text{by (vi)} \end{aligned}$$

for every $m \in M$. This implies that $m_0 \in L_M(x)$, which completes the proof.

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