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BOUNDEDNESS AND UNBOUNDEDNESS OF SOLUTIONS OF AN N-TH ORDER DIFFERENTIAL EQUATION WITH DELAYED ARGUMENT

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Consider an n-th order differential equation with delayed argument

(1) $$L_n y + a(t) f(y(g(t))) = b(t),$$

with $n \geqq 2$ and L_n a differential operator

$$L_n y = a_n(t)\,(a_{n-1}(t)\,(\dots(a_1(t)\,(a_0(t)\,y)')' \dots)')'.$$

Suppose that $a(t), b(t), g(t), a_0(t), \dots, a_n(t)$ are continuous on $\langle t_0, \infty)$ and $f(y)$ is continuous on $(-\infty, \infty)$.

We shall prove that certain conditions are necessary and sufficient for all solutions of (1) to be bounded. The sufficient conditions for nonoscillatory solutions of (1) are different from that given in paper [4].

Let us use the following notational conventions:

(2) (a) $L_0 y = a_0(t)\,y, \quad L_i y = a_i(t)\,(L_{i-1}y)', \quad i = 1, 2, \dots, n;$

(b) $I_0 = 1,$

$$I_k(t, s, a_{i_k}, \dots, a_{i_1}) = \int_s^t \frac{1}{a_{i_k}(r)} I_{k-1}(r, s, a_{i_{k-1}}, \dots, a_{i_1})\,\mathrm{d}r$$

$i_k \in \{1, \dots, n-1\}, \qquad 1 \leqq k \leqq n-1,\ t, s \in \langle t_0, \infty),\ s < t;$

(c) $$J_i(t, s) = \frac{1}{a_0(t)} I_i(t, s, a_1, \dots, a_i);$$

(d) $$K_i(t, s) = \frac{1}{a_n(t)} I_i(t, s, a_{n-1}, \dots, a_{n-i}).$$

It is easy to see that

$$I_k(t, s, a_{i_k}, \dots, a_{i_1}) = \int_s^t \frac{1}{a_{i_1}(r)} I_{k-1}(t, r, a_{i_k}, \dots, a_{i_2})\,\mathrm{d}r.$$

It will be supposed throughout that:

(3) (a) $\lim_{t\to\infty} g(t) = \infty$;

(b) $a(t) \geqq 0, \quad a_i(t) > 0, \quad$ for $i = 0, 1, \ldots, n$;

(c) $\lim_{t\to\infty} J_{n-1}(t, t_0) < \infty$.

We shall consider those solutions of (1) which exist on $\langle t_0, \infty)$.

Lemma 1. *Let $a_i(t) > 0$ on $\langle t_0, \infty)$. Then there exist constants α, β such that*

$$J_i(t, s) \leqq \alpha J_{n-1}(t, s),$$

$$K_i(t, s) \leqq \beta K_{n-1}(t, s) \qquad \text{for } i = 1, \ldots, n-2, s < t, s, t \in \langle t_0, \infty).$$

Proof. We have

$$J_{i+1}(t, s) = \frac{1}{a_0(t)} \int_s^t \frac{ds_1}{a_1(s_1)} \int_s^{s_1} \frac{ds_2}{a_2(s_2)} \cdots \int_s^{s_i} \frac{ds_{i+1}}{a_{i+1}(s_{i+1})} =$$

$$= \frac{1}{a_0(t)} \int_s^t \frac{ds_1}{a_1(s_1)} \int_s^{s_1} \frac{ds_2}{a_2(s_2)} \cdots \int_s^{s_{i-1}} \frac{ds_i}{a_i(s_i)} \left[\int_s^b \frac{ds_{i+1}}{a_{i+1}(s_{i+1})} + \int_b^{s_i} \frac{ds_{i+1}}{a_{i+1}(s_{i+1})} \right] \geqq$$

$$\geqq \int_s^b \frac{ds_{i+1}}{a_{i+1}(s_{i+1})} J_i(t, s),$$

hence

$$J_i(t, s) \leqq \alpha_i J_{i+1}(t, s)$$

and therefore, in particular

$$J_i(t, s) \leqq \alpha J_{n-1}(t, s)$$

for every $i = 1, \ldots, n-2$.

The proof of the statement for $K_i(t, s)$ is analogous. This completes the proof of Lemma 1.

Theorem 1. *Let conditions* (3) *be satisfied. Let $f(y)$ be bounded on $(-\infty, \infty)$. If*

(4) $$\lim_{t\to\infty} \int_{t_0}^t \frac{a(r) J_{n-1}(t, r)}{a_n(r)} dr < \infty$$

and

(5) $$\left| \lim_{t\to\infty} \int_{t_0}^t \frac{b(r) J_{n-1}(t, r)}{a_n(r)} dr \right| < \infty$$

then every solution of (1) *is bounded on* $\langle t_0, \infty)$.

Proof. Let $y(t)$ be a solution of (1) defined on $\langle t_0, \infty)$. There exists $T \geqq t_0$ such that $g(t) \geqq t_0$ for every $t \geqq T$. n – tuple integration from T to t, where (1) is multiplied by $\dfrac{1}{a_{n-i+1}(t)}$ before each integration, yields

(6)
$$a_0(t)\,y(t) = \sum_{i=0}^{n-1} c_i I_i(t, T, a_1, \dots, a_i) +$$
$$+ \int_T^t I_{n-1}(t, r, a_1, \dots, a_{n-1}) \frac{b(r) - a(r)\,f(y(g(r)))}{a_n(r)}\,\mathrm{d}r,$$

where c_i for $0 \leqq i \leqq n - 1$ are constants.

Owing to Lemma 1 and to the boundedness of $f(y)$, it follows that

$$|\,y(t)\,| \leqq cJ_{n-1}(t, t_0) + \left| \int_T^t \frac{b(r)\,J_{n-1}(t, r)}{a_n(r)}\,\mathrm{d}r \right| +$$
$$+ c_1^* \int_T^t \frac{a(r)\,J_{n-1}(t, r)}{a_n(r)}\,\mathrm{d}r,$$

and the statement of the theorem is immediately proved using (3c), (4) and (5).

Theorem 2. *Suppose that, in addition to* (4) *and* (3)

$$\left| \lim_{t \to \infty} \int_{t_0}^t \frac{b(r)\,J_{n-1}(t, r)}{a_n(r)}\,\mathrm{d}r \right| = \infty.$$

Then every solution of (1) *is unbounded on* $\langle t_0, \infty)$.

Proof. Let $y(t)$ be an arbitrary solution of (1) defined on $\langle t_0, \infty)$. Consider $T \geqq t_0$ such that, for every $t \geqq T$, $g(t) \geqq t_0$. If $y(t)$ is bounded, then because of the continuity of $f(y)$ there exists a constant K such that

$$\left| \int_T^t \frac{a(r)\,J_{n-1}(t, r)\,f(y(g(r)))}{a_n(r)}\,\mathrm{d}r \right| < K \int_T^t \frac{a(r)\,J_{n-1}(t, r)}{a_n(r)}\,\mathrm{d}r.$$

Together with the hypotheses of the theorem this can be used to prove that the right part of (6) is unbounded as $t \to \infty$ and so therefore we have $y(t)$. This completes the proof.

Theorem 3. *Let* $yf(y) > 0$ *for* $y \neq 0$. *If* (3) *and* (5) *hold, then every nonoscillatory solution of* (1) *is bounded on* $\langle t_0, \infty)$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1) defined on $\langle t_0, \infty)$ and suppose e.g. that $y(t) > 0$ for every $t \geqq t_1$. Owing to (3a) there exists $T \geqq t_1$ such that $y(g(t)) > 0$ for every $t \geqq T$. Since $yf(y) > 0$, $f(y(g(t))) > 0$ for every $t \geqq T$; therefore relation (6) yields

$$a_0(t)\,y(t) \leqq \sum_{i=0}^{n-1} c_i I_i(t, T, a_1, \dots, a_i) +$$
$$+ \int_T^t \frac{b(r)\,I_{n-1}(t, r, a_1, \dots, a_{n-1})}{a_n(r)}\,\mathrm{d}r.$$

Therefore $y(t)$ is bounded. The proof for $y(t) < 0$ is analogous. This completes the proof.

Remark 1. The sufficient condition for boundary of nonoscillatory solutions of the equation (1) stated in the Theorem 3 does not follow from the condition which was stated in the Theorem 1 in [4].

Example. Consider the equation

$$(t^2 y'(t))' + \frac{1}{t^2}[y(t)]^{-1} = \frac{1}{t}. \tag{7}$$

The assumptions of Theorem 3 are satisfied, but assumptions of Theorem 1 from the paper [4] are not satisfied. The equation (7) has nonoscillatory solution $y(t) = \frac{1}{t}$ bounded.

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