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OPERATIONS ON CYCLICALLY ORDERED SETS

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The aim of this paper is to define direct operations on the class of cyclically ordered sets and to show their basic properties. These properties are similar to those of cardinal (direct) operations on the class of ordered sets ([1], [2]).

1. Cyclically ordered sets

Let G be a set, let C be a ternary relation on G, i.e. $C \subseteq G^3$. This relation is called a *cyclic order* on G iff it is

(1) asymmetric, i.e. $(x, y, z) \in C \Rightarrow (z, y, x) \notin C$
(2) cyclic, i.e. $(x, y, z) \in C \Rightarrow (y, z, x) \in C$
(3) transitive, i.e. $(x, y, z) \in C, (x, z, u) \in C \Rightarrow (x, y, u) \in C$.

If G is a set and C is a cyclic order on G then the pair $\mathbf{G} = (G, C)$ is called a *cyclically ordered set*; the set G is called a *carrier* of $\mathbf{G}$ and denoted $\mathscr{C}(\mathbf{G})$, the set C is called a *relation* of $\mathbf{G}$ and denoted $\mathscr{R}(\mathbf{G})$.

A cyclically ordered set $\mathbf{G} = (G, C)$ is called a *cycle* iff the relation C is

(4) complete, i.e. $x, y, z \in G, x \neq y \neq z \neq x \Rightarrow (x, y, z) \in C$ or $(z, y, x) \in C$.

A cyclically ordered set $\mathbf{G}$ is called *discrete* iff $\mathscr{R}(\mathbf{G}) = \emptyset$.

An isomorphism of a cyclically ordered set $\mathbf{G} = (G, C)$ onto a cyclically ordered set $\mathbf{H} = (H, D)$ is a bijective mapping $f : G \to H$ with the property $(x, y, z) \in C \Leftrightarrow (f(x), f(y), f(z)) \in D$. Cyclically ordered sets $\mathbf{G}, \mathbf{H}$ are isomorphic iff there exists an isomorphism of $\mathbf{G}$ onto $\mathbf{H}$; we write in that case $\mathbf{G} \cong \mathbf{H}$.

Let us call a ternary relation T on a set G

(5) reflexive, iff $(x, x, x) \in T$ for any $x \in G$,
(6) antisymmetric, iff $(x, y, z) \in T, (z, y, x) \in T \Rightarrow x = y = z$.

For any ternary relation T on a set G let us denote $T^= = T \cup \{(x, x, x); x \in G\}$.

1.1 Lemma. *Let C be a cyclic order on a set G. Then the relation $C^=$ is reflexive, antisymmetric, cyclic and transitive.*

Proof. The reflexivity of $C^=$ follows from the definition. Let $(x, y, z) \in C^=$, $(z, y, x) \in C^=$ and suppose that $x = y = z$ does not hold. Then $(x, y, z) \in C$, $(z, y, x) \in C$ which contradicts the asymmetry of C. Thus $C^=$ is antisymmetric. Let $(x, y, z) \in C^=$. Then either $(x, y, z) \in C$ or $x = y = z$; in both cases it is $(y, z, x) \in C^=$ and $C^=$ is cyclic. Assume $(x, y, z) \in C^=$, $(x, z, u) \in C^=$. If $x = y = z$, then necessarily $x = z = u$ and vice versa. Thus we have only two possibilities: either $(x, y, z) \in C$, $(x, z, u) \in C$ or $x = y = z = u$. In both cases it is $(x, y, u) \in C^=$ and $C^=$ is transitive.

1.2 Lemma. *Let T be a reflexive, antisymmetric, cyclic and transitive ternary relation on a set G. Then the relation $C = T - \{(x, x, x);\ x \in G\}$ is a cyclic order on G.*

Proof. Assume $(x, y, z) \in C$, $(z, y, x) \in C$. Then also $(x, y, z) \in T$, $(z, y, x) \in T$, thus $x = y = z$ and this is a contradiction. Hence C is asymmetric. Let $(x, y, z) \in C$. Then $(x, y, z) \in T$ and $x = y = z$ does not hold. This implies $(y, z, x) \in T$, thus $(y, z, x) \in C$ and C is cyclic. Let $(x, y, z) \in C$, $(x, z, u) \in C$. Then $(x, y, z) \in T$, $(x, z, u) \in T$ and neither $x = y = z$ nor $x = z = u$. As T is transitive, we have $(x, y, u) \in T$. Assume $x = y = u$; then especially $x = u$ and we have $(x, z, x) \in T$. The antisymmetry of T then implies $x = z$ so that $x = y = z$ which is a contradiction. Hence $(x, y, u) \in C$ and C is transitive.

With respect to 1.1 and 1.2 a pair $\mathbf{G} = (G, C^=)$,where $C^=$ is a reflexive, antisymmetric, cyclic and transitive ternary relation on a set G, could be called a cyclically ordered set.

2. Sum and product

2.1 Definition. *Let I be a set, let $(\mathbf{G}_i;\ i \in I)$ be a family of cyclically ordered sets. Let $\mathscr{C}(\mathbf{G}_i) \cap \mathscr{C}(\mathbf{G}_j) = \emptyset$ for $i \neq j$. The direct sum $\sum_{i \in I} \mathbf{G}_i$ of this family is $\mathbf{G} = (G, C)$, where $G = \bigcup_{i \in I} \mathscr{C}(\mathbf{G}_i)$, $C = \bigcup_{i \in I} \mathscr{R}(\mathbf{G}_i)$.*

2.2 Lemma. *Let $(\mathbf{G}_i;\ i \in I)$ be a family of cyclically ordered sets and let $\mathscr{C}(\mathbf{G}_i) \cap \cap \mathscr{C}(\mathbf{G}_j) = \emptyset$ for $i \neq j$. Then $\sum_{i \in I} \mathbf{G}_i$ is a cyclically ordered set.*

Proof is trivial.

Especially, if $I = \{1, \ldots, n\}$, then the direct sum $\sum_{i \in I} \mathbf{G}_i$ is written as $\mathbf{G}_1 + \ldots \ldots + \mathbf{G}_n$.

2.3 Remark. Let $(\mathbf{G}_i;\ i \in I) = ((G_i, C_i);\ i \in I)$ be a family of cyclically ordered sets with $G_i \cap G_j = \emptyset$ for $i \neq j$. Then the canonical insertion $j_i : G_i \to \bigcup_{i \in I} G_i$ given by $j_i(x) = x$ for $x \in G_i$ is an isomorphic embedding of G_i into $\sum_{i \in I} G_i$.

2.4 Definition. *Let* $\mathbf{H} = (H, D)$ *be a cyclically ordered set, let* $(\mathbf{G}_i;\ i \in H) = ((G_i, C_i);\ i \in H)$ *be a family of cyclically* ordered sets. Put $G = \{(i, x);\ i \in H, x \in G_i\}$ and $((i_1, x_1), (i_2, x_2), (i_3, x_3)) \in C$ *iff either* $(i_1, i_2, i_3) \in D$ *or* $i_1 = i_2 = i_3$ *and* $(x_1, x_2, x_3) \in C_{i_1}$. *Then* $\mathbf{G} = (G, C)$ *is called an ordered sum of the family* $(\mathbf{G}_i;\ i \in H)$ *and we denote it* $\mathbf{G} = \dot{\sum\limits_{i \in H}} \mathbf{G}_i$.

2.5 Lemma. *Let* $\mathbf{H} = (H, D)$ *be a cyclically ordered set, let* $(\mathbf{G}_i;\ i \in H)$ *be a family of cyclically ordered sets. Then* $\dot{\sum\limits_{i \in H}} \mathbf{G}_i$ *is a cyclically ordered set.*

Proof. Denote $C_i = \mathscr{R}(\mathbf{G}_i)$, $C = \mathscr{R}(\dot{\sum\limits_{i \in H}} \mathbf{G}_i)$. Assume $((i_1, x_1), (i_2, x_2), (i_3, x_3)) \in C$. If $(i_1, i_2, i_3) \in D$, then neither $(i_3, i_2, i_1) \in D$ nor $i_1 = i_2 = i_3$ and it is $(i_2, i_3, i_1) \in D$. In this case $((i_3, x_3), (i_2, x_2), (i_1, x_1)) \mathrel{\bar{\in}} C$, $((i_2, x_2), (i_3, x_3), (i_1, x_1)) \in C$. If $i_1 = i_2 = i_3$, then $(x_1, x_2, x_3) \in C_{i_1}$, so that $(x_3, x_2, x_1) \mathrel{\bar{\in}} C_{i_1}$, $(x_2, x_3, x_1) \in C_{i_1}$ and $((i_3, x_3), (i_2, x_2), (i_1, x_1)) \mathrel{\bar{\in}} C$, $((i_2, x_2), (i_3, x_3), (i_1, x_1)) \in C$. We have shown that C is asymmetric and cyclic. Let $((i_1, x_1), (i_2, x_2), (i_3, x_3)) \in C$, $((i_1, x_1), (i_3, x_3), (i_4, x_4)) \in C$. If $(i_1, i_2, i_3) \in D$, then $i_1 \neq i_2 \neq i_3 \neq i_1$ so that necessarily $i_1 \neq i_3 \neq i_4 \neq i_1$. Thus $(i_1, i_3, i_4) \in D$, the transitivity of D implies $(i_1, i_2, i_4) \in D$ and from this $((i_1, x_1), (i_2, x_2), (i_4, x_4)) \in C$. If $i_1 = i_2 = i_3$, then also $i_1 = i_3 = i_4$ and $(x_1, x_2, x_3) \in C_{i_1}$, $(x_1, x_3, x_4) \in C_{i_1}$. This implies $(x_1, x_2, x_4) \in C_{i_1}$ and $((i_1, x_1), (i_2, x_2), (i_4, x_4)) \in C$. C is transitive and hence it is a cyclic order.

2.6 Remark. Let $\mathbf{H}$ be a discrete cyclically ordered set, let $(\mathbf{G}_i;\ i \in H)$ be a family of cyclically ordered sets and let $\mathscr{C}(\mathbf{G}_i) \cap \mathscr{C}(\mathbf{G}_j) = \emptyset$ for $i \neq j$. Then $\dot{\sum\limits_{i \in H}} \mathbf{G}_i \cong \sum\limits_{i \in H} \mathbf{G}_i$.

2.7 Definition. *Let* I *be a set, let* $(\mathbf{G}_i;\ i \in I) = ((G_i, C_i);\ i \in I)$ *be a family of cyclically ordered sets. Put* $G = \underset{i \in I}{\times} G_i$ *and for* $x, y, z \in G$ *let* $(x, y, z) \in C^=$ *iff* $(pr_i x, pr_i y, pr_i z) \in C_i^=$ *for every* $i \in I$. *Then* $\mathbf{G} = (G, C^=)$ *is called a direct product of the family* $(\mathbf{G}_i;\ i \in I)$ *and it is denoted by* $\mathbf{G} = \prod\limits_{i \in I} \mathbf{G}_i$.

Again, if $I = \{1, \ldots, n\}$ then we write $\prod\limits_{i \in I} \mathbf{G}_i = \mathbf{G}_1 \ldots \mathbf{G}_n$.

2.8 Lemma. *Let* $(\mathbf{G}_i;\ i \in I)$ *be a family of cyclically ordered sets. Then* $\prod\limits_{i \in I} \mathbf{G}_i$ *is a cyclically ordered set.*

Proof is trivial.

3. Homomorphisms

3.1 Definition. *Let* $\mathbf{G} = (G, C)$, $\mathbf{H} = (H, D)$ *be cyclically ordered sets, let* $f : G \to H$. *The mapping* f *is called a homomorphism of* $\mathbf{G}$ *into* $\mathbf{H}$ *iff it has the property*:

$$x, y, z \in G, \qquad (x, y, z) \in C^= \Rightarrow (f(x), f(y), f(z)) \in D^= .$$

3.2 Lemma. *Let* $(\mathbf{G}_i;\ i \in I) = ((G_i, C_i);\ i \in I)$ *be a family of cyclically ordered sets, let* $G_i \cap G_j = \emptyset$ *for* $i \neq j$ *and let* $\mathbf{G} = (G, C) = \sum_{i \in I} \mathbf{G}_i$. *Then* C *is the least cyclic order (with respect to the set inclusion) on the set* G *for which all canonical insertions* j_i *are homomorphisms of* $\mathbf{G}_i$ *into* $\mathbf{G}$.

Proof. Evidently all $j_i : G_i \to G$ are homomorphisms of $\mathbf{G}_i$ into $\mathbf{G}$. Let D be any cyclic order on the set G with the property that all j_i are homomorphisms. If $(x, y, z) \in C$, then there exists $i \in I$ with $x, y, z \in G_i$, $(x, y, z) \in C_i$. Then $(j_i(x), j_i(y), j_i(z)) = (x, y, z) \in D$ and thus $C \subseteq D$.

3.3 Lemma. *Let* $(\mathbf{G}_i;\ i \in I) = ((G_i, C_i);\ i \in I)$ *be a family of cyclically ordered sets and let* $\mathbf{G} = (G, C) = \prod_{i \in I} \mathbf{G}_i$. *Then* $C^=$ *is the greatest cyclic order on the set* G *for which all projections* pr_i *are homomorphisms of* $\mathbf{G}$ *onto* $\mathbf{G}_i$.

Proof. From the definition there follows that all projections pr_i are homomorphisms of $\mathbf{G}$ onto $\mathbf{G}_i$. Let $D^=$ be any cyclic order on the set G for which all pr_i are homomorphisms and let $(x, y, z) \in D^=$. Then $(pr_i x, pr_i y, pr_i z) \in C_i^=$ for all $i \in I$ which implies $(x, y, z) \in C^=$. Thus $D^= \subseteq C^=$.

3.4 Definition. *Let* $\mathbf{G} = (G, C)$, $\mathbf{H} = (H, D)$ *be cyclically ordered sets, let* $f : G \to H$. *The mapping* f *is called a strong homomorphism of* $\mathbf{G}$ *into* $\mathbf{H}$ *iff it has the property*:

$$(x, y, z) \in C \Rightarrow (f(x), f(y), f(z)) \in D^=,$$

$$(x, y, z) \mathrel{\bar{\in}} C \Rightarrow (f(x), f(y), f(z)) \mathrel{\bar{\in}} D.$$

For the purposes of the next theorem we now give an alternative definition of an ordered sum of cyclically ordered sets; namely, we need that „summands" $\mathbf{G}_j$ were substructures of the sum $\mathbf{G} = \dot{\sum}_{i \in H} \mathbf{G}_i$.

3.5 Remark. Let $\mathbf{H} = (H, D)$ be a cyclically ordered set, let $(\mathbf{G}_i;\ i \in H) = ((G_i, C_i);\ i \in H)$ be a family of cyclically ordered sets with $G_i \cap G_j = \emptyset$ for $i \neq j$. Put $G = \bigcup_{i \in H} G_i$ and for $x, y, z \in G$ let $(x, y, z) \in C$ be equivalent with either $x \in G_i$, $y \in G_j$, $z \in G_k$ and $(i, j, k) \in D$ or $x, y, z \in G_i$ and $(x, y, z) \in C_i$. Then the cyclically ordered set $\mathbf{G} = (G, C)$ is called an ordered sum of the family $(\mathbf{G}_i;\ i \in H)$ and is denoted by $\dot{\sum}_{i \in H} \mathbf{G}_i$.

3.6 Theorem. *Let* $\mathbf{G} = (G, C)$, $\mathbf{H} = (H, D)$ *be cyclically ordered sets, let* $f: G \to H$ *be a surjective mapping. For any* $i \in H$ *let us denote* $G_i = f^{-1}(\{i\})$, $C_i = C \cap G_i^3$, $\mathbf{G}_i = (G_i, C_i)$. *Then the following statements are equivalent*:

(A) $\mathbf{G} = \dot{\sum\limits_{i \in H}} \mathbf{G}_i$,

(B) f *is a strong homomorphism of* $\mathbf{G}$ *onto* $\mathbf{H}$.

Proof. 1. Let (A) hold and let $x_1, x_2, x_3 \in G$, $(x_1, x_2, x_3) \in C$. Then either $x_1 \in G_{i_1}, x_2 \in G_{i_2}, x_3 \in G_{i_3}$ and $(i_1, i_2, i_3) \in D$ or $x_1, x_2, x_3 \in G_i$ and $(x_1, x_2, x_3) \in C_i$. In the first case it is $(f(x_1), f(x_2), f(x_3)) = (i_1, i_2, i_3) \in D$, in the second one it is $f(x_1) = f(x_2) = f(x_3) = i$; thus it is $(f(x_1), f(x_2), f(x_3)) \in D^=$. Let $x_1, x_2, x_3 \in G$, $(x_1, x_2, x_3) \bar{\in} C$. Then either $x_1 \in G_{i_1}$, $x_2 \in G_{i_2}$, $x_3 \in G_{i_3}$ and $(i_1, i_2, i_3) \bar{\in} D^=$ or $x_1, x_2, x_3 \in G_i$ and $(x_1, x_2, x_3) \bar{\in} C_i$. In the first case we have $(f(x_1), f(x_2), f(x_3)) = (i_1, i_2, i_3) \bar{\in} D^=$, in the second one $(f(x_1), f(x_2), f(x_3)) = (i, i, i) \bar{\in} D$. Thus, in both cases $(f(x_1), f(x_2), f(x_3)) \bar{\in} D$, f is a strong homomorphism and (B) holds. 2. Let (B) hold. As $G = \bigcup\limits_{i \in H} G_i$ and the sets G_i are pairwise disjoint, it suffices to prove $C = \mathscr{R}(\dot{\sum\limits_{i \in H}} \mathbf{G}_i)$; denote briefly $\mathscr{R}(\dot{\sum\limits_{i \in H}} \mathbf{G}_i) = E$. Let $(x_1, x_2, x_3) \in C$. Then either $(f(x_1), f(x_2), f(x_3)) \in D$ or $f(x_1) = f(x_2) = f(x_3)$. In the first case denote $f(x_1) = i_1, f(x_2) = i_2, f(x_3) = i_3$; thus, $x_1 \in G_{i_1}, x_2 \in G_{i_2}, x_3 \in G_{i_3}$ and $(i_1, i_2, i_3) \in D$ so that $(x_1, x_2, x_3) \in E$. In the second case denote $f(x_1) = f(x_2) = f(x_3) = i$; thus, $x_1, x_2, x_3 \in G_i$ and $(x_1, x_2, x_3) \in C_i$ so that $(x_1, x_2, x_3) \in E$. We have proved $C \subseteq E$. Let $x_1, x_2, x_3 \in G$, $(x_1, x_2, x_3) \bar{\in} C$. Then $(f(x_1), f(x_2), f(x_3)) \bar{\in} D$. If $f(x_1) = f(x_2) = f(x_3) = i$, then $x_1, x_2, x_3 \in G_i$ and $(x_1, x_2, x_3) \bar{\in} C_i$ so that $(x_1, x_2, x_3) \bar{\in} E$. If $f(x_1) = f(x_2) = f(x_3)$ does not hold, denote $f(x_1) = i_1, f(x_2) = i_2, f(x_3) = i_3$; thus, $(i_1, i_2, i_3) \bar{\in} D^=$ and $x_1 \in G_{i_1}, x_2 \in G_{i_2}, x_3 \in G_{i_3}$ so that $(x_1, x_2, x_3) \bar{\in} E$. Hence $E \subseteq C$, thus $C = E$ and (A) holds.

4. Power

4.1 Definition. *Let* $\mathbf{G} = (G, C)$, $\mathbf{H} = (H, D)$ *be cyclically ordered sets. A power* $\mathbf{G}^{\mathbf{H}}$ *is a set of all homomorphisms* $f: \mathbf{H} \to \mathbf{G}$ *together with a ternary relation* $E^=$ *defined by* $(f, g, h) \in E^= \Leftrightarrow (f(x), g(x), h(x)) \in C^=$ for all $x \in H$.

4.2 Lemma. *Let* $\mathbf{G}, \mathbf{H}$ *be cyclically ordered sets. Then* $\mathbf{G}^{\mathbf{H}}$ *is a cyclically ordered set.*

Proof is trivial.

4.3 Example. For any positive integer $n \geq 3$ let $\boldsymbol{n}$ denote an n-elemented cycle, i.e. $\boldsymbol{n} = \{\{0, 1, \ldots, n - 1\}, \{(i, j, k);$ either $i < j < k$ or $j < k < i$ or $k < i < j$, $0 \leq i, j, k \leq n - 1\}\}$. Let $m \geq 3$, $n \geq 3$ be positive integers. Then it holds:

(a) if $m < n$, then $\boldsymbol{m}^{\boldsymbol{n}} = \boldsymbol{m}$,

(b) if $m = n$, then $\boldsymbol{n}^{\boldsymbol{n}} = \boldsymbol{n} + \boldsymbol{n}$,

(c) if $m > n$, then card $\mathscr{C}(\boldsymbol{m}^{\boldsymbol{n}}) = m + \binom{m}{n}$.

Proof. (a) If $m < n$, then the only homomorphisms $\boldsymbol{n} \to \boldsymbol{m}$ are constant mappings since if $f : \mathscr{C}(\boldsymbol{n}) \to \mathscr{C}(\boldsymbol{m})$ is not constant, then there exist element $i, j, k \in \mathscr{C}(\boldsymbol{n})$ with $(i, j, k) \in \mathscr{R}(\boldsymbol{n})$ and card $\{f(i), f(j), f(k)\} = 2$ so that $(f(i), f(j), f(k)) \bar{\in} \mathscr{R}(\boldsymbol{m})^{=}$. Thus f is not a homomorphism.

(b) $f : \mathscr{C}(\boldsymbol{n}) \to \mathscr{C}(\boldsymbol{n})$ is a homomorphism of $\boldsymbol{n}$ into $\boldsymbol{n}$ iff either f is a constant mapping or there exists $k \in \{0, 1, \ldots, n - 1\}$ such that $f(i) = i + k (\mathrm{mod}\, n)$. From this the assertion follows.

(c) If $m > n$, then homomorphisms $\boldsymbol{n} \to \boldsymbol{m}$ are constant mappings and the mappings of $\mathscr{C}(\boldsymbol{n})$ on n-elemented cycles in $\boldsymbol{m}$ constructed analogously as in (b). As there are $\binom{m}{n}$ n-elemented cycles in $\boldsymbol{m}$, we have the desired equality.

4.4 Theorem. *Let* $\mathbf{G} = (G, C)$, $\mathbf{H}_1 = (H_1, D_1)$, $\mathbf{H}_2 = (H_2, D_2)$ *be cyclically ordered sets and let* $H_1 \cap H_2 = \emptyset$. *Then* $\mathbf{G}^{\mathbf{H}_1 + \mathbf{H}_2} \cong \mathbf{G}^{\mathbf{H}_1} . \mathbf{G}^{\mathbf{H}_2}$.

Proof. For any $f \in \mathscr{C}(\mathbf{G}^{\mathbf{H}_1 + \mathbf{H}_2})$ denote $f_1 = f|_{H_1}, f_2 = f|_{H_2}$ ($f|_A$ denotes a restriction of the mapping f onto a set A). Evidently, $f_1 \in \mathscr{C}(\mathbf{G}^{\mathbf{H}_1})$, $f_2 \in \mathscr{C}(\mathbf{G}^{\mathbf{H}_2})$ so that $(f_1, f_2) \in \mathscr{C}(\mathbf{G}^{\mathbf{H}_1} . \mathbf{G}^{\mathbf{H}_2})$. Conversely, if $f_1 \in \mathscr{C}(\mathbf{G}^{\mathbf{H}_1})$, $f_2 \in \mathscr{C}(\mathbf{G}^{\mathbf{H}_2})$, then $f = f_1 \cup f_2 \in \mathscr{C}(\mathbf{G}^{\mathbf{H}_1 + \mathbf{H}_2})$ and $f_1 = f|_{H_1}$, $f_2 = f|_{H_2}$. This shows that the correspondence $f \to (f_1, f_2)$ is a bijection of $\mathscr{C}(\mathbf{G}^{\mathbf{H}_1 + \mathbf{H}_2})$ onto $\mathscr{C}(\mathbf{G}^{\mathbf{H}_1} . \mathbf{G}^{\mathbf{H}_2})$. But it is an isomorphism: if $f, g, h \in \mathscr{C}(\mathbf{G}^{\mathbf{H}_1 + \mathbf{H}_2})$, $(f, g, h) \in \mathscr{R}(\mathbf{G}^{\mathbf{H}_1 + \mathbf{H}_2})^{=}$, then $(f(x), g(x), h(x)) \in C^{=}$ for all $x \in H_1 \cup H_2$ which shows both $(f_1, g_1, h_1) \in \mathscr{R}(\mathbf{G}^{\mathbf{H}_1})^{=}$ and $(f_2, g_2, h_2) \in \mathscr{R}(\mathbf{G}^{\mathbf{H}_2})^{=}$ i.e. $((f_1, f_2), (g_1, g_2), (h_1, h_2)) \in \mathscr{R}(\mathbf{G}^{\mathbf{H}_1} . \mathbf{G}^{\mathbf{H}_2})^{=}$. This consideration can be turned vice versa which proves the assertion.

4.5 Theorem. *Let* $\mathbf{G}_1 = (G_1, C_1)$, $\mathbf{G}_2 = (G_2, C_2)$, $\mathbf{H} = (H, D)$ *be cyclically ordered sets. Then* $(\mathbf{G}_1 . \mathbf{G}_2)^{\mathbf{H}} \cong \mathbf{G}_1^{\mathbf{H}} . \mathbf{G}_2^{\mathbf{H}}$.

Proof. For any $f \in \mathscr{C}((\mathbf{G}_1 . \mathbf{G}_2)^{\mathbf{H}})$ denote $f_1 = pr_1 f$, $f_2 = pr_2 f$; then $f_1 \in \mathscr{C}(\mathbf{G}_1^{\mathbf{H}})$ $f_2 \in \mathscr{C}(\mathbf{G}_2^{\mathbf{H}})$. One can easily show that the assignment $f \to (f_1, f_2)$ is a bijection of $\mathscr{C}((\mathbf{G}_1 . \mathbf{G}_2)^{\mathbf{H}})$ onto $\mathscr{C}(\mathbf{G}_1^{\mathbf{H}} . \mathbf{G}_2^{\mathbf{H}})$ and, in fact, it is an isomorphism of $(\mathbf{G}_1 . \mathbf{G}_2)^{\mathbf{H}}$ onto $\mathbf{G}_1^{\mathbf{H}} . \mathbf{G}_2^{\mathbf{H}}$.

4.6 Theorem. *Let* $\mathbf{G} = (G, C)$, $\mathbf{H} = (H, D)$, $\mathbf{K} = (K, E)$ *be cyclically ordered sets. Then there exists an isomorphic embedding of* $\mathbf{G}^{\mathbf{H}.\mathbf{K}}$ *into* $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$.

Proof. Let $f \in \mathscr{C}(\mathbf{G}^{\mathbf{H}.\mathbf{K}})$ be any element. Thus, f is a mapping of $H \times K$ into G. Denote for any $y \in K$ by f_y a mapping $f_y : H \to G$ defined by $f_y(x) = f(x, y)$. We show that $f_y \in \mathscr{C}(\mathbf{G}^{\mathbf{H}})$. If $x_1, x_2, x_3 \in H$, $(x_1, x_2, x_3) \in D^{=}$, then $((x_1, y), (x_2, y), (x_3, y)) \in \mathscr{R}(\mathbf{H} . \mathbf{K})^{=}$ so that $(f(x_1, y), f(x_2, y), f(x_3, y)) \in C^{=}$, i.e. $(f_y(x_1), f_y(x_2), f_y(x_3)) \in C^{=}$. This means that f_y is a homomorphism of $\mathbf{H}$ into $\mathbf{G}$, i.e. $f_y \in \mathscr{C}(\mathbf{G}^{\mathbf{H}})$. Further, if $x \in H$ is any element and $y_1, y_2, y_3 \in K$, $(y_1, y_2, y_3) \in E^{=}$, then $((x, y_1), (x, y_2), (x, y_3)) \in \mathscr{R}(\mathbf{H} . \mathbf{K})^{=}$, thus $(f(x, y_1), f(x, y_2), f(x, y_3)) \in C^{=}$, i.e. $(f_{y_1}(x),$

$f_{y_2}(x), f_{y_3}(x)) \in C^=$ which means $(f_{y_1}, f_{y_2}, f_{y_3}) \in \mathscr{R}(\mathbf{G^H})$. In other words, the assignment $y \to f_y$ is a homomorphic mapping of $\mathbf{K}$ into $\mathbf{G^H}$, i.e. an element of $\mathscr{C}((\mathbf{G^H})^\mathbf{K})$. Denote it by $\varphi(f)$, i.e. $\varphi(f)(y) = f_y$ for $f \in \mathscr{C}(\mathbf{G^{H.K}})$ and $y \in K$. Then $\varphi : \mathscr{C}(\mathbf{G^{H.K}}) \to \to \mathscr{C}((\mathbf{G^H})^\mathbf{K})$; we show that φ is an isomorphism of $\mathbf{G^{H.K}}$ into $(\mathbf{G^H})^\mathbf{K}$. If $f, g \in \mathscr{C}\mathbf{G^{H.K}})$ and $f \neq g$, then $f(x, y) \neq g(x, y)$ for some $(x, y) \in H \times K$, i.e. $f_y(x) \neq g_y(x)$ for some $y \in K$ and some $x \in H$, i.e. $f_y \neq g_y$ for some $y \in K$ which shows that φ is an injection. If $(f, g, h) \in \mathscr{R}(\mathbf{G^{H.K}})^=$, then $(f(x, y), g(x, y), h(x, y)) \in C^=$ for any $(x, y) \in \in H \times K$, thus $(f_y(x), g_y(x), h_y(x)) \in C^=$ for any $x \in H$ and any $y \in K$, which means $(f_y, g_y, h_y) \in \mathscr{R}(\mathbf{G^H})^=$ for any $y \in K$ and $(\varphi(f), \varphi(g), \varphi(h)) \in \mathscr{R}((\mathbf{G^H})^\mathbf{K})^=$. Conversely, $(\varphi(f), \varphi(g), \varphi(h)) \in \mathscr{R}((\mathbf{G^H})^\mathbf{K})^=$ implies $(f, g, h) \in \mathscr{R}(\mathbf{G^{H.K}})^=$ and φ is an isomorphism of $\mathbf{G^{H.K}}$ into $(\mathbf{G^H})^\mathbf{K}$.

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