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Archivum Mathematicum, Vol. 20 (1984), No. 3, 101--112

Persistent URL: http://dml.cz/dmlcz/107193

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XX: 101–112, 1984

THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE DIFFERENTIAL EQUATION OF THE THIRD ORDER

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(Received April 18, 1983)

Consider the differential equation

$$y''' = f(t, y, y', y''), \tag{1}$$

where f, defined on $D = \{(t, x_1, x_2, x_3) : t \in [0, \infty), |x_i| < \infty\}$ satisfies the local Carathéodory-conditions and

$$f(t, x_1, x_2, x_3)\, x_1 \leqq 0. \tag{2}$$

By a solution of (1) we shall mean a function y which, along with its derivatives of the first, second order, is absolutely continuous on each segment of the interval $[0, \infty)$ and satisfies (1) for almost all t.

Put $N = \{1, 2, \ldots\}$. Let $L(t_0, \infty -)$ be the set of all functions that are summable on each finite segment of $[t_0, \infty)$.

In the present paper the behaviour of solutions of (1), (2) will be studied. This problem of oscillatory solutions was investigated in [1]. Many authors deal with the problem of the existence of solutions of (1), (2), see e. g. [2].

Definition 1. *The solution y of* (1), *defined on $[t_0, t_1) \in [0, \infty)$ is called noncontinuable if either $t_1 = \infty$ or $t_1 < \infty$ and $\limsup_{t \to t_1 -} \sum_{i=1}^{3} |y^{(i-1)}(t)| = \infty$ holds.*

Definition 2. *The solution y defined on $[t_0, \infty)$ is called proper if $\sup_{\tau \leqq t < \infty} |y(t)| > 0$ for an arbitrary number $\tau \in [t_0, \infty)$.*

Definition 3. *The solution y defined on $[t_0, b)$ is called oscillatory if there exists a sequence $\{t_k\}_1^\infty$ of zeros of y such that $\lim_{k \to \infty} t_k = b$.*

Definition 4. *The equation* (1) *has the property A_0 if every proper solution y is either oscillatory or $|y^{(i)}(t)| \downarrow 0$ when $t \uparrow \infty$, $i = 0, 1, 2$. The equation* (1) *has the* *property* A_i, $i = 1, 2$ *if every proper solution y is either oscillatory or* $\lim_{t\to\infty} y^{(k)}(t) = 0$. $k = i, \ldots, 2$.

The sufficient conditions for (1) which should have the property A_i are given in [2].

When investigating (1) we meet solutions of the types:

I. The solution y defined on $[t_0, t_1)$, $t_1 \leqq \infty$ is strongly oscillatory to the left: There exist sequence $\{t_k^i\}$, $i = 0, 1, 2$, $k \in N$ such that $\lim_{k\to\infty} t_k^i = t_1$ and

$$y^{(i)}(t_k^i) = 0, \qquad t_k^0 < t_k^2 < t_k^1 < t_{k+1}^0,$$

(3) $$y^{(i)}(t)\, y(t) > 0 \quad \text{on} \quad (t_k^0, t_k^i), \qquad i = 0, 1, 2$$

$$y^{(i)}(t)\, y(t) < 0 \quad \text{on} \quad (t_k^i, t_{k+1}^0) \quad \text{for } i = 0, 1,\ y'''(t)\, y(\ ) \leqq 0$$

$$\text{on } [t_k^i, t_{k+1}^0),\ k \in N \text{ holds.}$$

II. The solution y defined on $[t_0, \infty)$ is different from zero on (t_0, ∞) and there exists a number $\tau \in [t_0, \infty)$ such that.

(4) $$y(t)\, y'(t) \geqq 0, \qquad y(t)\, y''(t) \geqq 0, \qquad y'' \operatorname{sgn} y \text{ is nonincreasing on } [\tau, \infty).$$

III. The solution y, defined on $[t_1, t_2)$, $t_2 \leqq \infty$ is monotone and $(-1)^i y(t) y^{(i)}(t) > > 0$, $\lim_{t\to t_2} y^{(i)}(t) = 0$ for $i = 1, 2$, $\lim_{t\to t_2} y(t) = C$. Moreover, if $t_2 < \infty$ then $C \geqq 0$.

IV. The solution y, defined on $(t_0, t_1]$, $0 \leqq t_1$ is strongly oscillatory to the right, there exist sequences $\{t_k^i\}$, $i = 0, 1, 2$, $k = -1, -2, \ldots$ such that $\lim_{k\to-\infty} t_k^i = t_0$, (4) and $\lim_{t\to t_0} y^{(i)}(t) = 0$, $i = 0, 1, 2$ hold.

V. $y(t) = \pm(c_1 + c_2 t)^2$, c_1 and c_2 are suitable constants, $|c_1| + |c_2| \neq 0$, $t \in [t_0, t_1)$, $t_1 \leqq \infty$.

VI. $y(t) \equiv 0$ on $[t_0, t_1)$, $t_1 \leqq \infty$.

The following lemma can be proved directly from (2).

Lemma 1. *Let y be the solution of* (1), (2) *defined on* $[t_0, b)$. *Then the function* $y'' \operatorname{sgn} y$ *is nonincreasing on* $[t_0, b)$, $t \neq t^*$ *where* t^* *is a zero of y.*

The following function is of great importance

(5) $$F(t) = y'^2(t) - 2y(t)\, y''(t).$$

Lemma 2. *Let the solution y be defined on* $[t_0, b)$. *Then the function* (5) *is non-decreasing and*

(6) $F(t) \equiv 0$ *on* $[t_1, t_2]$, $t_0 \leqq t_1 < t_2 \Leftrightarrow y$ *is of the type* V *or* VI *on* $[t_1, t_2]$.

Proof. By virtue of (2)

$$F(t_2) - F(\ _1) = \int_{t_1}^{t_2} F'(t)\, dt = -\int_{t_1}^{t_2} 2y(\ )\, y'''(t)\, dt \geqq 0, \qquad t_1 \leqq t_2$$

holds and thus F is nondecreasing. The validity of the relation $\Leftarrow$ in (6) is trivial. Suppose that $F \equiv 0$ on $[t_1, t_2]$ and let y be not trivial. Let

$$(7) \qquad (-1)^i y(t) > 0,\ (-1)^j y'(t) > 0 \text{ on } J = (t_3, t_4) \subset [t_1, t_2],\ i, j = 1, 2.$$

According to the assumption

$$F(t) = y'^2(t) - 2y(t)\, y''(t) \equiv 0 \qquad \text{on } J$$

we get by integration

$$(8) \qquad y(t) = (-1)^i \left[(-1)^{i+j} \sqrt{|y(t_5)|} + \frac{1}{2} \sqrt{\frac{y'^2(t_5)}{|y(t_5)|}} (t - t_5) \right]^2. \quad t_5 \in J.$$

From this and from (7) if $y(t_k)\, y'(t_k) = 0$, then

$$y(t_k) = y'(t_k) = 0, \quad y''(t_k) = \frac{(-1)^i y'^2(t_5)}{2\,|y(t_5)|} \neq 0,$$

$k = 3, 4$.

From this and as according to (8) y is not oscillatory we can conclude that (8) is valid on the whole interval $[t_1, t_2]$. Thereby $t_5 \in (t_1, t_2)$, $y(t_5) \neq 0$, i, j must be taken from (7) for $t = t_5$ and if $y'(t_5) = 0$, then $j = 0$. The lemma is proved.

Lemma 3. *Let y be a non-continuable solution of* (1), (2), *defined on* $[t_0, b)$ *and let* $F(t_0) > 0$. *Then y is of the type* I *on* $[t_0, b)$ *or there exists only a finite number of zeros* t_k, $k = 1, \ldots, N$ *of y such that y is of the type* II *on* $[t_N, \infty)$.

Proof. Let y be the solution, defined by the Cauchy initial conditions $[t_0, y_0, y_0', y_0'']$. We shall investigate all possibilities which may occur, whereby we shall consider in each case the first possibility, the second one can be proved similarly.

$$1^\circ \qquad y_0 \geqq 0,\ y_0' \geqq 0,\ y_0'' > 0 \qquad \text{or} \qquad y_0 \leqq 0,\ y_0' \leqq 0,\ y_0'' < 0.$$

If y is such that

$$y(t) > 0,\ y'(t) > 0,\ y''(t) \geqq 0, \qquad y'' \text{ nonincreasing},\ t \in (t_0, b)$$

then with respect to the y being non-continuable $b = \infty$ must be valid and y is of the type II on $[t_0, \infty)$.

Suppose that y is not of the type II. Then there exists number t^2 such that $y''(t^2) = 0$, $y''(t) > 0$ on $[t_0, t^2)$. Moreover, according to 1° $y(t) > 0$, $y'(t) > 0$ on $(t_0, t^2]$.

Suppose the zero of y' does not exist on (t^2, b). In this case according to y being non-continuable $b = \infty$ and (see Lemma 1)

$$(9) \qquad y(t)\, y'(t) > 0,\ y''(t) \leqq 0, \qquad y'' \text{ nonincreasing},\ t \in (t^2, \infty).$$

As y is not of the type II it follows from (9) that there exist numbers $t_1 \in (t^2, \infty)$ and $K > 0$ such that

$$y''(t) < -K, \qquad t \in [t_1, \infty).$$

Then the relation

$$-y'(t_1) \leqq y'(t) - y'(t_1) = y''(\eta)(t - t_1) < -K(t - t_1), \qquad \eta \in (t_1, t)$$

gives us the contradiction for $t \to \infty$. Thus our assumption is not true and there exists a zero $t^1 \in (t^2, b)$ of the function y' and $y(t) > 0$, $y'(t) > 0$, $y''(t^1) < 0$. $t \in (t^2, t^1)$ and the zero t^1 is simple. The existence of zero t^0, $t^0 \in (t^1, b)$ of y can be proved similarly as for t^1. We can see that in t^0 $y(t_0) = 0$, $y'(t_0) < 0$, $y''(t_0) < 0$ holds and we have the same situation as at t_0. By repetition of this way we can conclude that y is of the type I. on some interval $[t_0, b_1)$, $b_1 \leqq b$. We prove by the indirect proof that $b_1 = b$. Let $b_1 < b$. As y is strongly oscillatory the solution can exist in b_1 only if

$$\lim_{t \to b_1 -} y'(t) = 0. \tag{10}$$

But by virtue of (5) and of Lemma 2

$$F(t_k^0) = y'^2(t_k^0) \geqq F(t_0) > 0,$$

where $\{t_k^0\}$, $\lim_{K \to \infty} t_k^0 = b_1$ is the sequence of zeros of y that contradicts (10). The emma is proved in this case.

2° $\qquad y_0 \geqq 0, y_0' > 0, y_0'' \leqq 0 \quad$ or $\quad y_0 \leqq 0, y_0' < 0, y_0'' \geqq 0.$

According to the fact, that in some right neighbourhood of t_0 the same conditions as in 1° for $t \in (t^2, t^1)$ are fulfilled, the behaviour of y is similar.

3° $\qquad y_0 \geqq 0, y_0' \leqq 0, y_0'' < 0 \quad$ or $\quad y_0 \leqq 0, y_0' \geqq 0, y_0'' > 0.$

The conditions $y'(t) < 0$, $y''(t) < 0$ hold in some right neighbourhood of t_0 and this situation was investigated in 1°, $t \in (t^1, t^0)$ or $t \in (t_0, t^2)$.

According to the assumption of lemma $F(t_0) > 0$, the last possible case is

4° $\qquad y_0 > 0, y_0' < 0, y_0'' \geqq 0 \quad$ or $\quad y_0 \leqq 0, y_0' > 0, y_0'' \leqq 0.$

First suppose that

$$y'(t) < 0 \qquad \text{on } [t_0, b). \tag{11}$$

Then it is clear that $\lim_{t \to \infty} y'(t) = \lim_{t \to \infty} y''(t) = 0$, $b = \infty$ (as y is non-continuable), But the relation

$$0 < F(t_0) \leqq \lim_{t \to \infty} F(t) = \lim_{t \to \infty} [y'^2(t) - 2y(t)\, y''(t)] = 0$$

gives us the contradiction. Thus (11) is not correct and there exists a number t_1 such that $y'(t_1) = 0$. According to the assumption $F(t_0) > 0$ and Lemma 2 $y(t_1)\,y''(t_1) < 0$ holds. Thus we have

$$y(t_1) > 0,\ y'(t_1) = 0,\ y''(t_1) < 0, \qquad \text{or } y(t_1) < 0,\ y'(t_1) = 0,\ y''(t_1) > 0$$

and there are given the same conditions at t_1 as in 3° for $t = t^0$.
The lemma is proved.

Theorem 1. *Let y be a non-continuable solution of* (1) *and* (2), *defined on* $[t_0, b)$, $b \leqq \infty$. *Then y is successively of the types* III, VI, IV *on the intervals* $[t_0, t_1)$. $[t_1, t_2)$, $(t_2, t_3]$ *and either of the type* I *on* $[t_3, b)$ *or of the type* II *on* $[t_3, \infty)$. *respectively. Here $t_0 \leqq t_1 \leqq t_2 \leqq t_3$ are suitable numbers. Some parts of y may be missing, the numbers $t_4 \in [t_0, t_3]$, $t_5 \in [t_1, t_2]$ may exists such that $t_4 = t_0$ or $t_5 = b = \infty$.*

Proof. Let y be given by Cauchy initial conditions $[t_0, y_0, y'_0, y''_0]$. The structure of y for $F(t_0) > 0$ was investigated in Lemma 3. Let

$$F(t_0) = 0. \tag{12}$$

If $F(t) = 0$, $t \in [t_0, b)$, then according to Lemma 2 y is of the type VI or II on $[t_0, b)$. In the opposite case the number $t_1 \in [t_0, b)$ exists such that $F(t_1) = 0$, $F(t) > 0$ on (t_1, b). The properties of y on (t_1, b) were investigated in Lemma 3. If there exists a seequence $\{t_k^0\}$ of zeros of y, $k = -1, -2, \ldots$ such that $\lim_{T \to -\infty} t_k^0 = t_1$, then (4) must be valid and with respect to the fact, that $y^{(i)}$ is continuous at t_1 for $i = 0, 1, 2$ we can conclude that y is of the type IV on some interval $(t_1, t_2) \subset \subset (t_1, b)$. In the opposite case $y^{(i)}(t) \neq 0$ in some right neighbourhood of t_1, $t \in (t_1, t_3)$, $i = 0, 1, 2$, $\sum_{i=0}^{2} | y^{(i)}(t_1) | \neq 0$. Let

$$F(t_0) < 0. \tag{13}$$

First, consider the case

1° $\qquad y_0 > 0,\ y'_0 \geqq 0,\ y''_0 > 0 \quad$ or $\quad y_0 < 0,\ y'_0 \leqq 0,\ y''_0 < 0.$

Put $y_0 > 0$ without the loss of generality. If there exists a number $\xi \in (t_0, b)$ such that $F(\xi) = 0$ holds, then the behaviour of y was studied above. Thereby $y^{(i)} > 0$ on (t_0, ξ), $i = 0, 1, 2$. In the opposite case y is of the type II.

In virtue of (13) we must still see the case

2° $\qquad y_0 > 0,\ y'_0 < 0,\ y''_0 > 0 \quad$ or $\quad y_0 < 0,\ y'_0 > 0,\ y''_0 < 0.$

The situation $F(t_1) = 0$ for a number $t_1 \in (t_0, b)$ was studied previously. Thus suppose that $F(t) < 0$, $t \in [t_0, b)$. Then according to (5)

$$y(t)\,y''(t) > 0, \qquad t \in [t_0, b).$$

If y is not of the type III, then exists a number $t_2 \in (t_0, b)$ such that $y'(t_2) = 0$, But this situation was met in 1°.
The theorem is proved.

Remark 1. Let y be given on $[t_0, b)$ and $y(t_0)\,y'(t_0) > 0$, $y(t_0)\,y''(t_0) > 0$, $y'^2(t_0) > 2y(t_0)\,y''(t_0)$. Then, according to the proof of Lemma 3 there exists a number $t_1 \in [t_0, b)$ such that y is of the type I on $[t_1, b)$ or of the type II on $[t_1, \infty)$.

Remark 2. According to the Definition 4 and the Remark 1 if (1) has the property A_0, then it has the strongly oscillatory solution of the type I.

Definition 5. *Denote*: $D(K, K_1) = \left\{(t, x_1, x_2, x_3): \frac{t^2}{K_1} \leqq |x_1| \leqq K_1 t^2, K \leqq \leqq |x_1|, \frac{t}{K_1} \leqq |x_2| \leqq K_1 t, K \leqq |x_2|, |x_3| \leqq K_1\right\}$, $D_1(K, K_1) = \left\{(t, x_1, x_2, x_3): |x_1| \geqq K, \frac{1}{K_1} t \leqq |x_1| \leqq K_1 t^2, \frac{1}{K_1} \leqq |x_2| \leqq K_1 t, |x_3| \leqq 1/K\right\}$.

Theorem 2. *Let the constants* $K > 0$, α, β *exist such that for an arbitrary* $C > 0$

(14) $$|f(t, x_1 x_2, x_3)| \geqq a_c(t)\,|x_1|^\alpha\,|x_2|^\beta \qquad \text{on } D(K, C),$$

(15) $$\int_0^\infty a_c(t)\,t^{2\alpha+\beta}\,dt = \infty$$

holds where $a_c \in L(0, \infty -)$.

Then for the solution y *of the type* II

(16) $$\lim_{t\to\infty} y''(t) = 0$$

holds and differential equation has the property A_2.

Proof. The property (16) will be proved by the indirect proof. Thus suppose that $\lim_{t\to\infty} |y''(t)| = K_2 > 0$. As y is of the type II there exists a number $\tau \in [t_0, \infty)$ such that $yy^{(i)} > 0$ on $[\tau, \infty)$, $i = 1, 2$ holds. As the function $y''\operatorname{sgn} y$ is non-increasing the following estimations hold for a suitable $t_2 \geqq \tau$

(17) $$C_1 t \leqq K_2(t - \tau) \leqq |y'(t)| \leqq |y'(\tau)| + |y''(\tau)|\,(t - \tau) \leqq C_2 t,$$

$$C_1 t^2 \leqq \frac{K_2}{2}(t - \tau)^2 \leqq |y(t)| \leqq |y(\tau)| + |y'(\tau)|\,(t - \tau) + + \frac{|y''(\tau)|}{2}(t - \tau)^2 \leqq C_2 t^2,$$

$$t \in [t_2, \infty), \qquad C_1 = \frac{K_2}{4}, \qquad C_2 = 2\,|y''(\tau)|.$$

Let $t_3 \in [t_2, \infty)$ be such that $|y(t)| \geqq K, |y'(t)| \geqq K$, $t \in [t_3, \infty)$ hold and let $C = \max\left(C_2, \frac{1}{C_1}\right)$. Then according to (14), (15) and (17) we have the following estimations

$$|y''(t_3)| \geqq \int_{t_3}^{\infty} |y'''(t)| \, dt \geqq \int_{t_3}^{\infty} a_c(t) |y(t)|^{\alpha} |y'(t)|^{\beta} \, dt \geqq$$

$$\geqq K_3 \int_{t_3}^{\infty} a_c(t) \, t^{2\alpha+\beta} \, dt = \infty,$$

where $K_3 > 0$ is a constant. The obtained contradiction proves the theorem.

Remark 3. Kiguradze [2] proved the following result: The differential equation (1) has the property A_k, $k = 1, 2$ if

$$f(t, x_1, x_2, x_3) \operatorname{sgn} x_1 \leqq -a(t) x_1^{\lambda_0} \prod_{j=1}^{k+1} (1 + |x_j|)^{\lambda_j} \qquad \text{on } D,$$

$$\lambda_0 > 0, \qquad \lambda_j \in R, \qquad j = 1, 2, \qquad \sum_{j=0}^{k+1} \lambda_j > 1, \qquad a \in L(0, \infty -), \qquad a \geqq 0,$$

$$\int_0^{\infty} a(t) \, t^{\gamma} \, dt = \infty, \qquad \gamma_k = 2 + k(\lambda_0 - 1) + \sum_{j=1}^{k} (k + 1 - j) \lambda_j.$$

For $k = 2$ the Theorem 2. generalizes this result.

Theorem 3. *Let exist constants $K > 0$, α, β such that for an arbitrary $C > 0$*

$$|f(t, x_1, x_2, x_3| \geqq a_c(t) |x_1|^{\alpha} (1 + |x_2|)^{\beta} \qquad \text{on } D_1(K, C),$$

$$\int_0^{\infty} a_c(t) \, t^{\gamma} \, dt = \infty$$

holds where γ is defined by one of the following possibilities

$1^0 \qquad \gamma = \frac{1}{2}\{[3 - \operatorname{sgn}(\alpha + \varepsilon)](\alpha + \varepsilon) + [1 - \operatorname{sgn}(\beta - 2\varepsilon)](\beta - 2\varepsilon)\},$

$2^0 \qquad \gamma = \frac{1}{2}[1 - \operatorname{sgn}(2\alpha + \beta)](2\alpha + \beta) \qquad$ *for* $\alpha > -1$,

$3^0 \qquad \gamma = \frac{1}{2}[3 - \operatorname{sgn}(\alpha + \beta/2)(\alpha + \beta/2) \qquad$ *for* $\beta < 2$.

Here $\varepsilon \in [0, 1)$, $a_c \in L(0, \infty -)$ is a non-negative function. Then the differential equation (1) *has the property A_1.*

P r o o f. By virtue of Theorem 1 we must prove that the solution y of (1), (2) of the type II does not exist, its derivative is not equal to zero identically in some neighbourhood of ∞. Thus suppose on the contrary that such a solution, defined on $[t_0, \infty)$ exists and let $y > 0$ on $[t_0, \infty)$ for simplicity.

It follows from the definition of y and Theorem 2 and its proof that

(18) $$y'(t) > 0,\ y' \text{ non-decreasing on } [t_0, \infty),\ \lim_{t\to\infty} y''(t) = 0$$

and

$$0 < y'(t_0) \leqq y'(t) \leqq y'(t_0) + y''(t_0)(t - t_0) \leqq 2y''(t_0)\, t,$$

(19) $$\frac{y'(t_0)}{2} t \leqq y'(t_0)(t - t_0) \leqq y(t) \leqq y(t_0) + y''(t_0)\, t^2 \leqq 2y''(t_0)\, t^2 \qquad t \in [t_1, \infty),$$

where $t_1 \in (t_0, \infty)$ is a suitable constant with the property $y(t) \geqq K$ for $t \in [t_1, \infty)$, $y''(t) \leqq 1/K$.

First we prove that there exists a constant $t_2 \geqq t_1$ such that

(20) $$F(t) = y'^2(t) - 2y(t)\, y''(t) > 0 \qquad \text{on } [t_2, \infty).$$

Suppose on the contrary that $F(t) \leqq 0$ on $[t_1, \infty)$. As

(21) $$\left(\frac{y(t)}{y'^2(t)}\right)' = \frac{F(t)}{y'^3},$$

then

$$y(t) \leqq My'^2(t), \qquad M = y(t_1)\,[y'^2(t_1)]^{-1}$$

and

$$0 \geqq F(t) = y'^2(t) - 2y(t)\, y''(t) \geqq y'^2(t)\,[1 - 2My''] \geqq$$
$$\geqq y'^2(t_1)\,[1 - 2My''(t)]$$

and we get the contradiction to (18) for $t \to \infty$. Thus (20) is valid and according to (21)

(22) $$y(t) \geqq M_1 y'^2(t), \qquad t \in [t_2, \infty), \qquad M_1 = y(t_2)\,[y'^2(t_2)]^{-1}.$$

Put $C = 2\max(y''(t_0), (y'(t_0))^{-1})$. Then according to the assumptions of the theorem

(23) $$y''(t_2)^{1-\varepsilon} = -\int_{t_2}^{\infty} \frac{y'''(t)}{y''(t)^{\varepsilon}}\,\mathrm{d}t \geqq \int_{t_2}^{\infty} \frac{a_c(t)\, y(t)^{\alpha}\, y'(t)^{\beta}}{y''(t)^{\varepsilon}}\,\mathrm{d}t \geqq$$
$$\geqq 2^{\varepsilon} \int_{t_2}^{\infty} a_c(t)\, y(t)^{\alpha+\varepsilon}\, y'(t)^{\beta-2\varepsilon}\,\mathrm{d}t = J.$$

If γ is defined according to 1°, then by use of (19) and (23) we get the contradiction:

$$y''(t_2)^{1-\varepsilon} = J \geqq 2^{\varepsilon} \int_{t_2}^{\infty} a_c(t)\, t^{\gamma} = \infty.$$

If 2° is valid, then put $\varepsilon = 0$ for $\alpha \geqq 0$, $\varepsilon = |\alpha|$ for $0 > \alpha > -1$ and according to (22), (19) and (23)

$$y''(t_2)^{1-\varepsilon} \geqq J \geqq 2^{\varepsilon} M_1^{\alpha+\varepsilon} \int_{t_2}^{\infty} a_c(t)\, y'(t)^{2\alpha+\beta}\,\mathrm{d}t = M_2 \int_{t_2}^{\infty} a_c(t)\, t^{\gamma}\,\mathrm{d}t = \infty.$$

M_2 is a constant. This contradiction proves the theorem in this case.

Let 3° be valid. Put $\varepsilon = 0$ for $\beta \leqq 0$ and $\varepsilon = \beta/2$ for $0 < \beta < 2$. If follows from (23), (22) and (19) that

$$y''(t_2)^{1-\varepsilon} \geqq J \geqq 2^{\varepsilon} M_1^{\beta/2-\varepsilon} \int_{t_2}^{\infty} a_c(t)\, y^{\alpha+\beta/2}\, dt = M_2 \int_{t_2}^{\infty} a_c(t)\, t^{\gamma}\, dt = \infty .$$

M_2 is a constant. This contradiction proves the theorem.

Remark 4. The theorem 3 generalizes the results obtained by Kiguradze [2], see Remark 3. For some special α and β the results by Kiguradze are more suitable.

Theorem 4. *Let the differential equation* (1) *have the property* A_1 *and let the constants* M, t_1 *and functions* $a \in L(t_1, \infty-)$, $g \in C_0(D_2)$ *exist such that* $g(x_1, x_2, x_3) > 0$ *for* $x_1 > 0$, $a \geqq 0$, $|f(t_1, x_1, x_2, x_3)| \geqq a(t)\, g(|x_1|, |x_2|, |x_3|)$ on D_2, $D_2 = \{(t, x_1, x_2, x_3) : t_1 \leqq t,\ 0 \leqq x_1,\ 0 \leqq x_i \leqq M,\ i = 2, 3\}$ *and*

$$\int_{t_1}^{\infty} \int_{\tau}^{\infty} \int_{x}^{\infty} a(t)\, dt\, dx\, d\tau = \infty$$

hold. Then (1) *has the property* A_0.

Proof. According to the Theorem 1 and the definition of the properties A_0 and A_1 we must prove that for the solution y with properties: $y^{(j)}$ monotone, $j = 0, 1, 2$ and

$$\lim_{t\to\infty} y^{(i)}(t) = 0, \qquad i = 1, 2, \qquad \lim_{t\to\infty} |y(t)| = C$$

the relation $C = 0$ holds. Suppose on the contrary that $C \neq 0$. Let $t_2 \in [t_1, \infty)$ be a number with the property

$$|y^{(i)}(t)| \leqq M, \qquad C/2 \leqq y(t) \leqq 2C, \qquad i = 1, 2.$$

Then

$$|y(t_2)| - C = \int_{t_2}^{\infty} |y'(t)|\, dt = \int_{t_2}^{\infty} \int_{\tau}^{\infty} |y''(x)|\, dx\, d\tau = \int_{t_2}^{\infty} \int_{\tau}^{\infty} \int_{x}^{\infty} |y'''(t)|\, dt\, dx\, d\tau \geqq$$

$$\geqq \int_{t_2}^{\infty} \int_{\tau}^{\infty} \int_{x}^{\infty} a(t)\, g(|y(t)|, |y'(t)|, |y''(t)|)\, dt\, dx\, d\tau \geqq$$

$$\geqq K \int_{t_2}^{\infty} \int_{\tau}^{\infty} \int_{x}^{\infty} a(t)\, dt\, dx\, d\tau = \infty,$$

$$K = \min g(x_1, x_2, x_3) > 0,$$

where the minimum is taken for $C/2 \leqq x_1 \leqq C$, $|x_i| \leqq M$, $i = 2, 3$.

The gained contradiction proves the theorem.

Theorem 5. *Let y be an oscillatory solution of the* I *type on* $[t_0, b)$ *and let constant* $M > 0$ *exist such that for* $t \in [t_0, b)$, $|x_1| \leqq M$, $x_2 \in R$

$$g_1(|x_1|, |x_2|, |x_3|) \leqq |f(t, x_1, x_2, x_3)|, \qquad x_3 \in R$$

and

$$|f(t, x_1, x_2, x_3)| \leqq g_2(|x_1|, |x_2|, |x_3|), \ |x_3| \leqq M$$

hold where g_i are continuous, $g_1(s_1, s_2, s_3) > 0$ for $s_1 > 0$. Then $\limsup_{t \to b-} |y'(t)| \not< \infty$.

Proof. We shall prove the statement of the theorem by the indirect proof. Suppose that

(24) $$|y'(t)| \leqq K < \infty, \qquad t \in [t_0, b).$$

According to (5), (24) and Lemma 2 F is non-decreasing and

(25) $$0 < F(t_k^i) = y'^2(t_k^i) \leqq K^2, \qquad \lim_{t \to b-} F(t) = K_1 \leqq K^2, \quad i = 0, 2.$$

First we investigate the case when

(26) $$\lim_{t \to b-} y(t) = 0$$

is valid. Let $k > k_0$, where k_0 is an integer with the property $|y(t)| \leqq M$ on $(t_{k_0}^0, b)$.

Put $\xi_k \in (t_k^2, t_k^1)$ such number that

(27) $$|y''(\xi_k)| = \frac{M}{K} |y'(\xi_k)|.$$

According to (3) such a number exists, by use of (25)

(28) $$|y''(t)| \leqq M \qquad \text{on} \qquad [t_k^2, \xi_k]$$

holds and with respect to the fact that $y' \operatorname{sgn} y$ is concave on $[t_k^2, \xi_k]$ we have

$$|y(\xi_k)| - |y(t_k^2)| = \int_{t^2_k}^{\xi_k} |y'(t)| \, dt \geqq \frac{|y'(t_k^2)|}{2} (\xi_k - t_k^2).$$

From this and from (26), (25)

(29) $$\lim_{K \to \infty} (\xi_k - t_k^2) = 0.$$

Further,

$$|y'(t_k^2)| - |y'(\xi_k)| = \int_{t^2_k}^{\xi_k} |y''(t)| \, dt \leqq |y''(\xi_k)| \, . \, (\xi_k - t_k^2) \leqq$$
$$\leqq M(\xi_k - t_k^2) \underset{k \to \infty}{\to} 0$$

and by virtue of (25) and (27)

$$\lim_{k \to \infty} |y'(\xi_k)| = \lim_{k \to \infty} |y'(t_k^2)| = \sqrt{K_1},$$

(30) $$\lim_{k \to \infty} |y''(\xi_k)| = \frac{M}{K} \sqrt{K_1}.$$

Finally,

$$\int_0^{|y''(\xi_k)|} \frac{ds}{g_3(s)} = \int_{t^2_k}^{\xi_k} \frac{|y'''(t)|\,dt}{g_3(|y''(t)|)} \leqq \int_{t^2_k}^{\xi_k} \frac{|y'''(t)|\,dt}{g_2(|y(t)|, |y'(t)|, |y''(t)|)} \leqq \xi_k - t_k^2,$$

$$g_3(s) = \max\{g_2(s_1, s_2, s) : 0 \leqq s_1 \leqq M, 0 \leqq s_2 \leqq K\},$$

that contradicts (29) and (30). Thus (26) is not correct and there exists an infinite set $N_1 \subset N$ and $K_2 > 0$ such that

(31) $$|y(t_k^1)| \geqq K_2, \qquad k \in N_1$$

holds. Let $\varepsilon > 0$ be such that $\varepsilon \leqq K_2$, $\varepsilon \leqq M$. Then according to (4) the sequence $\{\alpha_k\}$, $k \in N_1$ exists such that

$$|y(\alpha_k)| = \varepsilon, \qquad \alpha_k \in (t_k^0, t_k^1]$$

and by use of (31), (25) and (3)

(32) $$|y(t)| \leqq \varepsilon \qquad \text{on} \qquad [t_k^0, \alpha_k]$$

(33) $$|y''(t)| \leqq |y''(t_k^1)| = \frac{F(t_k^1)}{2|y(t_k^1)|} \leqq \frac{K_1}{K_2}, \qquad t \in [t_k^2, t_k^1]$$

hold. Define the sequence $\{\beta_k\}$, $k \in N_1$ in the following way

$$\beta_k \in [t_k^0, t_k^2),$$

(34) $$|y'(\beta_k)| = |y''(\beta_k)|$$

and $\beta_k = t_k^0$ if (34) has no solution β_k. As by virtue of (25) and (4)

$$0 \underset{k\to\infty}{\leftarrow} |y'(t_k^2)| - |y'(t_k^0)| \geqq |y'(\beta_k)| - |y'(t_k^0)| = \int_{t^0_k}^{\beta_k} |y''(t)|\,dt \geqq$$

$$\geqq |y''(\beta_k)|(\beta_k - t_k^0) \geqq |y'(\beta_k)|(\beta_k - t_k^0) \geqq \int_{t^0_k}^{\beta_k} |y'(t)|\,dt = |y(\beta_k)|,$$

(35) $$\lim_{k\to\infty} |y(\beta_k)| = 0, \qquad k \in N_1.$$

From this and from (3), (34), (33), (32) and (24) there exists on infinite set $N_2 \subset N_1$ such that

$$|y(\beta_k)| \leqq |y(t)| \leqq \varepsilon, \qquad |y''(t)| \leqq K_3 \qquad \text{on } [\beta_k, \alpha_k], \qquad k \in N_2,$$

where $K_3 = \max\left(\frac{K_1}{K_2}, M\right)$.

From this, finally,

$$F(\alpha_k) - F(\beta_k) = -\int_{\beta k}^{\alpha k} 2y'''(t)\,y(t)\,dt \geqq 2\int_{\beta k}^{\alpha k} g_1(|y(t)|, |y'(t)|, |y''(t)|) \times$$

$$\times |y(t)| \, dt \geqq \frac{2}{K} \int_{\beta k}^{\alpha k} g_4(|y(t)|) \, |y'(t)| \, dt \geqq \frac{2}{K} \int_{|y(\beta_k)|}^{\varepsilon} g_4(s) \, s \, ds,$$

$$g_4(s) = \min g_1(s, x_2, x_3),$$

where the minimum is taken for $0 \leqq x_2 \leqq M$, $0 \leqq x_3 \leqq K_3$ holds that contradicts (25) and (35). The theorem is proved.

Remarks 5. The Theorem 5 generalizes the similar result of [1].

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