Archivum Mathematicum

Magdalena Vencková

Oscillation of a forced nonlinear differential equation

Archivum Mathematicum, Vol. 20 (1984), No. 2, 59--68

Persistent URL: http://dml.cz/dmlcz/107188

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XX: 59–68, 1984

OSCILLATION OF A FORCED NONLINEAR DIFFERENTIAL EQUATION

MAGDALENA VENCKOVÁ, Bratislava

(Received February 20, 1981)

The paper is concerned with oscillatory character of solutions of the differential equation

$$L(y)(x) + g(x, y, y', \dots, y^{(n-1)}) = f(x), \tag{1}$$

where

$$L(y)(x) = 0 \tag{2}$$

is a linear differential equation of the n-th order ($n \geqq 2$), disconjugate on $I = [0, \infty)$, g is a continuous function on $(0, \infty) \times R^n$ and f is a continuous function on $(0, \infty)$. In the paper, theorems from paper (1) on oscillatory character of the second order differential equation are extended to the equation (1). Throughout the paper we shall suppose that every solution of the equation (1) exists on some interval $[x_0, \infty) \subset I^0 = (0, \infty)$ and by the oscillatory solution will be understood any solution (1), which vanishes on every interval $(a, \infty) \subset [x_0, \infty)$ at least once. Denote

$$L(y)(x) = y^{(n)}(x) + p_1(x) . y^{(n-1)}(x) + \dots + p_n(x)\, y(x),$$

where $p_1, \dots, p_n$ are continuous functions on I.

Supposing the differential equation $L(y)(x) = 0$ is disconjugate on I we get that the differential operator L can be factorized on $I^0 = (0, \infty)$ onto a product of n first order operators, that means $L(y)$ is the n-th quasi-derivative of the function y (2, Lemma 6, p. 93 and Theorem 2, p. 91). Regarding it, in the first part of the paper, we shall deal with the equation

$$L_n(y)(x) = f(x) \tag{3}$$

on the interval I^0, where $L_n(y)$ is the n-th quasi-derivative of the function y.

Let $n \geqq 2$ be a natural number, $a_0, a_1, \dots, a_n$ be continuous and positive functions on the interval I^0. Throughout the whole paper we shall use this hypothese and consider only the real functions. Then we shall call the expression $L_i(y)$

($i = 0, 1, \ldots, n$), where

$L_0(y)(x) = a_0(x)\, y(x), \quad L_i(y)(x) = a_i(x)\,(L_{i-1}(y)(x))', \quad x \in I^0, \quad i = 1, 2, \ldots, n$

the i-th quasi-derivative of the function y on I^0. Besides we suppose that the function y defined on I^0 is such that the expressions $L_1 y, \ldots, L_i y$ are well defined on I^0. We denote by $M_i(J)$ the set of all functions having the i-th quasi-derivative continuous on an interval $J \subset I^0$.

Consider now the functions $y_1, y_2, \ldots, y_n$ defined on $I^0 \times I^0$ by the relation

(4)
$$y_1(x, x_0) = \frac{1}{a_0(x)},$$

$$y_i(x, x_0) = \frac{1}{a_0(x)} \int_{x_0}^{x} \frac{1}{a_1(t_{i-1})} \int_{x_0}^{t_{i-1}} \frac{1}{a_2(t_{i-2})} \int_{x_0}^{t_{i-2}} \ldots \int_{x_0}^{t_2} \frac{1}{a_{i-1}(t_1)} \mathrm{d}t_1 \ldots \mathrm{d}t_{i-1}$$
$$(i = 2, \ldots, n, x, x_0 \in I^0).$$

The following Lemma shows their importance.

Lemma 1. *Let $x_0 \in I^0$, $i \in \{1, \ldots, n\}$. Then the function $y_i(., x_0)$ is the solution of the initial-value problem*

(5)
$$L_i(y)(x) = 0,$$
$$L_0(y)(x_0) = 0, L_1(y)(x_0) = 0, \ldots, L_{i-2}(y)(x_0) = 0, L_{i-1}(y)(x_0) = 1.$$

Proof: It follows from the substitution of (4) into (5).

Consider now the nonhomogeneous differential equation (3). Writing the equation in the form of a differential system and applying well-known facts to theories that system we get the statements:

I. For every $x_0 \in I^0$ and every point $(y_0, y_0', \ldots, y_0^{(n-1)}) \in R^n$ there exists exactly one solution y of the equation (3), which fulfils the initial conditions

$$L_0(y)(x_0) = y_0$$
$$L_1(y)(x_0) = y_0'$$
$$\vdots$$
$$L_{n-1}(y)(x_0) = y_0^{(n-1)}$$

II. Solutions $y_1, y_2, \ldots, y_n$ given by the relation (4) and by Lemma 1 satisfying (5) are linearly independent.

Further the following theorem holds.

Theorem 1. *Let $x_0 \in I^0$ and let the functions $y_i(., x_0)$ ($i = 1, \ldots, n$) be given by the relation* (4). *Let the function f be continuous on I^0. Then the solution of the equation* (3), *which satisfies the initial conditions*

(6)
$$L_0(y)(x_0) = y_0, L_1(y)(x_0) = y_0', \ldots, L_{n-1}(y)(x_0) = y_0^{(n-1)}$$

is of the form

(7) $$y(x) = y_0 y_1(x, x_0) + y_0' y_2(x, x_0) + \dots + y_0^{(n-1)} y_n(x, x_0) + \int_{x_0}^{x} y_n(x, t) \frac{f(t)}{a_n(t)} \, dt, \qquad x \in I^0.$$

(The generalized variation of constants formula.)

Proof. It follows from Lemma 1 that the function $y_0 y_1(., x_0) + y_0' y_2(., x_0) + \dots + y_0^{(n-1)} y_n(., x_0)$ is the solution of the equation

(8) $$L_n(y)(x) = 0,$$

which satisfies the initial conditions (6). Using the variation of constants formula we get that the solution of the equation (3), which satisfies the homogeneous conditions corresponding to (6), thus

(9) $$L_0(y)(x_0) = 0, \ L_1(y)(x_0) = 0, \dots, L_{n-1}(y)(x_0) = 0$$

is of the form

(10) $$y(x) = \frac{1}{a_0(x)} \int_{x_0}^{x} \frac{W(x, t)}{W(t)} \cdot \frac{f(t)}{a_n(t)} \, dt, \qquad (x \in I^0),$$

where (at a fixed x_0)

$$W(t) = \begin{vmatrix} L_0(y_1)(t, x_0) & \dots & L_0(y_n)(t, x_0) \\ L_1(y_1)(t, x_0) & \dots & L_1(y_n)(t, x_0) \\ \vdots & & \vdots \\ L_{n-1}(y_1)(t, x_0) & \dots & L_{n-1}(y_n)(t, x_0) \end{vmatrix}$$

and

$$W(x, t) = \begin{vmatrix} L_0(y_1)(t, x_0) & \dots & L_0(y_n)(t, x_0) \\ L_1(y_1)(t, x_0) & \dots & L_1(y_n)(t, x_0) \\ \vdots & & \vdots \\ L_{n-2}(y_1)(t, x_0) & \dots & L_{n-2}(y_n)(t, x_0) \\ L_0(y_1)(x, x_0) & \dots & L_0(y_n)(x, x_0) \end{vmatrix}$$

($x \in I^0$, *t lies between* x_0 *and* x).

By the relations (4) it follows that $W(t) = 1$, ($t \in I^0$). If $t \in I^0$ is fixed (and of course $x_0 \in I^0$ is fixed, too) we have

$$W(x, t) = \sum_{j=1}^{n} a_j L_0(y_j)(x, x_0) = L_0\Big(\sum_{j=1}^{n} a_j y_j(x, x_0)\Big).$$

This means that $W(x, t)$ can be written in the form

$$W(x, t) = L_0(y)(x) = a_0(x)\, y,$$

where $y = y(x)$ is the solution of the equation (8), which satisfies the initial conditions

$$L_0(y)(t) = 0, L_1(y)(t) = 0, \ldots, L_{n-2}(y)(t) = 0, L_{n-1}(y)(t) = 1.$$

Because the function $y_n(x, t)$ is the solution of the differential equation (8), which fulfils at the point t the same initial conditions, the uniqueness theorem implies that $y(x) = y_n(x, t)$ and thus

$$W(x, t) = L_0(y_n)(x, t) = a_0(x)\, y_n(x, t).$$

By this we have shown the relation (7).

Lemma 2. *Let*

$$\text{(11)} \qquad \int_{x_0}^{\infty} \frac{1}{a_1(t)} \mathrm{d}t = \infty, \int_{x_0}^{\infty} \frac{1}{a_2(t)} \mathrm{d}t = \infty, \ldots, \int_{x_0}^{\infty} \frac{1}{a_{n-1}(t)} \mathrm{d}t = \infty,$$

for an $x_0 \in I^0$. *Then the functions* $y_1(., x_0), y_2(., x_0), \ldots, y_n(., x_0)$ *given by the relation* (4) *form such a fundamental system of solutions of the equation* (8) *that*

$$\text{(12)} \qquad \lim_{x\to\infty} \frac{y_i(x, x_0)}{y_{i+1}(x, x_0)} = 0, \qquad i = 1, 2, \ldots, n-1$$

holds.

Proof. With respect to (11) we get that

$$\lim_{x\to\infty} \frac{y_1(x, x_0)}{y_2(x, x_0)} = \lim_{x\to\infty} \frac{1}{\int_{x_0}^{x} \frac{1}{a_1(t_1)} \mathrm{d}t_1} = 0$$

and applying L'Hospital principle

$$\lim_{x\to\infty} \frac{y_3(x, x_0)}{y_2(x, x_0)} = \lim_{x\to\infty} \frac{\frac{1}{a_1(x)} \int_{x_0}^{x} \frac{1}{a_2(t_1)} \mathrm{d}t_1}{\frac{1}{a_1(x)}} = \lim_{x\to\infty} \int_{x_0}^{x} \frac{1}{a_2(t_1)} \mathrm{d}t_1 = \infty.$$

Thus (12) holds for $i = 1, 2$. Suppose now that

$$\lim_{x\to\infty} \frac{y_i(x, x_0)}{y_{i+1}(x, x_0)} = 0, \qquad i = 1, 2, \ldots, k (k \leq n-1) \qquad \text{is true.}$$

Then

$$\lim_{x\to\infty} \frac{y_{k+1}(x, x_0)}{y_k(x, x_0)} = \lim_{x\to\infty} \frac{\int_{x_0}^{x} \frac{1}{a_1(t_k)} \int_{x_0}^{t_k} \frac{1}{a_2(t_{k-1})} \int_{x_0}^{t_{k-1}} \ldots \int_{x_0}^{t_2} \frac{1}{a_k(t_1)} \mathrm{d}t_1 \ldots \mathrm{d}t_k}{\int_{x_0}^{x} \frac{1}{a_1(t_{k-1})} \int_{x_0}^{t_{k-1}} \frac{1}{a_2(t_{k-2})} \int_{x_0}^{t_{k-2}} \ldots \int_{x_0}^{t_2} \frac{1}{a_{k-1}(t_1)} \mathrm{d}t_1 \ldots \mathrm{d}t_{k-1}}.$$

As $\lim_{x\to\infty} a_0(x)\, y_i(x, x_0) = \infty$ $(i = 2, \ldots, n)$ follows from (11), we can use the L'Hospital rule $k - 1$ times and we get

$$\lim_{x\to\infty} \frac{y_{k+1}(x, x_0)}{y_k(x, x_0)} = \lim_{x\to\infty} \int_{x_0}^{x} \frac{1}{a_k(t_k)}\,\mathrm{d}t_k = \infty.$$

Thus it holds that $\lim_{x\to\infty} \dfrac{y_i(x, x_0)}{y_{i+1}(x, x_0)} = 0$ $(i = 1, 2, \ldots, n - 1)$.

Lemma 2 states that the solutions $y_1(., x_0), \ldots, y_n(., x_0)$ of the equation (8) given by the relation (4) form a hierarchic fundamental system.

Remark. Let $x_0 \in I^0$ and let y be the solution of the initial problem

$$L_n(y)(x) = 1, \qquad (x \in I^0),$$
$$L_0(y)(x_0) = 0, L_1(y)(x_0) = 0, \ldots, L_{n-1}(y)(x_0) = 0.$$

Then y fulfils

$$L_{n+1}(y)(x) = 0, \qquad (x \in I^0),$$
$$L_0(y)(x_0) = 0, L_1(y)(x_0) = 0, \ldots, L_{n-1}(y)(x_0) = 0, L_n(y)(x_0) = 1$$

and hence $y(x) = y_{n+1}(x, x_0)$.

The proof follows by differentiating as well as by the uniqueness of the solution of the initial-value problem.

Consider now the differential equation (1). In what follows we shall assume that

$$(13) \qquad a_i \in C_{n-i}(I^0) \qquad (i = 0, 1, \ldots, n),$$

further relations (11) hold, f is continuous on I^0, g is continuous on $I^0 \times R^n$ and it enjoys the property

$$(14) \qquad y_1 g(x, y_1, \ldots, y_n) \geqq 0 \qquad (x \in I^0, (y_1, y_2, \ldots, y_n) \in R^{n-1}).$$

Theorem 2. *If*

$$(15) \qquad \underline{\lim}_{x\to\infty} \frac{1}{y_n(x, x_0)} \int_{x_0}^{x} y_n(x, t) \frac{f(t)}{a_n(t)}\,\mathrm{d}t = -\infty$$

and

$$(16) \qquad \overline{\lim}_{x\to\infty} \frac{1}{y_n(x, x_0)} \int_{x_0}^{x} y_n(x, t) \frac{f(t)}{a_n(t)}\,\mathrm{d}t = +\infty$$

for each $x_0 > 0$ sufficiently large, then every solution y of the differential equation (1) *is oscillatory.*

Proof: Suppose that y is such a solution of the equation (1) that $y > 0$ on (a, ∞), for an $a > 0$. Choose $x_0 > a$. Then y fulfils initial conditions of the form (6) at the point x_0. It follows from Theorem 1 that this solution can be written as

$$y(x) = y_0 y_1(x, x_0) + y_0' y_2(x, x_0) + \dots + y_0^{(n-1)} y_n(x, x_0) +$$
$$+ \int_{x_0}^{x} y_n(x, t)\left[-g(t, y, y', \dots, y^{(n-1)}) + f(t)\right] \frac{1}{a_n(t)}\,dt, \qquad (x \in [x_0, \infty))$$

and further on the basis of (14)

$$y(x) \leqq y_0 y_1(x, x_0) + \dots + y_0^{(n-1)} y_n(x, x_0) + \int_{x_0}^{x} y_n(x, t) \frac{f(t)}{a_n(t)}\,dt, \qquad (x \in [x_0, \infty)).$$

Consider the ratio $\dfrac{y(x)}{y_n(x, x_0)}$ on $[x_0, \infty)$. Then

$$\frac{y(x)}{y_n(x, x_0)} \leqq y_0 \frac{y_1(x, x_0)}{y_n(x, x_0)} + y_0' \frac{y_2(x, x_0)}{y_n(x, x_0)} + \dots + y_0^{(n-1)} \frac{y_n(x, x_0)}{y_n(x, x_0)} +$$
$$+ \frac{1}{y_n(x, x_0)} \int_{x_0}^{x} y_n(x, t) \frac{f(t)}{a_n(t)}\,dt.$$

Using the assumption (15) and on the basis of Lemma 2 we get that $\varliminf_{x \to \infty} \dfrac{y(x)}{y_n(x, x_0)} = -\infty$ and thus y cannot be positive on (a, ∞). In the case that $y < 0$ we use the assumption (16).

Theorem 3. *If*

(17) $$\varliminf_{x \to \infty} \int_{x_0}^{x} \frac{f(t)}{a_n(t)}\,dt = -\infty \qquad \textit{for every } x_0 \textit{ sufficiently large,}$$

(18) $$\varlimsup_{x \to \infty} \int_{x_0}^{x} \frac{f(t)}{a_n(t)}\,dt = +\infty \qquad \textit{for every } x_0 \textit{ sufficiently large,}$$

(19) $$\left| \frac{1}{y_n(x, x_0)} \int_{x_0}^{x} y_n(x, t) \frac{f(t)}{a_n(t)}\,dt \right| < M,$$

for each $x \geqq x_0$ and every x_0 sufficiently large. Then every solution of the equation (1) *is oscillatory.*

Proof: We suppose again that there exists a solution y of the equation (1) which is positive on a certain interval (a, ∞), $a > 0$. We choose $x_0 > a$. From the equation (1) we get that this solution fulfils the relation

$$L_{n-1}(y)(x) - L_{n-1}(y)(x_0) = \int_{x_0}^{x} \frac{f(t)}{a_n(t)}\,dt - \int_{x_0}^{x} \frac{g(t, y(t), \dots, y^{(n-1)}(t))}{a_n(t)}\,dt$$

and thus also the inequality

$$L_{n-1}(y)(x) \leqq L_{n-1}(y)(x_0) + \int_{x_0}^{x} \frac{f(t)}{a_n(t)}\,dt \qquad (x \geqq x_0).$$

Therefore

(20) $$\varliminf_{x \to \infty} L_{n-1}(y)(x) = -\infty$$

holds according to (17). Now we choose $K > 0$ such that $M - K < 0$. From (20) it follows that there exists $x_1 > x_0$ such that $L_{n-1}(y)(x_1) < -K$. The solution y fulfils at the point x_1 some initial conditions of the form (6). It follows from Theorem 1 that this solution can be written as follows

$$y(x) = y_0 y_1(x, x_1) + y_0' y_2(x, x_1) + \dots + y_0^{(n-1)} y_n(x, x_1) +$$

$$+ \int_{x_1}^{x} y_n(x, t)\left[-g(t, y(t), y'(t), \dots, y^{(n-1)}(t)) + f(t)\right] \frac{1}{a_n(t)}\,\mathrm{d}t,$$

$$(x \in [x_1, \infty))$$

and further regarding (14)

$$y(x) \leqq y_0 y_1(x, x_1) + \dots + y_0^{(n-1)} y_n(x, x_1) + \int_{x_1}^{x} y_n(x, t) \frac{f(t)}{a_n(t)}\,\mathrm{d}t \qquad (x \in [x_1, \infty)).$$

Consider the ratio $\dfrac{y(x)}{y_n(x, x_1)}$ on $[x_1, \infty)$. Then

$$\frac{y(x)}{y_n(x, x_1)} \leqq$$

$$\leqq y_0 \frac{y_1(x, x_1)}{y_n(x, x_1)} + \dots + y_0^{(n-1)} \frac{y_n(x, x_1)}{y_n(x, x_1)} + \frac{1}{y_n(x, x_1)} \int_{x_1}^{x} y_n(x, t) \frac{f(t)}{a_n(t)}\,\mathrm{d}t \leqq$$

$$\leqq y_0 \frac{y_1(x, x_1)}{y_n(x, x_1)} + \dots + y_0^{(n-1)} + M.$$

Thus

$$\overline{\lim_{x \to \infty}} \frac{y(x)}{y_n(x, x_1)} \leqq L_{n-1}(y)(x_1) + M < -K + M < 0$$

and y cannot be positive on (a, ∞).

Lemma 3. *Let k be a natural number, $K > 0$, c be real numbers and let the function $y \in M_k((c, \infty))$ satisfy*

(21) $$L_k(y) \leqq -K \qquad \textit{on the interval } (c, \infty)$$

or

(22) $$L_k(y) \geqq K \qquad \textit{on the interval } (c, \infty).$$

Let further the relations (11) (*for $n = k$*) *be fulfilled and let $m > 0$ exist such that*

(23) $$a_0(x) \leqq m \qquad \textit{on the interval } (c, \infty).$$

Then

$$\lim_{x \to \infty} y(x) = -\infty \qquad (\lim_{x \to \infty} y(x) = +\infty).$$

Proof. Consider only the case (21). Denote

$$f_1(x) = L_k(y)(x) + K \qquad (x \in (c, \infty)).$$

Then y fulfils the equation

$$L_k(y) = -K + f_1(x), \qquad \text{whereby} \qquad f_1(x) \leqq 0 \quad \text{on} \quad (c, \infty).$$

Regarding Theorem 1 and Remark following this theorem we can write

$$\begin{aligned} y(x) &= y_0 y_1(x, x_0) + y_0' y_2(x, x_0) + \dots + y_0^{(k-1)} y_k(x, x_0) + \\ &\quad + \int_{x_0}^{x} y_k(x, t) \frac{-K + f_1(t)}{a_k(t)} \mathrm{d}t \leqq y_0 y_1(x, x_0) + \dots + \\ &\quad + y_0^{(k-1)} y_k(x, x_0) + (-K) \int_{x_0}^{x} \frac{y_k(x, t)}{a_k(t)} \mathrm{d}t = \\ &= y_0 y_1(x, x_0) + \dots + y_0^{(k-1)} y_k(x, x_0) - K y_{k+1}(x, x_0) = \\ &= y_{k+1}(x, x_0) \left(y_0 \frac{y_1(x, x_0)}{y_{k+1}(x, x_0)} + \dots + y_0^{(k-1)} \frac{y_k(x, x_0)}{y_{k+1}(x, x_0)} - K \right). \end{aligned}$$

Using Lemma 2 and the assumption (23), leads to the conclusion $\lim_{x \to \infty} y(x) = -\infty$.

Theorem 4. *Let n be even, let* (23) *be true and let g have the property:*

(i) *To every $\beta > 0$ there exists such a $\beta_1 > 0$ that if $y_1 > \beta > 0$, then $g(x, y_1, y_2, \dots, y_n) > \beta_1 > 0$ for all $(y_2, \dots, y_n) \in R^{n-1}$ and if $y_1 < -\beta < 0$, then $g(x, y_1, \dots, y_n) < -\beta_1 < 0$ for all $(y_2, \dots, y_n) \in R^{n-1}$. Let*

(ii) $$\overline{\lim_{x \to \infty}} \int_{x_0}^{x} \frac{1}{a_n(t)} (1 + \lambda f(t)) \mathrm{d}t = \infty \qquad \textit{for all } x_0 > 0$$

and for every $\lambda \neq 0$.

(iii) *Let there exist $a, b > 0$ such that*

$$h_1(x) = \int_{a}^{x} y_n(x, t) \frac{f(t)}{a_n(t)} \mathrm{d}t \geqq 0 \qquad \textit{for all } x \geqq a,$$

$$h_2(x) = \int_{b}^{x} y_n(x, t) \frac{f(t)}{a_n(t)} \mathrm{d}t \leqq 0 \qquad \textit{for all } x \geqq b,$$

(iv) *h_1, h_2 have arbitrarily large zero-points,*

(v)

$$\left| \frac{1}{y_n(x, x_0)} \int_{x_0}^{x} y_n(x, t) \frac{f(t)}{a_n(t)} \mathrm{d}t \right| < M \qquad \textit{for all } x_0 \leqq x,$$

then every solution of the equation (1) *is oscillatory.*

Proof. Suppose there exists such a solution y of the equation (1) which is not oscillatory. Thus let it be positive on the interval (c, ∞), for some $c > a$. In the case $y < 0$ we shall proceed analogically. Regarding the fact that by Theorem 1 h_1 is a solution of the differential equation

$$L_n(h_1) = f(x)$$

we have

$$L_n(y - h_1) + g(x, y, y', \dots, y^{(n-1)}) = 0 \qquad \text{on } (c, \infty).$$

Because $y > 0$ on this interval, by (14) $g(x, y, y', \dots, y^{(n-1)}) > 0$ and thus $L_n(y - h_1)(x) < 0$ for all $x > c$. It follows from this inequality that the function $L_{n-1}(y - h_1)$ is decreasing on (c, ∞). If on some subinterval (c_1, ∞), $c_1 > c$, $L_{n-1}(y - h_1)$ were negative, then there would exist such a $K > 0$, that $L_{n-1}(y - h_1) < -K$ for $x \in (c_2, \infty)$, $c_2 > c_1$. According to Lemma 3 $y(x) - h_1(x) < 0$ would hold which would together with the assumption (iv) lead to a contradiction with the fact that y is positive on (c, ∞). Thus $L_{n-1}(y - h_1)(x) > 0$ on (c, ∞). We claim that there exists such a $c_3 > c$ that

$$L_1(y - h_1)(x) > 0 \tag{24}$$

holds for every $x \in (c_3, \infty)$.

It follows from the inequality $L_{n-1}(y - h_1)(x) > 0$ on (c, ∞) that either $L_{n-2}(y - h_1)(x) > K_1 > 0$ on some interval (c_2, ∞), $c_2 > c$ or $L_{n-2}(y - h_1)(x) < 0$ on the whole interval (c, ∞). According to Lemma 3 the first case leads to (24), while in the second case the situation is analogical to that as for $L_n(y - h_1) < 0$, but we have lowered the order of the quasi-derivative by 2. Using this procedure we come to the alternative: Either some of the quasi-derivatives $L_{n-2k}(y - h_1) > K > 0$ on some interval (c_2, ∞), $c_2 > c$ and this leads to (24) or the inequalities $L_n(y - h_1) < 0$, $L_{n-1}(y - h_1) > 0$, $L_{n-2}(y - h_1) < 0, \dots, L_2(y - h_1) < 0$, $L_1(y - h_1) > 0$ hold on (c, ∞). Then (24) is true again.

Integrating (24) from x_1 to x where $x_1 > c_3$ and x_1 is according to (iv) such that

$$y(x_1) - h_1(x_1) > 0, \tag{25}$$

we get

$$a_0(x)\, y(x) > a_0(x_1)\, y(x_1) + a_0(x)\, h_1(x) - a_0(x_1)\, h_1(x_1) \geqq$$
$$\geqq a_0(x_1)\,(y(x_1) - h_1(x_1))$$

and thus taking into consideration (23) and (25)

$$y(x) > \frac{1}{m}\, a_0(x_1)\,[y(x_1) - h_1(x_1)] = \beta > 0 \qquad \text{for all } x > x_1. \tag{26}$$

The integration of the equation (1) from x_2 to $x > x_2$ for an $x_2 > x_1$ gives that

$$L_{n-1}(y)(x) = L_{n-1}(y)(x_2) - \int_{x_2}^{x} \frac{g(t, y(t), y'(t), \dots, y^{(n-1)}(t)) - f(t)}{a_n(t)}\, dt.$$

From (26) and according to (i) the last expression equals or is smaller than

$$L_{n-1}(y)(x_2) - \int_{x_2}^{x} \frac{\beta_1 - f(t)}{a_n(t)}\, dt,$$

thus

$$L_{n-1}(y)(x) \leqq L_{n-1}(y)(x_2) - \beta_1 \int_{x_2}^{x} \frac{1}{a_n(t)} \left(1 - \frac{1}{\beta_1} f(t)\right) dt.$$

Then the condition (ii) implies that

$$\lim_{x \to \infty} L_{n-1}(y)(x) = -\infty.$$

Let $K_2 > 0$ be such that $M - K_2 < 0$. Then there exists such an $x_3 > x_2$ that $L_{n-1}(y)(x_3) < -K_1$. Further by the same proceeding as in the proof of Theorem 3, just x_1 is replaced by x_3 and K by K_2 we get that

$$\overline{\lim_{x \to \infty}} \frac{y(x)}{y_n(x, x_3)} < 0,$$

which contradicts the fact that y is positive on (c, ∞). This completes the proof of Theorem 4.

REFERENCES

[1] G D. Jones e S. M. Rankin: *III, Oscillation of a forced non-linear second order differential equation,* Lincei-Rend. Sc. fis. mat. e nat. — Vol. LIX — dicembre 1975.

[2] W. A. Coppel: *Disconjugary,* Springer Verlag Berlin—Heidel New York 1971.

[3] A. J. Levin: *Neoscillacija rešenij uravnenja* $x^{(n)} + p_1(t) x^{(n-1)} + \ldots + p_n(t) x = 0$, Uspechi matematičeskich nauk, tom XXIV, vypusk 2 (146), 1969.

M. Vencková
Matematicko-fyzikálna fakulta Univerzity Komenského
842 15 Bratislava, Mlynská dolina
Czechoslovakia