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Jan Chvalina; Květoslava Matoušková

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COREGULARITY OF ENDOMORPHISM MONOIDS OF UNARS

JAN CHVALINA, KVĚTOSLAVA MATOUŠKOVÁ, Brno
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Monounary algebras (called also unars [10]) – which are pairs (A, f), where A is a nonempty set and f is a selfmap of the set A – with regular and inverse endomorphism monoids have been characterized by L. A. Skornjakov in his paper [11]. This note aims to describe unars with coregular and anti-inverse endomorphism monoids. Coregular semigroups have been introduced and studied in [1]. In [8] there is defined a notion of an anti-regular semigroup, studied further in [9] and [2]. In the last paper such semigroups are called anti-inverse. Coregular and anti-inverse semigroups form subclasses of the class of regular semigroups. Identities satisfied in the both above mentioned classes (see below) enforce a very simple structure of set transformations f the centralizers of which (with respect to the complete transformation monoid) – i.e. in other words endomorphism monoids of corresponding unars (A, f) – belong to the mentioned classes. Although coregular semigroups form a subclass of the class of anti-inverse semigroups, strictly containing the commutative anti-inverse semigroup class [1], these properties coincide in the case of endomorphism monoids of connected unars and the first two coincide in the general case (of disconnected unars) as well.

Concerning basic notions from the semigroup theory see [4]. A semigroup S is called *anti-inverse* if for each element $a \in S$ there is an element $b \in S$ such that $aba = b$ and $bab = a$. The elements a and b are then called *anti-inverses* ([8], [2] and [9]). It is to be noted that the just mentioned semigroups are called anti-regular in [8] and [9]. Since every such a semigroup is regular and the above definition is also a modification of the definition of an inverse semigroup, we agree on the terminology of the paper [2].

An element a of a semigroup S is called *coregular* and b its *coinverse*, if $a = aba = bab$. A semigroup S is said to be *coregular* if each element of S is coregular (cf. [1]).

Concerning monounary algebras we use the notation and the terminology of [10] and [11] except some minor changes. The composition of mappings $f : A \to$

$\to B$, $g : B \to C$ is denoted by gf and the n-th iteration of a selfmap f of a set A is defined by $f^0 = \mathrm{id}_A$, $f^n = ff^{n-1}$. A subunar of a unar (A,f) is a pair (B, g) such that $\emptyset \neq B \subset A, f(B) \subset B$ and $g = f \mid B$. A unar (A,f) is said to be connected if for any pair of elements $a, b \in A$ there exist nonnegative integers m, n such that $f^n(a) = f^m(b)$. A maximal connected subunar of a unar (A,f) is a component of (A,f). If $\{(A_\iota, f_\iota) \mid \iota \in I\}$ is the system of all components of (A,f) we write simply $(A,f) = \sum_{\iota \in I} (A_\iota, f_\iota)$ or $(A,f) = (A_1, f_1) + (A_2, f_2)$ in the case $I = \{1, 2\}$. A construction of all homomorphisms between suitable unars (especially endomorphisms of a unar) has been developed in [5] and [6]. Results of these papers enable to define immediately endomorphisms desirable for our purposes. The monoid of all endomorphisms (with the composition of mappings as a binary operation) of a unar (A,f) will be denoted by $\mathrm{End}(A,f)$. An element a of the unar (A,f) is said to be cyclic if there is a positive integer n such that $f^n(a) = a$. The subunar consisting of all cyclic elements of a connected unar is called a cycle; its carrier set (called also a cycle) is denoted by $A_f^{\infty 2}$. An order of a cycle is the number of all its elements. A cycle with short tails is a connected unar (A,f) containing some noncyclic elements with the property $f(a)$ is cyclic for all $a \in A$ (cf. [11]). A sentence: „(A,f) has at most short tails" means that $f(a)$ is cyclic for all $a \in A$. A cycle which is a singleton is also called a loop.

The following results will be used in our considerations:

Proposition 1. ([2], Theorem 2.1). *A semigroup S is anti-inverse iff for any $a \in S$ there exists $b \in S$ such that $a^2 = b^2$, $ba = a^3$, $a^3 = a$.*

Proposition 2. ([9], Lemma 2). *Let S be an anti-inverse semigroup, and let a and b be anti-inverses in S. The following conditions are equivalent:*

1° $a^3 = a$,

2° *a^2 is an idempotent,*

3° $ab = ba$.

Proposition 3. ([3], Proposition 5). *Let (A,f) be a connected unar.* $\mathrm{End}(A,f)$ *is an anti-inverse semigroup iff (A,f) is a cycle of the order* 1 *or* 2 *with at most one short tail.*

Proposition 4. ([1], Theorem 3). *For a semigroup S the following conditions are equivalent:*

1° *S is coregular,*

2° *$a^3 = a$ for every element a of S,*

3° *S is the union of disjoint groups, the elements of which are of the order at most two.*

Consider first connected unars.

Proposition 5. *Let (A,f) be a connected unar. The following conditions are equivalent*

1° End(A,f) *is coregular.*

2° End(A,f) *is anti-inverse.*

3° (A,f) *is at most a two-element cycle with at most one short tail.*

Proof. The implication 1° ⇒ 2° follows immediately from Proposition 4. The equivalence of conditions 2°, 3° has been established in Proposition 3. Suppose the condition 3° is satisfied. Let $g \in \mathrm{End}(A,f)$ be an arbitrary element. It is easy to see that either g is the identity selfmap of A (where card $A \leqq 3$) or (A, g) is a two-element cycle with at most one short tail or it is a one-element cycle (i.e. a loop) with one short tail or finally (A, g) is formed by two one-element cycles, one of which has one short tail. In the all possible cases $g^3 = g$, thus again by Proposition 4 we have the condition 1° satisfied, q.e.d.

Remark. It is easy to verify (which also follows immediately from results of the paper [7]) that each of the following conditions 1°′, 2°′ is also equivalent to every condition from 1°, 2°, 3° given above:

1°′ End(A,f) is commutative and coregular.

2°′ End(A,f) is commutative and anti-inverse.

Lemma 1. *Let (A,f) be a unar such that $f^3 = f$ and there exists a positive integer $n \geqq 3$ with the property $g^n = g$ for all $g \in \mathrm{End}(A,f)$. Then (A,f) has at most two components, each of those has a nonempty cycle of the order at most 2, and (A,f) contains at most one noncyclic element.*

Proof. Suppose $(A,f) = \sum_{\iota \in I} (A_\iota, f_\iota)$. Let $\iota_0 \in I$ be an arbitrary index. Put $g(x) = f(x)$ for every $x \in A_{\iota_0}$ and $g(x) = x$ for all $x \in A \setminus A_{\iota_0}$. Since $f^3 = f$ we have $g^3 = g$ and thus each component of (A,f) has a nonempty cycle of the order at most 2 and (A,f) has only short tails. Admit card $I \geqq 3$ and choose a three-element subset $\{\iota, \varkappa, \lambda\} \subset I$. We have under a suitable notation $1 \leqq \mathrm{card}\, A_\iota^{\infty 2} \leqq \mathrm{card}\, A_\varkappa^{\infty 2} \leqq \mathrm{card}\, A_\lambda^{\infty 2} \leqq 2$. By [6], Theorem 2.14, there exist homomorphisms $h_\lambda : (A_\lambda, f_\lambda) \to (A_\varkappa, f_\varkappa)$, $h_\varkappa : (A_\varkappa, f_\varkappa) \to (A_\iota, f_\iota)$ and $h_\iota : (A_\iota, f_\iota) \to (A_\iota, f_\iota)$. Put $h(x) = h_\alpha(x)$ for any $x \in A_\alpha$, $\alpha \in \{\iota, \varkappa, \lambda\}$ and $h(x) = x$ for each $x \in A \setminus (A_\iota \cup A_\varkappa \cup A_\lambda)$. Then $h \in \mathrm{End}(A,f)$, but $h(a) \in A_\varkappa$, $h^2(a) \in A_\iota$ for any $a \in A_\lambda$, thus $h^n \neq h$ for any integer $n \geqq 2$, which contradicts the assumption. Therefore card $I \leqq 2$. Consequently the unar (A,f) has a form $(A_1, f_1) + (A_2, f_2)$. Now admit there are at least two different elements $a_1, a_2 \in A$ such that $f^{-1}(a_1) = f^{-1}(a_2) = \emptyset$. Without loss of generality we can suppose $\mathrm{card}\, A_1^{\infty 2} \leqq \mathrm{card}\, A_2^{\infty 2}$ and $a_1 \in A_1$. There exists an endomorphism $p \in \mathrm{End}(A,f)$ such that $p(a_2) = a_1, p(a_1) = f(a_1) \in A_1^{\infty 2}$ and $p \mid A_1^{\infty 2} = f \mid A_1^{\infty 2}$ (cf. [6], Theorem 2.14). Then $p^n \neq p$ for every $n \geqq 2$, which is a contradiction again. Hence (A,f) has at most one short tail, q.e.d.

Using Lemma 1 and some of the above quoted results we get the theorem characterizing unars with coregular endomorphism monoids.

Theorem 1. *Let* $(A,f) = \sum_{\iota \in I} (A_\iota, f_\iota)$ *be a unar. Then* End (A,f) *is coregular iff the following conditions are satisfied:*

1° card $A \leqq 4$ *and* card $I \leqq 2$.

2° (A,f) *has at most one noncyclic element.*

3° (A,f) *has at most two-element cycles, number of those is at most one.*

Proof. Assume (A,f) satisfies the conditions 1°–3°. Let $g \in \text{End}\,(A,f)$ be an arbitrary element. From the form of the unar (A,f) it follows that every cycle of the unar (A,g) has at most two elements and if $a \in A$ does not belong to any cycle of (A,g) then $g^{-1}(a) = \emptyset$. Thus $g^3 = g$ and by Proposition 4 the monoid End (A,f) is coregular.

Now suppose End (A,f) is coregular. Assume (A,f) is disconnected (for the opposite case see Proposition 5). Then $g^3 = g$ for any $g \in \text{End}\,(A,f)$ (Proposition 3). By the above Lemma 1 we have $(A,f) = (A_1,f_1) + (A_2,f_2)$, where (under a suitable notation) $1 \leqq \text{card}\, A_1^{\infty 2} \leqq \text{card}\, A_2^{\infty 2} \leqq 2$ and (A,f) has at most one short tail. It remains to show that the unar (A,f) has at most one two-element cycle. Admit card $A_1^{\infty 2} = \text{card}\, A_2^{\infty 2} = 2$. Then for an endomorphism h of (A,f) such that $(A_1^{\infty 2} \cup A_2^{\infty 2}, h \mid (A_1^{\infty 2} \cup A_2^{\infty 2}))$ is a four-element cycle we have $h^3 \neq h$, which contradicts the assumption of coregularity of the unar (A,f). The proof is complete.

Theorem 2. *Let* (A,f) *be a unar. Then* End (A,f) *is coregular iff it is anti-inverse.*

Proof. Since from the coregularity of End (A,f) it follows that this monoid is anti-inverse (compare e.g. Proposition 4(2°) with the definition of an anti-inverse semigroup) we prove the opposite implication. The case of a connected unar is included in Proposition 5, thus (A,f) is supposed to be disconnected.

Assume End (A,f) is anti-inverse. By Proposition 1 we have $g^5 = g$ for any $g \in \text{End}\,(A,f)$. Let $h \in \text{End}\,(A,f)$ be an anti-inverse element of f in the monoid End (A,f), i.e. f, h are mutual anti-inverses. Since $fh = hf$, by Proposition 2 we have $f^3 = f$. Hence in virtue of Lemma 1 we can write $(A,f) = (A_1,f_1) + (A_2,f_2)$, where $1 \leqq \text{card}\, A_1^{\infty 2} \leqq \text{card}\, A_2^{\infty 2} \leqq 2$ and (A,f) has at most one short tail. We show that the unar (A,f) has at most one two-element cycle. To this aim consider these two cases:

I. (A_1,f_1), (A_2,f_2) are two-element cycles,

II. (A_1,f_1) is a two-element cycle and (A_2,f_2) is a two-element cycle with one short tail.

Admit (A,f) has the form I.; say $A_1 = \{a_1, b_1\}$, $A_2 = \{a_2, b_2\}$. Define a permutation g of the set A in this way:

$$g = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ a_2 & b_2 & b_1 & a_1 \end{pmatrix}.$$

It is easy to see that $g \in \text{End}\,(A,f)$ and (A,g) is a four-element cycle. Denote by h an anti-inverse element of g in the monoid End (A,f), i.e. $g = hgh$, $h = ghg$. Since g is bijective there exists a mapping $g^{-1} \in \text{End}(A,f)$ inverse to g. From the equality $h = ghg$ it follows $hg = g^{-1}h$, i.e. h is a homomorphism of the unar (A, g) onto the unar (A, g^{-1}) (g maps the four-element cycle onto the four-element cycle) thus it is an isomorphism. Let $a \in A$ be an arbitrary element. There is an integer $k \in \{0, 1, 2, 3\}$ such that $h(a) = g^k(a)$. From here $g^{-1}h(a) = g^{k-1}(a)$, thus $g^{k+1}(a) = hg(a) = g^{-1}h(a) = g^{k-1}(a)$. Consequently $a = g^{1-k}g^{k+1}(a) = g^2(a)$, which is a contradiction for $g^5 = g$, $g^m \neq g$, where $2 \leqq m \leqq 4$.

Consider the case II. Suppose (A_1, f_1) is defined as above, $A_2 = \{a_0, a_2, b_2\}$, $f_2(a_0) = f_2(b_2) = a_2$, $f_2(a_2) = b_2$. Further, $g_1 : A \to A$ is defined by putting $g_1 \mid (A_1^{\infty 2} \cup A_2^{\infty 2}) = g$ (defined above) and $g_1(a_0) = g^3(a_2) = a_1$. Let $h_1 \in \text{End}(A,f)$ be an anti-inverse to g_1. Since $h_1(A_1^{\infty 2} \cup A_2^{\infty 2}) \subseteq A_1^{\infty 2} \cup A_2^{\infty 2}$, hence for $g = g_1 \mid (A_1^{\infty 2} \cup A_2^{\infty 2})$ and $h_2 = h_1 \mid (A_1^{\infty 2} \cup A_2^{\infty 2})$ we have $g = h_2 g h_2$, $h_2 = g h_2 g$. Moreover g is an automorphism of the unar $(A_1^{\infty 2} \cup A_2^{\infty 2}, f \mid (A_1^{\infty 2} \cup A_2^{\infty 2}))$. Now in the same way as above (where h_2 is considered instead of h) we get a contradiction again; consequently both cases are not admissible. The reference to Theorem 1 completes the proof.

From Theorem 1 and Theorem 2 we get immediately:

Corollary. *Let $\mathcal{T}_A$ be the complete transformation monoid of a set A. The following conditions are equivalent:*

1° *$\mathcal{T}_A$ is coregular.*

2° *$\mathcal{T}_A$ is anti inverse.*

3° card $A \leqq 2$.

The just formulated assertion is completed by the following result:

Proposition 6. *Let A be an infinite set. There exists a coregular commutative subsemigroup $\mathcal{S}_A$ of the monoid $\mathcal{T}_A$ (not generated by idempotents only) such that* card $\mathcal{S}_A$ = card A.

Proof. Let $\{A_\iota \mid \iota \in I\}$ be a decomposition of the set A such that $2 \leqq \text{card } A_\iota \leqq 3$ for every $\iota \in I$ and $\{\iota \mid \iota \in I, \text{card } A_\iota = 3\} \neq \emptyset$. For any $\iota \in I$ define $f_\iota : A \to A$ in this way: If card $A_\iota = 3$, say $A_\iota = \{a, b, c\}$, we put $f_\iota(a) = b, f_\iota(b) = c, f_\iota(c) = b$ and $f_\iota(x) = x$ for all $x \in A \setminus A_\iota$. If card $A_\iota = 2$, say $A_\iota = \{a, b\}$, we put $f_\iota(a) = f_\iota(b) = b$ and $f_\iota(x) = x$ for all $x \in A \setminus A_\iota$, again. We have $f_\varkappa f_\lambda = f_\lambda f_\varkappa$ for every pair $\varkappa, \lambda \in I$. Denote by $\mathcal{S}_A$ the subsemigroup of $\mathcal{T}_A$ generated by the set $\{f_\iota \mid \iota \in I\} \subset \mathcal{T}_A$. Evidently, card $\mathcal{S}_A$ = card I = card A. Further, for any $f \in \mathcal{S}_A$ there exists an n-tupple $\{\iota_1, \dots, \iota_n\}$ of elements of I such that $f = f_{\iota_1} \dots f_{\iota_n}$. Then $f^3 = (f_{\iota_1} \dots f_{\iota_n})^3 = f_{\iota_1}^3 \dots f_{\iota_n}^3 = f$, thus each element of $\mathcal{S}_A$ is self-coinverse, consequently $\mathcal{S}_A$ is coregular. Since elements of the generating set of $\mathcal{S}_A$ commute, the semigroup $\mathcal{S}_A$ is commutative.

REFERENCES

[1] G. Bijev, K. Todorov: *Coregular semigroups.* Notes on Semigroups VI, Budapest 1980 – 4, 1–11.

[2] S. Bogdanović, S. Milić and V. Pavlović: *Anti-inverse semigroups.* Publ. Inst. Math. Beograd, T. 24 (38), (1978), 19–28.

[3] J. Chvalina: *On connected unars with regular endomorphism monoids.* Arch. Math. (Brno) 16 (1980), 7–14.

[4] A. H. Clifford, G. B. Preston: *The Algebraic Theory of Semigroups.* Amer. Math. Soc., Providence 1964, transl. Алгебраическая теория полугрупп, Мир, Москва 1972.

[5] M. Novotný: *Sur un problème de la théorie des applications.* Publ. Fac. Sci. Univ. Masaryk, 344 (1953), 53–64.

[6] M. Novotný: *Über Abbildungen von Mengen.* Pacific J. Math. 13 (1963), 1359–1369.

[7] O. Reimer: *Commutativity of endomorphism monoids of monounary algebras,* manuscript. To appear.

[8] J. C. Sharp: *Anti-regular semigroups.* Notices Amer. Math. Soc. 24: 2 (1977) p. A–266.

[9] J. C. Sharp, Jr.: *Anti-regular semigroups.* Publ. Inst. Math. Beograd, 24 (1978), 147–150.

[10] L. A. Skornjakov: *Unars.* Coll. Math. Soc. J. Bolyai 29. Universal Algebra, Esztergom (Hungary) (1977), 735–743.

[11] L. A. Skornjakov: *Unary algebras with regular endomorphism monoids.* Acta Sci. Math. 40 (1978), 375–381.

J. Chvalina, K. Matoušková
Pedagogical faculty Department of Mathematics
662 80 Brno, Poříčí 31
Czechoslovakia