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LINEAR POSITIVE OPERATORS AND THEIR APPLICATIONS TO DIFFERENTIAL EQUATIONS

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In the present paper we shall deal with linear positive operators constructed in [2].

Let us consider two real functions α and g which are holomorphic functions defined in the disks $|x| < R_1$ and $|x| < R_2$. It is supposed that coefficients of the corresponding developments in power series are non-negative and that $\alpha(0) \neq 0$. We define the sequence α_n, $n = 1, 2, \ldots$, by the relation

$$\alpha_n(x) = \exp n \int_0^x \alpha'(s)\, g(s)\, ds, \qquad x \in [0, R),\ R = \min(R_1 R_2). \tag{1}$$

In this case the function α_n admits a development in power series with the convergence radius equal to R, thus

$$\alpha_n(x) = \sum_{\nu=0}^{\infty} c_{n\nu} x^\nu$$

and the coefficients $c_{n\nu}$ are non-negative, $c_{n0} = 1$.

Now we establish some formulas which will be usefull in what follows.

Let

$$\int_0^x \alpha'(s)\, g(s)\, ds = \sum_{\nu=0}^{\infty} a_\nu x^\nu, \qquad x \in [0, R) \tag{2}$$

where the coefficients a_k are non-negative.

By differentiating α_n we obtain

$$\alpha_n'(x) = n\alpha'(x)\, g(x) \,.\, \alpha_n(x)$$

thus

$$\sum_{\nu=1}^{\infty} \nu c_{n\nu} x^{\nu-1} = n\Big(\sum_{\nu=1}^{\infty} \nu a_\nu x^{\nu-1}\Big)\Big(\sum_{\nu=0}^{\infty} c_{n\nu} x^\nu\Big), \qquad x \in [0, R).$$

This formula implies

$$\begin{aligned} na_1 &= c_{n1} \\ 2na_2 + na_1c_{n1} &= 2c_{n2} \\ 3na_3 + 2na_2c_{n1} + na_1c_{n2} &= 3c_{n3} \end{aligned}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

hence

$$\nu c_{n\nu} = n\sum_{k=0}^{\nu-1} c_{nk}a_{\nu-k}(\nu - k), \qquad \nu = 1, 2, 3, \dots$$

(3) $$c_{n0} = 1$$

Further from the definition of the function α_n it follows

$$\alpha_{n+1}(x) = \alpha_1(x) \, . \, \alpha_n(x)$$

which means that

$$\sum_{\nu=0}^{\infty} c_{n+1,\nu}x^\nu = (\sum_{\nu=0}^{\infty} c_{n\nu}x^\nu)(\sum_{\nu=0}^{\infty} c_{1\nu}x^\nu).$$

Thus we may conclude for coefficient $c_{n\nu}$:

$$\begin{aligned} c_{n+1,0} &= c_{n,0}c_{1,0} \\ c_{n+1,1} &= c_{n0}c_{11} + c_{n1}c_{10} \end{aligned}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

(4) $$c_{n+1,\nu} = \sum_{k=0}^{\nu} c_{nk}c_{1,\nu-k}, \qquad \nu = 1, 2, \dots$$

Let $Q[a, b]$ be the set of all real functions defined and bounded on the interval $[0, \infty)$ and continuous on the interval $[a, b]$, continuous to the left in $x = a$ and continuous to the right in $x = b$. For $n = 1, 2, 3, \dots$ we define operators L_n by the relations:

$$L_n(f; x) = \frac{1}{\alpha_n(x)} \sum_{\nu=0}^{\infty} c_{n\nu}x^\nu f\left(\frac{\nu}{n}\right).$$

These operators are defined for each function which is bounded for $x \geqq 0$.

Further we consider the function

$$\tau(x) = x\alpha'(x)\, g(x).$$

From our conditions for α and g it follows that τ is an absolutely monotone function on the interval $[0, R)$.

In [2] it is shown that operators $L_n(f; x)$ satisfy the following conditions:

$$\begin{aligned} L_n(f; x) &= 1, \\ L_n(t; x) &= \tau(x), \\ L_n(t^2; x) &= \tau^2(x) + \frac{1}{n}[x^2\alpha'(x)\, g'(x) + x^2\alpha''(x)\, g(x) + \tau(x)]. \end{aligned}$$

In the same paper there is proved the following theorem:

Theorem. *Let* $a \in (0, R)$ *and let* $a^* = \tau(a)$. *If* $f \in Q[0, a^*]$ *then the sequence* $\{L_n(f; x)\}$, $n = 1, 2, \ldots$, *converges uniformly towards the function* $f(\tau(x))$ *on the interval* $[0, a]$.

It is known that

$$L_n(t; x) = \tau(x) \qquad \text{for all } n = 1, 2, \ldots,$$

consequently

$$\frac{1}{\alpha_{n+1}(x)} \sum_{\nu=\epsilon}^{\infty} c_{n+1,\nu} x^\nu \frac{\nu}{n+1} = \frac{1}{\alpha_n(x)} \sum_{n=\epsilon}^{\infty} c_{n,\nu} x^\nu \frac{\nu}{n}$$

and

$$\sum_{\nu=0}^{\infty} c_{n+1,\nu} x^\nu \frac{\nu}{n+1} = \alpha_1(x) \sum_{\nu=0}^{\infty} c_{n\nu} x^\nu \frac{\nu}{n}$$

hence

$$\frac{\nu}{n+1} c_{n+1,\nu} = \sum_{k=0}^{\nu} c_{nk} c_{1,\nu-k} \frac{k}{n}, \qquad \nu = 1, 2, \ldots \tag{5}$$

We recall the following definitions:

Definition 1. *A real function* f *is called convex, non-concave, polynomial, non-convex, concave of the k-th order on the interval* $[a, b]$, *if*

$$[x_1, x_2, \ldots, x_{k+2}; f] > 0, \geqq 0, = 0, \leqq 0, < 0,$$

respectively, for any system of $k + 2$ *knots from* $[a, b]$; $[x_1, \ldots, x_{k+2}; f]$ *is the* $(k + 1)$*st-order divided difference of the function* f *on the knots* $x_1, \ldots, x_{k+2}$.

Definition 2. *A linear functional* T *defined on* $C[a, b]$ *is of the exactness degree* k *or* T *is said to be in* $\mathscr{E}_k$ *if*

$$T[x^j] = 0, \qquad j = 0, 1, \ldots, k \qquad \text{and} \qquad T[x^{k+1}] \neq 0.$$

Definition 3. *A linear functional* T *defined on* $C[a, b]$ *has the simple form of the k-th-order and in this case we say that* $T \in \mathscr{A}_k$. *if for all* $f \in C[a, b]$ *it is*

$$T[f] = B[x_1, \ldots, x_{k+2}; f],$$

where $B \neq 0$ *is independent of* $f(x)$ *and the distinct knots* $x_1, \ldots, x_{k+2}$ *depend generally on the choice of* $f(x)$.

In the rest we shall use the following theorem [3]:

Theorem. (*T. Popoviciu*). *Let* T *be a linear functional defined on* $C[a, b]$. *Then* $T \in \mathscr{A}_k$ *if and only if* $T \in \mathscr{E}_k$ *and* $T[f] \neq 0$ *for any function convex of the k-th-order on* $[a, b]$.

Remark. *For* $x = 0$ *it is*

$$f(0) = L_n(f; 0) \qquad n = 1, 2, 3, \ldots$$

Now, we shall prove the following theorem:

Theorem 1. *Let f be convex of the first order on the interval* $[0, \infty)$.
Then the sequence $\{L_n(f; x)\}$, $n = 1, 2, \ldots$, *is decreasing on the interval* $(0, a]$, *i.e.*

$$L_n(f; x) > L_{n+1}(f; x), \qquad x \in (0, a], n = 1, 2, 3, \ldots$$

Proof

$$L_n(f; x) - L_{n+1}(f; x) = \frac{1}{\alpha_{n+1}(x)} \left\{ \alpha_1(x) \sum_{\nu=0}^{\infty} c_{n\nu} x^\nu f\left(\frac{\nu}{n}\right) - \sum_{\nu=0}^{\infty} c_{n+1,\nu} x^\nu f\left(\frac{\nu}{n+1}\right) \right\}_\beta$$

Using the Taylor's series for α_1 and carrying out the multiplication by Cauchy's rule, we obtain for the expression in brackets:

$$\sum_{\nu=0}^{\infty} \left[\sum_{k=0}^{\nu} c_{nk} c_{1,\nu-k} f\left(\frac{k}{n}\right) - c_{n+1,\nu} f\left(\frac{\nu}{n+1}\right) \right] x^\nu.$$

Then it suffices to establish

$$\sum_{k=0}^{\nu} c_{nk} c_{1,\nu-k} f\left(\frac{k}{n}\right) \geqq c_{n+1,\nu} f\left(\frac{\nu}{n+1}\right)$$

This is, however, a direct consequence of convexity since f is convex and relations (3) and (4) are valid.

Remark. *If f is non-concave, polynomial, non-convex. concave of the first order on* $[0, \infty)$, *then the sequence* $\{L_n(f; x)\}$ *is non-increasing, stationary, non-decreasing, increasing on the interval* $(0, a]$, *respectively.*

Corollary. *Let f be convex, non-concave, polynomial, non-convex, concave of the 1-st-order on* $[0, \infty)$.
Then

$$L_n(f; x) > f(\tau(x)), \qquad L_n(f; x) \geqq f(\tau(x)), \qquad L_n(f; x) = f(\tau(x)),$$
$$L_n(f; x) \leqq f(\tau(x)), \qquad L_n(f; x) < f(\tau(x)), \qquad x \in (0, a],$$

respectively.

Let x be a fixed point in $(0, a]$. Let T_{nx} be a functional defined on $C[0, \infty)$. by the relation:

$$T_{nx}[f] = L_{n+1}(f; x) - L_n(f; x).$$

These functionals are in $\mathscr{E}_1$ since

$$T_{nx}[1] = 0,$$
$$T_{nx}[t] = 0,$$
$$T_{nx}[t^2] = -\frac{x\tau'(x)}{n(n+1)}.$$

These functionals take negative values for any function convex of the first order on $[0, \infty)$. We see that T_{nx} satisfies the conditions of the Popoviciu's theorem and these functionals have simple forms of the first order, namely,

$$T_{nx}[f] = c_n(x)\,[\xi_{1n}, \xi_{2n}, \xi_{3n}; f]. \tag{6}$$

The value $c_n(x)$ can be determined by

$$T_{nx}[t^2] = \frac{-x\tau'(x)}{n(n+1)} = c_n(x)\,[\xi_{1n}, \xi_{2n}, \xi_{3n}; t^2].$$

From this

$$c_n(x) = -\frac{x\tau'(x)}{n(n+1)}. \tag{7}$$

Now, we define functionals R_{nx}, $x \in (0, a]$, $n = 1, 2, \ldots$, according to relations

$$R_{nx}[f] = L_n(f; x) - f(\tau(x)).$$

These functionals are in $\mathscr{E}_1$ since

$$R_{nx}[1] = R_{nx}[t] = 0,$$

$$R_{nx}[t^2] = \frac{1}{n}\,x\tau'(x).$$

According to the corollary we can see that

$$R_{nx}[f] < 0$$

for any function convex of the first order on $[0, \infty)$. Functionals R_{nx} satisfy the Popoviciu's theorem and have the following simple forms:

$$R_{nx}[f] = A_n(x)\,[\eta_{1n}, \eta_{2n}, \eta_{3n}; f], \tag{8}$$

where

$$A_n(x) = \frac{1}{n}\,x\tau'(x). \tag{9}$$

Remark. *If f'' is continuous on the interval $[0, \infty)$ and $|f''(x)| \leq M$ for al $x \in [0, \infty)$ then*

$$|R_{nx}[f]| \leq \frac{M}{2n}\,x\tau'(x), \tag{10}$$

$$|T_{nx}[f]| \leq \frac{M}{2n(n+1)}\,x\tau'(x). \tag{11}$$

Next we shall study the sequence formed by the first order derivatives of the operators $L_n(f; x)$.

We know

$$L_n'(f;x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\alpha_n(x)}\sum_{\nu=0}^{\infty} c_{n\nu}x^{\nu}f\left(\frac{\nu}{n}\right)\right\} =$$

$$= \frac{-\alpha_n'(x)}{\alpha_n(x)}\sum_{\nu=0}^{\infty} c_{n\nu}x^{\nu}f\left(\frac{\nu}{n}\right) + \frac{1}{\alpha_n(x)}\sum_{n=1}^{\infty} \nu c_{n\nu}x^{\nu-1}f\left(\frac{\nu}{n}\right) =$$

$$= \frac{1}{\alpha_n(x)}\left\{\sum_{\nu=1}^{\infty} \nu c_{n\nu}x^{\nu}f\left(\frac{\nu}{n}\right) - \frac{\alpha_n'(x)}{\alpha_n(x)}\sum_{\nu=0}^{\infty} c_{n\nu}x^{\nu}f\left(\frac{\nu}{n}\right)\right\}.$$

It is obvious that

$$\frac{\alpha_n'(x)}{\alpha_n(x)} = n\sum_{k=1}^{\infty} k a_k x^{k-1}.$$

Then

$$L_n'(f;x) = \frac{1}{\alpha_n(x)}\left\{\sum_{\nu=1}^{\infty} \nu c_{n\nu}x^{\nu-1}f\left(\frac{\nu}{n}\right) - n\sum_{\nu=1}^{\infty}\left[\sum_{k=0}^{\nu-1} a_{\nu-k}c_{nk}(\nu-k)\,f\left(\frac{k}{n}\right)\right]\right\}x^{\nu-1}.$$

For the expression in brackets we use the relation (3):

$$\left(\sum_{k=0}^{\nu-1} a_{\nu-k}c_{nk}(\nu-k)\right)f\left(\frac{\nu}{n}\right) - \sum_{k=0}^{\nu-1} c_{nk}a_{\nu-k}(\nu-k)\,f\left(\frac{k}{n}\right) =$$

$$= \sum_{k=0}^{\nu-1} a_{\nu-k}c_{nk}(\nu-k)\left(f\left(\frac{\nu}{n}\right) - f\left(\frac{k}{n}\right)\right) =$$

$$= \frac{1}{n}\sum_{k=0}^{\nu-1} a_{\nu-k}c_{nk}(\nu-k)^2\left[\frac{k}{n}, \frac{\nu}{n}; f\right].$$

Consequently, for $L_n'(f;x)$ we have:

$$L_n'(f;x) = \frac{1}{\alpha_n(x)}\sum_{\nu=1}^{\infty}\left(\sum_{k=0}^{\nu-1} a_{\nu-k}c_{nk}(\nu-k)^2\left[\frac{k}{n}, \frac{\nu}{n}; f\right]\right)x^{\nu-1}.$$

Next, let us suppose that f satisfies the Lipschitz' condition on the interval $[0, \infty)$ with a constant K, i.e.

(12) $$|f(x_1) - f(x_2)| \leqq K|x_1 - x_2|, \quad x_1 x_2 \in [0, \infty).$$

Then the absolute value of the divided difference $\left[\frac{k}{n}, \frac{\nu}{n}; f\right]$ is bounded by number K.

Then

$$|L_n'(f;x)| \leqq \frac{K}{\alpha_n(x)}\sum_{\nu=1}^{\infty}\left(\sum_{k=0}^{\nu-1} a_{\nu-k}c_{nk}(\nu-k)^2\right)x^{\nu-1} =$$

$$= \frac{K}{\alpha_n(x)}\left(\sum_{k=1}^{\infty} k^2 a_k x^{k-1}\right)\left(\sum_{k=0}^{\infty} c_{nk}x^{k}\right) = K \,.\, \tau'(x).$$

Thus the following lemma is proved:

Lemma. *Let a function f satisfy the condition* (12). *Then*

$$(13) \qquad |L_n'(f; x)| \leqq K\tau'(x) \quad \textit{for all } x \in [0, a].$$

Remark. *If $f'(x)$ is continuous and bounded on the interval* $[0, \infty)$, *then* (13) *is valid.*

Now, we can prove the following theorem concerning application to differential equations.

Theorem 2. *Let an initial value problem be given*

$$(14) \qquad y' = f(x, y), y(0) = y_0, \qquad x \in [0, a), a \leqq 1.$$

Let $f(x, y)$ satisfy the Lipschitz' condition in the strip $0 \leqq x < a, \ -\infty < y < +\infty$

$$|f(x, y_1) - f(x, y_2)| \leqq \lambda |y_1 - y_2| \qquad \textit{with } \lambda \in [0, 1).$$

Let $f(x, y)$ and its first two partial derivatives be continuous and bounded in the domain $0 \leqq x < \infty, \ -\infty < y < \infty$.

Then the functions $y_n(x)$ defined recursively by

$$(15) \qquad y_0(x) = y_0, \qquad y_n(x) = y_0 + \int_0^x L_n\{f(t, y_{n-1}, (t)); s\}\, ds$$

converge uniformly towards the solution $y(x)$ of the initial value problem (14).

Proof. As mentioned in [1] we shall show that the series

$$y_0 + \sum_{n=0}^{\infty} (y_{n+1}(x) - y_n(x))$$

converges uniformly for $x \in [0, a)$.

Let us put

$$\varepsilon_n(x) = y_{n+1}(x) - y_n(x), \qquad y_n'(x) = f(x, y_n(x)).$$

Then

$$|\varepsilon_n(x)| = |y_{n+1}(x) - y_n(x)| = \left|\int_0^x L_{n+1}(y_n', s)\, ds - \int_0^x L_n(y_{n-1}', s)\, ds\right| \leqq$$

$$\leqq \int_0^x |L_{n+1}(y_n'; s) - L_n(y_n', s)|\, ds + \int_0^x |L_n(y_n', s) - L_n(y_{n-1}', s)|\, ds = E_1 + E_2,$$

where

$$E_1 = \int_0^x |L_{n+1}(y_n', s) - L_n(y_n', s)|\, ds,$$

$$E_2 = \int_0^x |L_n(y_n', s) - L_n(y_{n-1}', s)|\, ds.$$

By using (11) it follows

$$E_1 \leqq \frac{ka^2\tau'(a)}{2n(n+1)},$$

where

$$k = \sup_{\Omega} \left| \frac{d^2 f(x, y)}{dx^2} \right|, \qquad \Omega = \{0 \leqq x < \infty, -\infty < y < \infty\}.$$

In the same way as in [1] it is shown

$$\left| \frac{d^2 f(x, y)}{dx^2} \right| \leqq k < \infty.$$

To estimate E_2 we use the Lipschitz' condition:

$$E_2 = \int_0^x | L_n(y'_n, s) - L_n(y'_{n-1}, s) | \, ds \leqq$$

$$\leqq \lambda x \sup_{0 \leqq t < a} | \varepsilon_{n-1}(t) | \leqq \lambda a \sup_{0 \leqq t < a} | \varepsilon_{n-1}(t) |.$$

The conclusion of this proof is the same as in [1].

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