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Archivum Mathematicum, Vol. 19 (1983), No. 4, 199--208

Persistent URL: `http://dml.cz/dmlcz/107174`

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIX: 199—208, 1983

REGULARITY OF ALGEBRAS WITH APPLICATIONS TO CONGRUENCE CLASS GEOMETRY

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(Received August 20, 1981)

1. Introduction

The concept of regular congruence relation was introduced in A. I. Mal'cev [11] as follows: A congruence Θ on an algebra $\mathfrak{A} = \langle A, F \rangle$ is said to be *regular* if it is uniquely determined by any of its classes $[a]\,\Theta$, $a \in A$. *Regular algebras*, i.e. algebras with regular congruences only, were investigated by H. A. Thurston[16]; *varieties of regular algebras* were studied in B. Csákány [2, 3], G. Grätzer [9], R. Wille [20] and J. Hagemann [10]. Some recent results on this topic can be found in I. Chajda [1] and also in [5, 6]. Geometrical properties of regularity were discussed in H. Werner and R. Wille [19], R. Wille [20] and in A. Pasini [13, 14, 15].

The organization of the material is as follows. Firstly, the concept of regularity is introduced in a more precise form and, simultaneously, some useful characterizations of this concept are derived. Secondly, we show the relationship between regular algebras and the parallelism of congruence class geometries. Finally, some applications for varieties of regular, weakly regular and subregular algebras are achieved in section 4.

2. Regularity in universal algebra: Regular elements, regular congruences and regular algebras

Let $\mathfrak{A} = \langle A, F \rangle$ be an algebra and let $S \subseteq A$, $R \subseteq A \times A$ be arbitrary subsets. Then the symbol

$\Theta(R)$ denotes the smallest congruence on $\mathfrak{A}$ containing R;

$\Theta[S]$ denotes the smallest congruence on $\mathfrak{A}$ collapsing S, i.e. $\Theta[S] = \Theta(S \times S)$.

Now, consider the equality $\Psi = \Theta[[a]\,\Psi]$ for $a \in A$, $\Psi \in \mathrm{Con}\,\mathfrak{A}$. Using the universal quantifier $\forall$, the following three concepts may be easily introduced.

Definition 1. *Let $\mathfrak{A} = \langle A, F \rangle$ be an algebra. Then*

(i) *An element* $a \in A$ *is said to be regular if* $\Psi = \Theta[[a]\, \Psi]$ *holds for every* $\Psi \in Con\, \mathfrak{A}$;

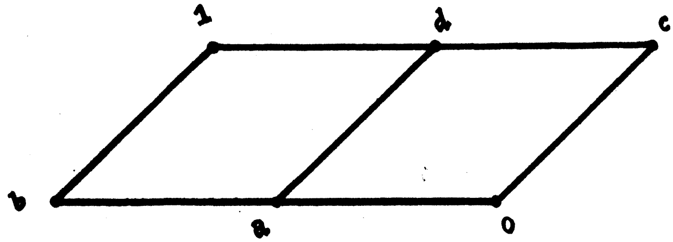
(ii) *A congruence relation* $\Psi \in Con\, \mathfrak{A}$ *is said to be regular if* $\Psi = \Theta[[a]\, \Psi]$ *holds for every* $a \in A$;

(iii) $\mathfrak{A}$ *is called regular if* $\Psi = \Theta[[a]\, \Psi]$ *holds for every* $a \in A$, $\Psi \in Con\, \mathfrak{A}$.

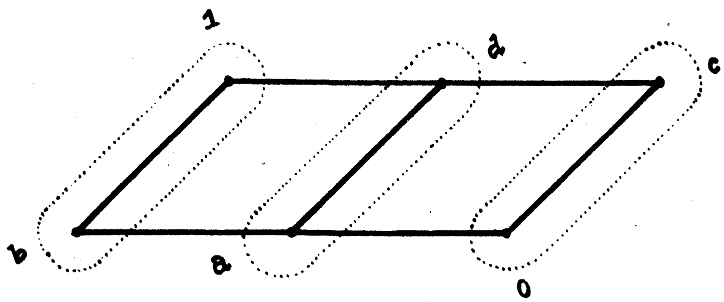
For illustration we present the following three

Examples. (i) Let $\mathfrak{A}$ be an *implicative semilattice,* i.e. a meet-semilattice with 1 and with the (binary) operation of relative pseudocomplementation $a * b = \max\{x \in A;\, x \wedge a \leqq b\}$, see, e.g. [12] or [4]. Then it is well-known that the distinguished nullary operation 1 is a regular element.

(ii) Consider the lattice $\mathfrak{A}$ represented by the following diagram



Then it is easily seen that the congruence relation $\{\{0, c\}, \{a, d\}, \{b, 1\}\}$ is regular and, on the contrary, $\{\{0, c\}, \{a, d, b, 1\}\}$ is not regular, see



Obviously, $\omega_A = \{\langle a, a\rangle;\, a \in A\}$ as well as $\iota_A = A \times A$ are regular congruences on any algebra $\mathfrak{A} = \langle A, F\rangle$.

(iii) Many algebras "from the life", e.g. quasigroups and hence also groups, rings, modules, Boolean algebras, etc. are regular.

Theorem 1. *Let* $\mathfrak{A} = \langle A, F\rangle$ *be an algebra. Then for any element* $a \in A$ *and for any congruences* Ψ, Ψ_i, $i \in I$, *the following three conditions hold:*

(1) $\Psi = \Theta[[a]\, \Psi]$ *if and only if* $\Psi = \Theta(\{a\} \times S)$ *for some subset* $S \subseteq A$.

(2) $\Psi = \Theta[[a]\,\Psi]$ *and* Ψ *is compact* (*in the congruence lattice Con* $\mathfrak{A}$) *if and only if* $\Psi = \Theta(\{a\} \times F)$ *for some finite subset* $F \subseteq A$.

(3) $\bigvee_{i \in I} \Psi_i = \Theta[[a] \bigvee_{i \in I} \Psi_i]$ *whenever* $\Psi_i = \Theta[[a]\,\Psi_i]$ *for every* $i \in I$.

Proof. (1) Suppose $\Psi = \Theta[[a]\,\Psi]$. Since $\Theta[[a]\,\Psi] = \Theta(\{a\} \times [a]\,\Psi)$ the subset $S = [a]\,\Psi$ has the desired property: $\Psi = \Theta(\{a\} \times S)$.

Conversely, let $\Psi = \Theta(\{a\} \times S)$ for some $S \subseteq A$. Then $\Psi = \Theta(\{a\} \times S) \supseteq \supseteq \{a\} \times S$ and, consequently, $[a]\,\Psi \times [a]\,\Psi \supseteq \{a\} \times S$. In summary, $\Theta[[a]\,\Psi] = = \Theta([a]\,\Psi \times [a]\,\Psi) \supseteq \Theta(\{a\} \times S) = \Psi$. The inverse inclusion is trivial concluding $\Psi = \Theta[[a]\,\Psi]$.

(2) It is well-known (and trivial) that a congruence is compact in the congruence lattice if and only if it is finitely generated. Combining this fact with the preceding part (1) we immediately get (2).

(3) By Theorem 1(1), for any $i \in I$, there is a subset $S_i \subseteq A$ satisfying $\Psi_i = = \Theta(\{a\} \times S_i)$. We claim that $\bigvee_{i \in I} \Theta(\{a\} \times S_i) = \Theta(\{a\} \times \bigcup_{i \in I} S_i)$. Clearly, it suffices to verify the inclusion $\bigvee_{i \in I} \Theta(\{a\} \times S_i) \supseteq \Theta(\{a\} \times \bigcup_{i \in I} S_i)$: Since $\{a\} \times \bigcup_{i \in I} S_i = = \bigcup_{i \in I} (\{a\} \times S_i) \subseteq \bigcup_{i \in I} \Theta(\{a\} \times S_i) \subseteq \bigvee_{i \in I} \Theta(\{a\} \times S_i)$, also $\Theta(\{a\} \times \bigcup_{i \in I} S_i) \subseteq \subseteq \bigvee_{i \in I} \Theta(\{a\} \times S_i)$ is true. So we have $\bigvee_{i \in I} \Psi_i = \Theta(\{a\} \times \bigcup_{i \in I} S_i)$ and thus, by Theorem 1(1), $\bigvee_{i \in I} \Psi_i = \Theta[[a] \bigvee_{i \in I} \Psi_i]$. This completes the proof.

Corollary 1. (1) *Regular congruences are closed with respect to arbitrary suprema.*

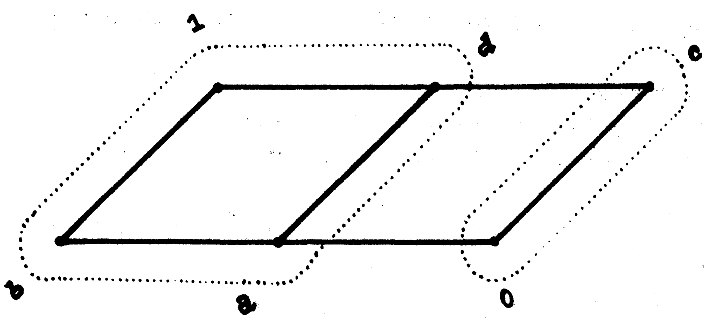
(2) *Any congruence* Ψ *is regular whenever the principal congruences* $\Theta(a, b)$, $\langle a, b \rangle \in \Psi$, *are regular.*

Proof. (1) Follows directly from Theorem 1 (3).

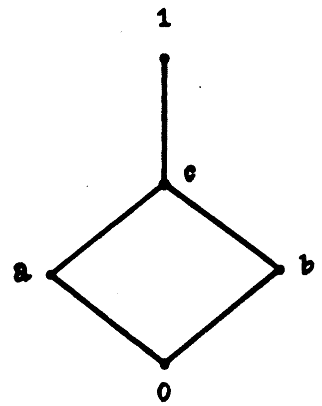
(2) Combining Theorem 1 (3) with the well-known fact that $\Psi = \bigvee \{\Theta(a, b); \langle a, b \rangle \in \Psi\}$ we immediately get the desired result.

Remark. Part (1) of the preceding Corollary 1 gives rise to a problem: Are regular congruences closed under arbitrary infima, i.e. under arbitrary intersections?

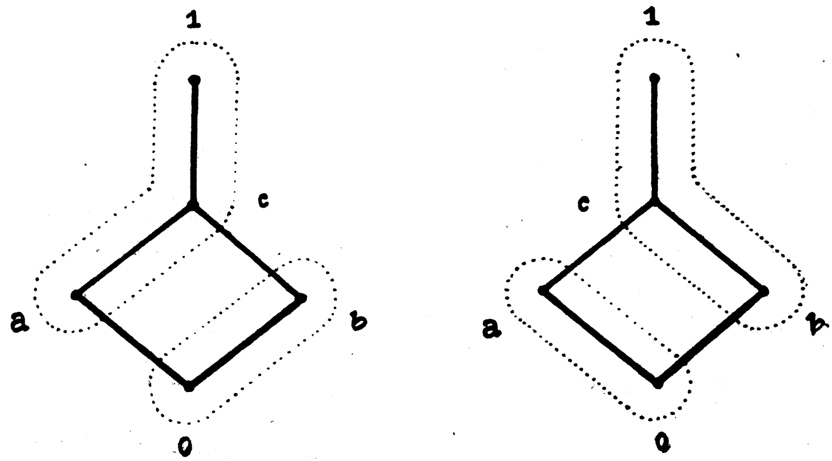
The following example answers this question in the negative. Let $\mathfrak{A}$ be an implicative semilattice represented by the diagram



Then it is easily seen that $\{\{0, a\}, \{b, c, 1\}\}$ and $\{\{0, b\}, \{a, c, 1\}\}$



are regular congruences, however, their intersection $\{\{0\}, \{a\}, \{b\}, \{c, 1\}\}$



is not regular.

Now, we state the main theorem of this section.

Theorem 2. (1) *For any algebra* $\mathfrak{A} = \langle A, F\rangle$ *the following three conditions are equivalent*:

(i) $\mathfrak{A}$ *is regular*;

(ii) *Every principal congruence on* $\mathfrak{A}$ *is regular*;

(iii) *For any elements* $a, b, c \in A$, *there exists a finite subset* $F \subseteq A$ *such that* $\Theta(b, c) = \Theta(\{a\} \times F)$.

(2) *For any element* a *of an algebra* $\mathfrak{A} = \langle A, F\rangle$, *the following three conditions are equivalent*:

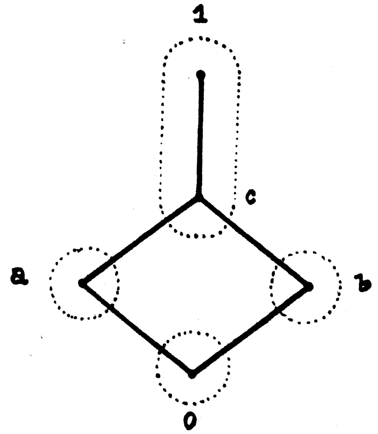
(i) a *is regular*;

(ii) *a is regular with respect to principal congruences on* $\mathfrak{A}$, *i.e.* $\Theta(b, c) = \Theta[[a]\,\Theta(b, c)]$ *holds for any elements* $b, c \in A$;

(iii) *For any elements* $b, c \in A$, *there exists a finite subset* $F \subseteq A$ *such that* $\Theta(b, c) = \Theta(\{a\} \times F)$.

Proof. (1) (i) $\Rightarrow$ (ii) is trivial. The equivalence (ii) $\Leftrightarrow$ (iii) follows immediately from Theorem 1 (2). (ii) $\Rightarrow$ (i) is a direct consequence of Corollary 1 (2).

(2) The proof goes along the same line as that of part (1) and is therefore omitted.



3. Application: Regular algebras and parallelism of R. Wille

The basic results concerning congruence class geometries were given by R. Wille in [20]. Further investigations of this topic were realized by H. Werner [18], H. Werner and R. Wille [19] and by A. Pasini, see [13, 14, 15] and references there.

For the purposes of this paper we recall the definition from [18; pp. 118–119]: A pair (A, Π) is called a *congruence class geometry* if the mapping $\Pi : A \times 2^A \to 2^A$ satisfies the following four axioms:

(Π 1) $\Pi(x \mid \emptyset) = \{x\}$

(Π 2) $\Pi(x \mid \Pi(y \mid M)) \subseteq \Pi(x \mid M)$

(Π 3) $y \in \Pi(x \mid x, y)$

(Π 4) $\Pi(x \mid M) = \bigcup \{\Pi(x \mid F);\ F \text{ is finite subset of } M\}$.

Having an algebra $\mathfrak{A} = \langle A, F\rangle$ the mapping $\Pi : A \times 2^A \to 2^A$ defined via $\Pi(x \mid M) = [x]\,\Theta[M]$, $x \in A$, $M \subseteq A$, evidently satisfies the above axioms (Π 1), ..., (Π 4) and so (A, Π), denoted by $\Gamma(\mathfrak{A})$, is called a *congruence class geometry of algebra* $\mathfrak{A}$. Further, for any congruence class geometry (A, Π), there is a binary relation π on the set $\mathcal{T}(A, \Pi) = \{\Pi(x \mid M);\ x \in A, M \subseteq A\}$ defined by the rule: $T_1 \pi T_2$ if and only if $T_2 = \Pi(t_2 \mid T_1)$ for some (or every) $t_2 \in T_2$. It is well-known, see [18; p. 121] and [20; p. 29] that π satisfies the axioms of the so-called *weak parallelism*, see [20; pp. 14–15] for this concept. Moreover, π is said to be *parallelism* whenever it is an equivalence relation on $\mathscr{T}(A, \Pi)$. So π is a weak parallelism for any congruence class geometry $\Gamma(\mathfrak{A})$ and it remains to characterize algebras for which π is a parallelism. This problem was solved (under some assumptions) in [20] and [14]; before giving the full description we need a preliminary lemma.

Lemma 1. (1) *For any subset S of an algebra $\mathfrak{A} = \langle A, F\rangle$, the following two conditions are equivalent:*

(i) $\Theta[S]$ *is regular;*

(ii) $\Pi(x \mid \Pi(y \mid S)) = \Pi(x \mid S)$ *holds for every $x, y \in A$.*

(2) *For any element a of an algebra $\mathfrak{A} = \langle A, F\rangle$, the following three conditions are equivalent:*

(i) *a is regular;*

(ii) $\Pi(x \mid \Pi(a \mid S)) = \Pi(x \mid S)$ *holds for every $x \in A$, $S \subseteq A$;*

(iii) $\Pi(x \mid \Pi(a \mid x, z)) = \Pi(x \mid x, z)$ $(= [x, z]$ *the line of the geometry* $\Gamma(\mathfrak{A})$ *generated by elements $x, z)$ holds for every $x, z \in A$;*

(iv) $z \in \Pi(x \mid \Pi(a \mid x, z))$ *holds for every $x, z \in A$.*

Proof. (1) (i) $\Rightarrow$ (ii). Clearly, $\Pi(x \mid \Pi(y \mid S)) = [x]\,\Theta[[y]\,\Theta[S]] = [x]\Theta[S] = \Pi(x \mid S)$ follows immediately from the regularity of $\Theta[S]$.

(ii) $\Rightarrow$ (i). By hypothesis, the equality $[x]\,\Theta[[y]\,\Theta[S]] = [x]\,\Theta[S]$ holds for any elements $x, y \in A$. Consequently, $\Theta[[y]\,\Theta[S]] = \Theta[S]$ is true for any $y \in A$ proving the regularity of $\Theta[S]$.

(2) The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are straightforward; (iv) $\Rightarrow$ (i) easily follows from Theorem 2 (2).

Now, we are ready to prove the main result of this section.

Theorem 3. *For any algebra $\mathfrak{A} = \langle A, F\rangle$, the following seven conditions are equivalent:*

(1) $\mathfrak{A}$ ***is regular;***

(2) π ***is a quasiordering on the set*** $\mathscr{T}\Gamma(\mathfrak{A})$;

(3) π ***is a parallelism, i.e.*** π ***is an equivalence on*** $\mathscr{T}\Gamma(\mathfrak{A})$;

(4) $T_1 \pi T_2$ ***if and only if*** $\Theta[T_1] = \Theta[T_2]$ ***holds for any*** $T_1, T_2 \in \mathscr{T}\Gamma(\mathfrak{A})$;

(5) $\Pi(x \mid \Pi(y \mid S)) = \Pi(x \mid S)$ ***holds for every*** $x, y \in A$, $S \subseteq A$;

(6) $\Pi(x \mid \Pi(y \mid u, v)) = \Pi(x \mid u, v)$ ***holds for every*** $x, y, u, v \in A$;

(7) $z \in \Pi(x \mid \Pi(y \mid x, z))$ ***holds for every*** $x, y, z \in A$.

Proof. The proof goes along the following diagram:

$$\begin{tikzcd}
(7) \arrow[r, Rightarrow] & (1) \arrow[dd, Rightarrow] & (2) \arrow[l, Rightarrow] & \\
 & & & (3) \arrow[ul, Rightarrow] \\
(6) \arrow[uu, Rightarrow] & (5) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (4) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

(1) $\Rightarrow$ (5). It follows immediately from the regularity of $\Theta[S]$. Implications (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are trivial.

(7) $\Rightarrow$ (1). Combining hypothesis with Lemma 1 (2), we get that every element $y \in A$ is regular. This proves the regularity of $\mathfrak{A}$.

(5) $\Rightarrow$ (4). It suffices to verify that $T_1 \pi T_2$ implies $\Theta[T_1] = \Theta[T_2]$. So let $T_2 = \Pi(t_2 \mid T_1)$ and suppose (5). Then for any $x \in A$, $[x]\,\Theta[T_2] = \Pi(x \mid T_2) = \Pi(x \mid \Pi(t_2 \mid T_1)) = \Pi(x \mid T_1) = [x]\,\Theta[T_1]$. Since element $x \in A$ was chosen arbitrarily, we get $\Theta[T_2] = \Theta[T_1]$ which was to be proved.

(4) $\Rightarrow$ (3). Immediate.

(3) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (1). Let $T \in \mathscr{T}\Gamma(\mathfrak{A})$ and let $r, s \in A$. Defining $S = \Pi(s \mid T)$ and $R = \Pi(r \mid S)$ we have $T\pi S$ and $S\pi R$. Then, by hypothesis, also $T\pi R$ is true, i.e. $R = \Pi(r \mid T)$. In summary, the equality $\Pi(r \mid T) = \Pi(r \mid \Pi(s \mid T))$ holds for any $r, s \in A$ and thus, by Lemma 1 (1), $\Theta[T]$ is regular for any $T \in \mathscr{T}\Gamma(\mathfrak{A})$.

Take a congruence $\Psi \in Con\,\mathfrak{A}$. Then, as was already proved, $\Theta[[a]\,\Psi]$ is regular for any element $a \in A$. Further, $[a]\,\Psi \times [a]\,\Psi \subseteq \Theta[[a]\,\Psi] \subseteq \Psi$ and thus also $\bigcup_{a \in A} ([a]\,\Psi \times [a]\,\Psi) \subseteq \bigcup_{a \in A} \Theta[[a]\,\Psi] \subseteq \Psi$ which yields $\Psi = \bigcup_{a \in A} \Theta[[a]\,\Psi]$ since $\bigcup_{a \in A} ([a]\,\Psi \times [a]\,\Psi) = \Psi$. Consequently, $\Psi = \bigvee_{a \in A} \Theta[[a]\,\Psi]$, i.e. Ψ is a supremum of regular congruences $\Theta[[a]\,\Psi]$, $a \in A$. Corollary 1 (1) completes the proof.

Remark. Theorem 3 sufficiently describes those algebras for which the weak parallelism π is a parallelism. Moreover, this result can be strengthened in the following way:

(i) One can easily verify that all the above results on regularity hold true for partial algebras, too (since only the fact that any congruence lattice is an algebraic lattice is used).

(ii) It is well-known, see [18; p. 121], that any congruence class geometry (A, Π) is *affine coordinatized* by a suitable partial algebra $\mathfrak{A} = \langle A, F \rangle$; i.e. $(A, \Pi) = \Gamma(\mathfrak{A})$ for some partial algebra $\mathfrak{A} = \langle A, F \rangle$.

Combining these two facts we get that congruence class geometries with parallelism are affine coordinatized exactly by regular partial algebras. Simultaneously, these congruence class geometries are definable by axioms (Π 1), (Π 3), (Π 4) and by any of the equivalent conditions (5), (6) or (7) from Theorem 3.

4. Application: Regular, weakly regular and subregular varieties and their Mal'cev characterizations

In this section we apply the results on regularity, see section 2, to *varieties of regular algebras* (briefly: *regular varieties*), to *varieties of weakly regular algebras* (briefly: *weakly regular varieties*) and to *varieties of subregular algebras* (briefly: *subregular varieties*).

Regular varieties were investigated by B. Csákány [2, 3], G. Grätzer [9] and and by R. Whille [20]. Some recent results were achieved by J. Hagemann [10] and I. Chajda [1]. Part (3) of the following Corollary 2 simplifies the identities exhibited in the former papers.

Corollary 2. *For any variety V, the following four conditions are equivalent*:

(1) *V is regular*;

(2) *For any elements x, y, z of an algebra $\mathfrak{A} \in V$, there is a finite subset $\{p_i; 1 \leqq i \leqq m\}$ of $\mathfrak{A}$ such that $\Theta(x, y) = \Theta(\{z\} \times \{p_i; 1 \leqq i \leqq m\})$*;

(3) *There exist ternary polynomials $\mathbf{p}_1, \ldots, \mathbf{p}_n$ and 4-ary polynomials $\mathbf{r}_1, \ldots, \mathbf{r}_n$ such that*

$$\mathbf{x} = \mathbf{r}_1(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z})$$

$$\mathbf{r}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{p}_i(\mathbf{x}, \mathbf{y}, \mathbf{z})) = \mathbf{r}_{i+1}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}) \qquad \text{for } 1 \leqq i \leqq n$$

$$\mathbf{y} = \mathbf{r}_n(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{p}_n(\mathbf{x}, \mathbf{y}, \mathbf{z}))$$

$$\mathbf{z} = \mathbf{p}_i(\mathbf{x}, \mathbf{x}, \mathbf{z}) \qquad \text{for } 1 \leqq i \leqq n;$$

(4) (B. Csákány [2]) *There exist ternary polynomials $\mathbf{p}_1, \ldots, \mathbf{p}_n$ such that*

$$(\mathbf{z} = \mathbf{p}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}),\ 1 \leqq i \leqq n) \Leftrightarrow \mathbf{x} = \mathbf{y}.$$

Sketch of proof. (1) $\Rightarrow$ (2) follows immediately from Theorem 2 (1).

(2) $\Rightarrow$ (3). See [6].

(3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1). See B. Csákány [2].

For varieties with distinguished nullary operations, say $\mathbf{c}_1, \ldots, \mathbf{c}_k$, the concept of regularity was generalized to that of weak regularity as follows: A variety V with nullary operations $\mathbf{c}_1, \ldots, \mathbf{c}_k$ is said to be *weakly regular with respect to* $\mathbf{c}_1, \ldots, \mathbf{c}_k$ if any congruence Θ on an algebra $\mathfrak{A} \in V$ is uniquely determined by its classes $[\mathbf{c}_i]\,\Theta$, $1 \leqq i \leqq k$.

Weakly regular varieties were investigated by G. Grätzer [9], K. Fichtner [7] and J. Hagemann [10]. Again the identities from Corollary 3 (3) below simplify the former results. Condition (4) is a new criterion.

For the sake of brevity the following statement is fomulated for varieties with one nullary operation $\mathbf{c}$ only.

Corollary 3. *For any variety V with nullary operation $\mathbf{c}$, the following four conditions are equivalent*:

(1) *V is weakly regular with respect to $\mathbf{c}$*;

(2) *For any elements x, y of an algebra $\mathfrak{A} \in V$, there is a finite subset $\{q_i; 1 \leqq i \leqq m\}$ of $\mathfrak{A}$ such that $\Theta(x, y) = \Theta(\{c\} \times \{q_i; 1 \leqq i \leqq m\})$*;

(3) *There exist binary polynomials $\mathrm{p}_1, \ldots, \mathrm{p}_n$ and ternary polynomials $w_1, \ldots, w_n$ such that*

$$\mathbf{x} = \mathbf{w}_1(\mathbf{x}, \mathbf{y}, \mathbf{c})$$

$$\mathbf{w}_i(\mathbf{x}, \mathbf{y}, \mathbf{q}_i(\mathbf{x}, \mathbf{y})) = \mathbf{w}_{i+1}(\mathbf{x}, \mathbf{y}, \mathbf{c}) \qquad \text{for } 1 \leqq i < n$$

$$\mathbf{y} = \mathbf{w}_n(\mathbf{x}, \mathbf{y}, \mathbf{q}_n(\mathbf{x}, \mathbf{y}))$$

$$\mathbf{c} = \mathbf{q}_i(\mathbf{x}, \mathbf{x}) \qquad \text{for } 1 \leqq i \leqq n;$$

(4) *There exist binary polynomials $\mathbf{q}_1, \ldots, \mathbf{q}_n$ such that*

$(c = q_i(x, y), 1 \leqq i \leqq n) \Leftrightarrow x = y.$

Sketch of proof. (1) $\Rightarrow$ (2) is a direct consequence of Theorem 2 (2). For (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1), see [6]. (3) $\Rightarrow$ (4) is immediate.

In [17] J. Timm has introduced the concept of subregular algebra as follows: An algebra $\mathfrak{A} = \langle A, F \rangle$ is called *subregular* if any congruence Θ on $\mathfrak{A}$ is uniquely determined by its classes $[b]\,\Theta$, $b \in B$, for every subalgebra $\mathfrak{B} = \langle B, F \rangle$ of $\mathfrak{A}$. A variety V is called subregular provided each algebra $\mathfrak{A} \in V$ is subregular.

The following theorem states that also subregular varieties are Mal'cev definable. The proof goes along the same line as those of the preceding two corollaries and is therefore omitted.

Corollary 4. *For any variety V, the following four conditions are equivalent*:

(1) *V is subregular*;

(2) *For any elements x, y of an algebra $\mathfrak{A} \in V$ and every subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, $\Theta(x, y) = \Theta(\bigcup_{i<m} (\{b_i\} \times \{p_{ij};\ 1 \leqq j \leqq k\}))$ holds for some elements $b_1, \dots, b_m$ of $\mathfrak{B}$ and for some finite subsets $\{p_{ij};\ 1 \leqq j \leqq k\}$ of $\mathfrak{A}$, $1 \leqq i \leqq m$*;

(3) *There exist unary polynomials $u_1, \dots, u_n$, ternary polynomials $p_1, \dots, p_n$ and 4-ary polynomials $s_1, \dots, s_n$ such that*

$$x = s_1(x, y, z, u_1(z))$$
$$s_i(x, y, z, p_i(x, y, z)) = s_{i+1}(x, y, z, u_{i+1}(z)) \qquad \text{for } 1 \leqq i < n$$
$$y = s_n(x, y, z, p_n(x, y, z))$$
$$u_i(z) = p_i(x, x, z) \qquad \text{for } 1 \leqq i \leqq n;$$

(4) *There exist unary polynomials $u_1, \dots, u_n$ and ternary polynomials $p_1, \dots, p_n$ such that*

$$(u_i(z) = p_i(x, y, z),\ 1 \leqq i \leqq n) \Leftrightarrow x = y.$$

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