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A NOTE ON ARGUESIAN LATTICES

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§ 0 INTRODUCTION

In [4] Jónsson introduced the Arguesian lattice identity which reflects precisely Desargues' Theorem for projective spaces qua lattices. In this note we "geometrize" Jónsson's equation by proving that one can assume more geometrical facts about the lattice variables in the equational or the (equivalent) implicational form. An interesting consequence of this is the fact that 2-distributive (modular) lattices are Arguesian. This fact reflects the property that one dimensional projective spaces geometrically trivial.

§ 1 PRELIMINARIES

The Arguesian identity and its equivalents have been developed by Jónsson et al in a series of papers (see especially [3], [6] and [7]). An early result, [5], was that Arguesian lattices are modular. Therefore we will assume throughout this paper that all lattices are modular.

Let L be a (modular) lattice. A triangle (or trilateral) in L is an arbitrary triple $\boldsymbol{a} = (a_0, a_1, a_2) \in L^3$. For two triangles, $\boldsymbol{a}$ and $\boldsymbol{b}$, in L we require certain derived polynomials and statements

1. Definition. For $\boldsymbol{a}, \boldsymbol{b}$ in L

(a) $p(\boldsymbol{a}, \boldsymbol{b}) = (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2)$

(b) $c_i(\boldsymbol{a}, \boldsymbol{b}) = (a_j \vee a_k) \wedge (b_j \vee b_k), \qquad \{i, j, k\} = \{0, 1, 2\}$

(c) $\mathrm{CP}(\boldsymbol{a}, \boldsymbol{b}) : (a_0 \vee b_0) \wedge (a_1 \vee b_1) \leqq a_2 \vee b_2$

(d) $\mathrm{AP}(\boldsymbol{a}, \boldsymbol{b}) : c_2(\boldsymbol{a}, \boldsymbol{b}) \leqq c_0(\boldsymbol{a}, \boldsymbol{b}) \vee c_1(\boldsymbol{a}, \boldsymbol{b})$

$\mathrm{CP}(\boldsymbol{a}, \boldsymbol{b})$ is an abbreviation of central perspectivity for the triangles (of points in a projective plane) $\boldsymbol{a}$ and $\boldsymbol{b}$. $\mathrm{AP}(\boldsymbol{a}, \boldsymbol{b})$ is an abbreviation for the axial perspectivity of these triangles (of points).

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2. Definition. A lattice $(L; \vee, \wedge)$ is called:

(a) Arguesian if for any pair of triangles $\boldsymbol{a}, \boldsymbol{b}$ in L, $p(\boldsymbol{a}, \boldsymbol{b}) \leqq a_0 \vee \vee (b_0 \wedge (b_1 \vee (c_2 \wedge (c_0 \vee c_1))))$.

(b) Desarguean if for any pair of triangles $\boldsymbol{a}, \boldsymbol{b}$ in L, CP$(\boldsymbol{a}, \boldsymbol{b})$ implies AP$(\boldsymbol{a}, \boldsymbol{b})$.

(c) Arguesian (Desarguean) for a given class of triangles if (a) (respectively (b)) holds for that particular class of triangles.

The following theorem is a synopsis of the pertient results that appear in aforementioned papers.

3. Theorem. ([3], [6], and [7]): *Let L be a modular lattice, then the following are equivalent*:

(1) *L is Arguesian.*

(2) *L is Desarguean.*

(3) *L is Desarguean for triangles* $\boldsymbol{a}, \boldsymbol{b}$ *in L satisfying* (A): $a_i \vee p = b_i \vee p = = a_i \vee b_i$, $i = 0, 1, 2$.

(4) *L is Arguesian for triangles* $\boldsymbol{a}, \boldsymbol{b}$ *in L satisfying* (A): $a_i \vee p = b_i \vee p = a_i \vee b_i$, $i = 0, 1, 2$.

The other notion we need is that of an n-diamond. This notion reflects the geo metrical property of $n + 1$ points in general position in a projective space o- dimension $\geqq n - 1$.

4. Definition. (i) A sequence $\boldsymbol{d} = (d_0, d_1, \ldots, d_n)$ in a (modular) lattice L is called an *n-diamond* if

$$(D_n1) \bigvee_{j \neq i}^{0,n} d_j = v, \qquad i \leqq n$$

$$(D_n2)\ d_i \wedge \bigvee_{\substack{k \neq i \\ k \neq j}}^{0,n} d_k = u, \qquad i \neq j \leqq n.$$

We will call a sequence, $\boldsymbol{d}$ in L, *a lower n-diamond* (*upper n-diamond*) if it satisfies (D_n2) (respectively (D_n1)).

(ii) A lattice is called *n-distributive* if it does not contain an *n-diamond.*

By Huhn [2] and Hermann & Huhn [1] we know that (modular) n-distributive lattices form an equational class of (modular) lattices and that an n-diamond is either non-trivial ($d_i \neq d_j$ for $i \neq j$) or completely trivial (i.e. $d_0 = d_1 = \ldots = d_n$).

The last notion we need is that of an *independent set* $\{a_1, \ldots, a_n\}$ in L.

$$\perp (a_1, \ldots, a_n) \qquad \text{iff} \qquad u = a_i \wedge \bigvee_{j \neq i}^{l,n} a_j \qquad \text{for all } i \leqq n.$$

§2 THE RESULTS

Our goal is to discover what other restrictions one can place on the triangles and still ensure the Desarguean implication. Basically, we are attempting to "regeometrize" the Desarguean implication by removing all "degenerate" cases of variable substitutions. Our first result is an easy reduction to independent triangles.

1. Lemma. *For a modular lattice L, L is Arguesian if and only if L is Desarguean for triangles **a** and **b** in L satisfying* (A) $a_i \vee p = b_i \vee p \ (= a_i \vee b_i)$ $i \leqq 2$ *and (I)* $\perp(a_0, a_1, a_2)$ *and* $\perp(b_0, b_1, b_2)$.

Proof: The condition is clearly necessary so let **a** and **b** be arbitrary centrally perspective triangles in L satisfying (A). By defining

$$x_i' = (x_i \vee x_j) \wedge (x_i \vee x_k)$$

for $x \in \{a, b\}$ and $\{i, j, k\} = \{0, 1, 2\}$ the reader can easily check that $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are centrally perspective triangles in L that satisfy (A) and (I). By the condition we infer AP($\boldsymbol{a}'$, $\boldsymbol{b}'$) and since $c_i' = c_i$ for all i we obtain AP($\boldsymbol{a}$, $\boldsymbol{b}$).

Later results will be proven in a similar way as we add more conditions on our triangles. Let us first note however that we may always assume that the triangles **a** and **b** are *strictly* independent. That is x_i and x_j will be incomparable if $i \neq j$, $x \in \{a, b\}$,

2. Lemma. *Let **a** and **b** be centrally perspective triangles in a modular lattice L. If either* $\{a_0, a_1, a_2\}$ *or* $\{b_0, b_1, b_2\}$ *contains comparable elements, then **a** and **b** are axially perspective.*

Proof: By symmetry we need only consider the cases $a_0 \leqq a_1$, $a_0 \leqq a_2$ and $a_0 \leqq a_1$. If $a_0 \leqq a_1$ then CP($\boldsymbol{a}$, $\boldsymbol{b}$) gives us

$$a_0 \leqq (a_0 \vee b_0) \wedge (a_1 \vee b_1) \leqq a_2 \vee b_2$$

and

$$\begin{aligned} c_0 \vee c_1 &= (a_1 \vee a_2) \wedge (b_1 \vee b_2 \vee c_1) = \\ &= (a_1 \vee a_2) \wedge (b_1 \vee ((b_2 \vee a_0 \vee a_2) \wedge (b_0 \vee b_2))) \geqq \\ &\geqq (a_1 \vee a_2) \wedge (b_1 \vee ((a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (b_0 \vee b_2))) = \\ &= (a_1 \vee a_2) \wedge (a_1 \vee b_1) \wedge (b_1 \vee ((a_0 \vee b_0) \wedge (b_0 \vee b_2))) \geqq \\ &\geqq a_1 \wedge (b_1 \vee b_0) = \\ &= (a_0 \vee a_1) \wedge (b_0 \vee b_1) = \\ &= c_2 . \end{aligned}$$

If $a_2 \leqq a_0$ then CP($\boldsymbol{a}$, $\boldsymbol{b}$) gives us

$$(a_2 \vee a_1 \vee b_1) \wedge (a_0 \vee b_0) \leqq a_2 \vee b_2$$

and

$$
\begin{aligned}
c_0 \vee c_1 &= c_0 \vee (a_0 \wedge (b_0 \vee b_2)) = \\
&= (a_0 \wedge (b_0 \vee b_2)) \vee (a_2 \cap b_0 \vee b_2)) \vee c_0 = \\
&= (a_0 \wedge (b_0 \vee b_2)) \vee \{(a_1 \vee a_2) \wedge [b_1 \vee b_2 \vee (a_2 \wedge (b_0 \vee b_2))]\} = \\
&= (a_0 \wedge (b_0 \vee b_2)) \vee \{(a_1 \vee a_2) \wedge [b_1 \vee ((b_2 \vee a_2) \wedge (b_0 \vee b_2))]\} \geqq \\
\geqq (a_0 \wedge (b_0 \vee b_2)) &\vee \{(a_1 \vee a_2) \wedge [b_1 \vee ((a_2 \vee a_1 \vee b_1) \wedge (a_0 \vee b_0) \wedge (b_0 \vee b_2))]\} = \\
&= (a_0 \wedge (b_0 \vee b_2)) \vee \{(a_1 \vee a_2) \wedge [b_1 \vee b_0 \vee (a_0 \wedge (b_0 \vee b_2))]\} = \\
&= [a_1 \vee a_2 \vee (a_0 \wedge (b_0 \vee b_2))] \wedge (b_0 \vee b_1 \vee (a_0 \wedge (b_0 \vee b_2)) \geqq \\
&\geqq [a_1 \vee ((a_2 \vee b_0 \vee b_2) \wedge a_0)] \wedge (b_0 \vee b_1) \geqq \\
&\geqq [a_1 \vee [a_0 \vee (b_0 \vee ((a_0 \vee b_0) \wedge (a_1 \vee b_1)))]] \wedge (b_0 \vee b_1) = \\
&= [a_1 \vee (a_0 \wedge (a_1 \vee b_0 \vee b_1))] \wedge (b_0 \vee b_1) = \\
&= c_2 .
\end{aligned}
$$

The case where $a_0 \leqq a_2$ is left for the reader.

Now let $\boldsymbol{a}$ and $\boldsymbol{b}$ be centrally perspective triangles in L satisfying (A) and (I) define

and(i) $s_a = (a_0 \wedge (a_2 \vee b_2)) \vee (a_1 \wedge (a_2 \vee b_2))$

(ii) $s_b = (b_0 \wedge (a_2 \vee b_2)) \vee (b_1 \wedge (a_2 \vee b_2))$

(iii) $a_i' = a_i \vee s_a$ and $b_i' = b_i \vee s_b$

Easy calculations give us that $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are centrally perspective triangles satisfying (A) and (I). Moreover $c_2' = c_2$, $c_1' = c_1 \vee (c_0 \wedge (a_2 \vee b_2))$ and $c_0' = c_0 \vee (c_1 \wedge (a_2 \vee b_2))$. Therefore AP($\boldsymbol{a}, \boldsymbol{b}$) holds if and only if AP($\boldsymbol{a}', \boldsymbol{b}'$) holds.

The above allows us to add the conditions $a_0 \wedge (a_2 \vee b_2) = a_1 \wedge (a_2 \vee b_2) = 0_a$ and $b_0 \wedge (a_2 \vee b_2) = b_1 \wedge (a_2 \vee b_2) = 0_b$ ($0_a = a_0 \wedge a_1 = a_0 \wedge a_2 = a_1 \wedge a_2$ and similarly for 0_b) to (A) and (I). We are however more interested in certain consequences of these conditions.

3. Theorem. *A modular lattice, L, is Arguesian if and only if L is Desarguean for triangles* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *in L satisfying* (A), (I) *and* (U): $p = (a_i \vee b_i) \wedge (a_j \vee b_j)$, $i \neq j$.

Proof: Using the notation above the theorem statement we get

$$a_0' \wedge (a_2' \vee b_2') = 0_a \vee s_a \leqq p'.$$

Therefore $p' = p' \vee (a_0' \wedge (a_2' \vee b_2')) = (p' \vee a_0') \wedge (a_2' \vee b_2')$ since $\mathrm{CP}(\boldsymbol{a}', \boldsymbol{b}') = (a_0' \vee b_0') \wedge (a_2' \vee b_2')$ by (A).

We want one more condition on our triangles. Let $\boldsymbol{a}$, $\boldsymbol{b}$ be triangles in L satisfying (A), (I) and (U) and define

(i) $u_a = \bigvee_i^{0,2} (a_i \wedge p)$, $u_b = \bigvee_i^{0,2} (b_i \wedge p)$,

(ii) $a_i' = a_i \vee u_a$, $b_i' = b_i \vee u_b$.

We have by easy calculations:

(1) $a'_i \vee b'_i = a_i \vee b_i$

(2) $p' = p$ and $\boldsymbol{a}'$, $\boldsymbol{b}'$ satisfy (U)

(3) $a'_i \vee a'_j = a_i \vee a_j \vee (a_k \wedge (a_i \vee b_i)) = a_i \vee a_j \vee (a_k \wedge (a_j \vee b_j))$

(4) $(a'_i \vee a'_j) \wedge (a'_i \vee a'_k) = [(a_i \vee a_j) \wedge (a_i \vee a_k)] \vee u_a$

(5) $\boldsymbol{a}$, $\boldsymbol{b}$ satisfy (I)

(6) $a'_i \vee p' = a'_i \vee b'_i = b'_i \vee p'$ hence $\boldsymbol{a}$, $\boldsymbol{b}$ satisfy (A)

(7) $c'_i = c_i \vee (c_j \wedge (a_k \vee b_k)) = c_i \vee (c_k \wedge (a_j \vee b_j))$

(8) AP($\boldsymbol{a}$, $\boldsymbol{b}$) iff AP($\boldsymbol{a}'$, $\boldsymbol{b}'$)

Therefore it behooves one to determine what other special properties hold for these triangles.

4. Lemma. *(a'_0, a'_1, a'_2, p') and (b'_0, b'_1, b'_2, p') are lower 3-diamonds with bottom element u_a* (resp. u_b).

Proof:

$$\begin{aligned} a'_0 \wedge (a'_1 \vee a'_2) &= (a_0 \vee (a_1 \wedge p) \vee (a_2 \wedge p)) \wedge (a_1 \vee a_2 \vee (a_0 \wedge p)) = \\ &= (a_0 \wedge (a_1 \vee a_2)) \vee u_a = \\ &= u_a \end{aligned}$$

$$\begin{aligned} a'_0 \wedge (a'_1 \vee p') &= (a_0 \vee (a_1 \wedge p) \vee (a_2 \wedge p)) \wedge (a_1 \vee p) = \\ &= u_a \vee (a_0 \wedge (a_1 \vee p)) = \\ &= u_a \vee (a_0 \wedge (a_1 \vee b_1) \wedge (a_1 \vee a_0 \vee b_0)) = \\ &= u_a \vee (a_0 \wedge (a_1 \vee b_1)) = \\ &= u_a \vee (a_0 \wedge p) = \\ &= u_a \end{aligned}$$

$$\begin{aligned} p' \wedge (a'_0 \vee a'_1) &= p' \wedge (a_0 \vee a_1 \vee (a_2 \wedge p)) = \\ &= (a_2 \wedge p) \vee [(a_0 \vee a_1) \wedge (a_0 \vee b_0) \wedge (a_1 \vee b_1)] = \\ &= u_a \end{aligned}$$

5. Lemma. *For each $\{i, j, k\} = \{0, 1, 2\}$, (a'_i, b'_i, c'_j, c'_k) is a lower 3-diamond.*

Proof:

$$\begin{aligned} c'_j \wedge (a'_i \vee b'_i) &= (a_i \vee b_i) \wedge (c_j \vee (c_k \wedge (a_i \vee b_i))) = \\ &= [c_j \wedge (a_i \vee b_i)] \wedge [c_k \wedge (a_i \vee b_i)] = \\ &= c'_k \wedge (a'_i \vee b'_i) \end{aligned}$$

$$\begin{aligned} a'_i \wedge (c'_j \vee c'_k) &= [a_i \vee (a_j \wedge p) \vee (a_k \wedge p)] \wedge [c_j \vee c_k] = \\ &= [a_i \vee (a_j \wedge (a_i \vee b_i)) \vee (a_k \wedge (a_i \vee b_i))] \wedge [c_j \vee c_k] \\ &= [a_i \vee (b_j \wedge (a_i \vee a_j)) \vee (b_i \wedge (a_i \vee a_k))] \wedge [c_j \vee c_k] = \\ &= (b_j \wedge (a_i \vee a_j)) \vee (b_i \wedge (a_i \vee a_k)) \vee (a_i \wedge (c_j \vee c_k)) \end{aligned}$$

$$= (b_j \wedge (a_i \vee a_j)) \vee (b_i \wedge (a_i \vee a_k)) \vee (a_i \wedge (b_i \vee b_k)) \vee$$
$$\vee (a_i \wedge (b_i \vee b_j)) =$$
$$= [c_j \wedge (a_i \vee b_i)] \vee [c_k \wedge (a_i \vee b_i)]$$

The other calculations are left to the reader.

6. Theorem. *A modular lattice, L, is Arguesian if and only if any triangles* **a**, **b** *in L satisfying*:

(A) $a_i \vee p = b_i \vee p = a_i \vee b_i$

(I) $\perp(a_0, a_1, a_2)$ *and* $\perp(b_0, b_1, b_2)$

(U) $p = (a_i \vee b_i) \wedge (a_j \vee b_j),\ i \neq j$

(LF) (a_0, a_1, a_2, p), (b_0, b_1, b_2, p), (a_i, b_i, c_j, c_k) *are lower* 3-*diamonds are axially perspective. Moreover* $c_i \vee c_j = c_i \vee c_k = c_j \vee c_k$.

7. Corollary. 2-*distributive* (*modular*) *lattices are Arguesian.*

Proof: If L is 2-distributive and **a** and **b** are triangles in L satisfying (A), (I), (U) and (LF) then the 3-diamond generated by the lower 3-diamond (a_2, b_2, c_0, c_1) must be trivial. This gives

$$[c_0 \wedge (a_2 \vee b_2)] \vee [c_1 \wedge (a_2 \vee b_2)] = q_{02} \wedge q_{12} \wedge a \wedge b$$

where

$$q_{ij} = a_i \vee a_j \vee p, \qquad a = a_0 \vee a_1 \vee a_2 \qquad \text{and} \qquad b = b_0 \vee b_1 \vee b_2.$$

Therefore

$$\begin{aligned} c_0 \vee c_1 &= c_0 \vee c_1 \vee (q_{01} \wedge q_{02} \wedge a \wedge b) = \\ &= [c_0 \vee c_1 \vee (q_{01} \wedge q_{02})] \wedge a \wedge b = \\ &= [c_0 \vee (q_{02} \wedge (c_1 \vee q_{01}))] \wedge a \wedge b \\ &= (c_0 \vee q_{02}) \wedge a \wedge b = \\ &= a \wedge b \geqq \\ &\geqq c_2. \end{aligned}$$

§ 3 DISCUSSION

One would obviously have liked to show that we can assume $0_a = 0_b$ and $a = b$. This however is impossible for as Jónsson (private communication) has observed, 2-dimensional Hall–Dilworth gluings of vector space lattices can produce both Arguesian and non-Arguesian lattices. In the non Arguesian case one needs $0_a \neq 0_b$.

Andras Huhn has asked if there exists a partial (modular) lattice configuration whose exclusion would characterize Arguesian lattices. The author cannot answer this question at present. Its answer should be connected with the projectivity of the "correct" version of the Desarguean implication. Our version seems to lack that property.

Finally the author feels that one *should* be able to add $a_i \vee a_j \vee p = a_i \vee a_k \vee p$ to the conditions on the triangles. He (obviously) has no proof of this at this time.

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