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INFINITESIMAL AFFINE DEFORMATIONS OF SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD*

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0. Let M^m be an m-dimensional submanifold of an n-dimensional Riemannian manifold M^n.

In the present paper we study the infinitesimal affine deformations of submanifolds of a Riemannian manifold.

In Theorem 1 and Theorem 3 we answer the following question: when an infinitesimal affine deformation of a submanifold M^m is infinitesimal isometric or infinitesimal volume preserving.

In Theorem 4 and Theorem 5, conditions have been found in which a hypersurface M^m does not allow non-trivial infinitesimal affine deformations.

All manifolds, tensors and maps are assumed to be C^∞.

All manifolds are assumed connected.

1. Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{U, x^h\}$. Let means g_{ij}, Γ^k_{ij}, ∇_i, R^h_{ijk} and R_{ij}, the metric tensor, the Christoffel symbols formed with g_{ij}, the operator of covariant differentiation with respect to Γ^i_{kj}, the curvature tensor and the Ricci tensor of M^n respectively. The indices $i, j, k, \ldots$ assume the values $1, 2, \ldots, n$.

Let M^m be an m-dimensional Riemannian manifold, covered by a system of coordinate neighbourhoods $\{V, u^\alpha\}$ and let by $g_{\alpha\beta}$, $\Gamma^\alpha_{\delta\beta}$, ∇_α, $R^\delta_{\alpha\beta\gamma}$ and $R_{\alpha\beta}$ the corresponding quantities of M^m be denoted. The indices $\alpha, \beta, \gamma, \delta, \ldots$ run over the range $1, 2, \ldots, m$.

We suppose that the manifold M^m is isometrically immersed in M^n by the immersion $r\colon M^m \to M^n$ and we identify $r(M^m)$ with M^m.

We represent the immersion r by

$$x^h = x^h(u^\alpha) \tag{1.1}$$

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and denote

(1.2) $$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha},$$

B^i_α are m linearly independent vectors of M^n tangent to M^m.

Since the immersion is isometric, we have

(1.3) $$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta.$$

We denote by N^h_λ $(\lambda = m+1, m+2, \ldots, n)$ $n-m$ mutually orthogonal unit normals to M^m, and by $D: I \times M^m \to M^n$, $I = (-\varepsilon, \varepsilon)$ $\varepsilon > 0$ an arbitrary deformation of M^m. Then the field z^h of the deformation D can be represented as:

(1.4) $$z^h = \zeta^\alpha B^h_\alpha + \zeta^\lambda N^h_\lambda,$$

where $\zeta^\alpha(\alpha = 1, 2, \ldots, m)$ and $\zeta^\lambda(\lambda = m+1, \ldots, n)$ are tangential and normal components of the field of deformation z^h, respectively.

We call a deformation D of the submanifold M^m trivial, when the field of the deformation z^h is identically equal to zero.

If the deformation vector z^h is tangent to the submanifold, we say that the deformation is tangential (i.e. $\zeta^\lambda = 0$).

If the deformation vector z^h is normal to the submanifold, we say that the deformation is normal (i.e. $\zeta^\alpha = 0$).

A deformation D of M^m is then and only then [2]

a) infinitesimal isometric, when the components ζ^α and ζ^λ of the field of deformation z^h satisfy the following system of equations:

(1.5) $$\nabla_\alpha\zeta_\beta + \nabla_\beta\zeta_\alpha - 2h_{\alpha\beta\lambda}\zeta^\lambda = 0,$$

where $h^\lambda_{\alpha\beta}$ are the second fundamental tensors of M^m with respect to the normals N^h_λ; $h^\beta_{\alpha\lambda} = g^{\beta\delta}h_{\alpha\delta\lambda}$; $h_\lambda = h^\alpha_{\alpha\lambda} = g^{\alpha\beta}h_{\alpha\beta\lambda}$.

b) infinitesimal affine, when ξ^α and ξ^λ satisfy the system of equations:

(1.6) $$\nabla_\gamma\nabla_\beta\zeta_\alpha + R_{\varepsilon\gamma\beta\alpha}\zeta^\varepsilon = \nabla_\gamma(h_{\beta\alpha\lambda}\zeta^\lambda) + \nabla_\beta(h_{\alpha\gamma\lambda}\zeta^\lambda) - \nabla_\alpha(h_{\beta\gamma\lambda}\zeta^\lambda).$$

c) infinitesimal volume preserving, if ζ^α and ζ^λ satisfy:

(1.7) $$\nabla_\alpha\zeta^\alpha = h_\lambda\zeta^\lambda.$$

2. Theorem 1. *If an infinitesimal affine deformation of a submanifold M^m of a Riemannian manifold M^n is infinitesimal isometric at least at one point of M^m, then this affine deformation is isometric on the whole M^m.*

Proof: From equation (1.6) and

(2.1) $$\nabla_\gamma\nabla_\alpha\zeta_\beta + R_{\varepsilon\gamma\alpha\beta}\zeta^\varepsilon = \nabla_\gamma(h_{\alpha\beta\lambda}\zeta^\lambda) + \nabla_\alpha(h_{\gamma\beta\lambda}\zeta^\lambda) - \nabla_\beta(h_{\alpha\gamma\lambda}\zeta^\lambda)$$

in view of $-R_{\varepsilon\gamma\beta\alpha} = R_{\varepsilon\gamma\alpha\beta}$ we obtain

(2.2) $$\nabla_\gamma(\nabla_\beta\zeta_\alpha + \nabla_\alpha\zeta_\beta - 2h_{\alpha\beta\lambda}\zeta^\lambda) = 0.$$

If we denote

(2.3) $$T_{\alpha\beta} = \nabla_\beta \zeta_\alpha + \nabla_\alpha \zeta_\beta - 2h_{\alpha\beta\lambda}\zeta^\lambda,$$

then

(2.4) $$\nabla_\gamma T_{\alpha\beta} = 0, \qquad T_{\alpha\beta} = T_{\beta\alpha} \qquad \text{and} \qquad T^{\alpha\beta} = T_{\varepsilon\delta} g^{\varepsilon\alpha} g^{\delta\beta}.$$

We multiply (1.6) by $T^{\alpha\beta}$

(2.5) $$T^{\alpha\beta}\, \nabla_\gamma(\nabla_\beta \zeta_\alpha - h_{\alpha\beta\lambda}\zeta^\lambda) = 0.$$

From (2.4) and (2.5) we have

(2.6.) $$T^{\alpha\beta}(\nabla_\beta \zeta_\alpha - h_{\alpha\beta\lambda}\zeta^\lambda) = C_1,$$

where C_1 is a global constant, since M^m is connected.

Since $T^{\alpha\beta} = T^{\beta\alpha}$, we can write (2.6) in the form

(2.7) $$T^{\alpha\beta} T_{\alpha\beta} = 2C_1.$$

The rest of the proof follows easily from the assumptions.

From this theorem we obtain some corollaries.

Corollary 1. *If $\zeta^h = \zeta^\alpha B^h_\alpha + \zeta^\lambda N^h_\lambda$ is a deformation vector field of an infinitesimal affine deformation, then the tensor $T_{\alpha\beta}$ has a constant length.*

Corollary 2. *If $z^h = \zeta^\alpha B^h_\alpha + \zeta^\lambda N^h_\lambda$ is a deformation vector field of an infinitesimal affine deformation, then*

(2.8) $$\frac{1}{2}(\nabla_\alpha \zeta_\beta + \nabla_\beta \zeta_\alpha)(\nabla^\alpha \zeta^\beta + \nabla^\beta \zeta^\alpha) \geqq 4h_{\alpha\beta\lambda}\zeta^\lambda \nabla^\alpha \zeta^\beta - 2h_{\alpha\beta\lambda} h^{\alpha\beta}_\mu \zeta^\lambda \zeta^\mu.$$

The equality is valid only if the deformation is infinitesimally isometric.

Theorem 2. *If $z^h = \zeta^\alpha B^h_\alpha + \zeta^\lambda N^h_\lambda$ is a deformation vector of an infinitesimal affine, deformation of a non-totally geodesic compact orientable submanifold M^m of an orientable Riemannian manifold M^n, then*

(2.9) $$\int_{M^n} h_{\alpha\beta\lambda} h^{\alpha\beta}_m \zeta^\lambda \zeta^m \,\mathrm{d}V \geqq \int_{M^n} h_{\alpha\beta\lambda}\zeta^\lambda \nabla^\alpha \zeta^\beta \,\mathrm{d}V.$$

The equality is fulfilled only if the deformation is infinitesimally isometric.

Proof: By Green's theorem and equality (2.4) it follows

(2.10) $$0 = \int_{M^m} \nabla^\beta (T_{\alpha\beta}\zeta^\alpha)\,\mathrm{d}V = \int_{M^m} T_{\alpha\beta} \nabla^\alpha \zeta^\beta \,\mathrm{d}V.$$

From this equality in view of (2.6) and (2.7) we have

(2.11) $$\int_{M^m} \frac{1}{2} T_{\alpha\beta} T^{\alpha\beta}\,\mathrm{d}V = \int_{M^m} \{2h_{\alpha\beta\lambda} h^{\alpha\beta}_m \zeta^\lambda \zeta^m - 2h_{\alpha\beta\lambda}\zeta^\lambda \nabla^\alpha \zeta^\beta\}\,\mathrm{d}V.$$

From (2.11) it follows that

$$\int_{M^m} 2h_{\alpha\beta\lambda}h_m^{\alpha\beta}\zeta^\lambda\zeta^m \,\mathrm{d}V \geqq \int_{M^n} 2h_{\alpha\beta\lambda}\zeta^\lambda \nabla^\alpha\zeta^\beta \,\mathrm{d}V.$$

The theorem is proved.

If we take into consideration

$$(2.12)\quad \frac{1}{2}T_{\alpha\beta}T^{\alpha\beta} = \frac{1}{2}(\nabla_\alpha\zeta_\beta + \nabla_\beta\zeta_\alpha)(\nabla^\alpha\zeta^\beta + \nabla^\beta\zeta^\alpha) - 4h_{\alpha\beta\lambda}\zeta^\lambda\nabla^\alpha\zeta^\beta + 2h_{\alpha\beta\lambda}h_\mu^{\alpha\beta}\zeta^\lambda\zeta^\mu,$$

then from (2.11) we have

Corollary 1. *If z^h is deformation vector of an infinitesimal affine deformation of a compact orientable submanifold M^m of an orientable Riemannian manifold M^n, then*

$$(2.13)\qquad \int_{M^n} h_{\alpha\beta\lambda}\zeta^\lambda\nabla^\alpha\zeta^\beta \,\mathrm{d}V \geqq 0.$$

Theorem 3. *An infinitesimal affine deformation of a minimal compact orientable submanifold M^m of an orientable Riemannian manifold M^n is necessarily infinitesimal volume preserving.*

Proof: If a submanifold is minimal, then

$$(2.14)\qquad h^\alpha_{\alpha\lambda} = h_\lambda = 0.$$

From equation (1.6) we can get the following equalities:

$$(2.15)\qquad \nabla^\beta\nabla_\beta\zeta_\alpha + R_{\beta\alpha}\zeta^\beta = 2\nabla^\beta(h_{\alpha\beta\lambda}\zeta^\lambda) - \nabla_\alpha(h_\lambda\zeta^\lambda),$$

$$(2.16)\qquad \nabla_\alpha\zeta^\alpha = h_\lambda\zeta^\lambda + C,$$

where C is a global constant, since M^m is connected.

From (2.14) and (2.16) it follows

$$(2.17)\qquad \nabla_\alpha\zeta^\alpha = C.$$

Since the submanifold M^m is compact and orientable, then

$$(2.18)\qquad \int_{M^m} \nabla_\alpha\zeta^\alpha \,\mathrm{d}V = 0.$$

From (2.17) and (2.18) we obtain that $C \equiv 0.$

Theorem 4. *Let M^m be a non-minimal compact orientable hypersurface of an orientable Riemannian manifold M^n. If the submanifold M^m satisfies the conditions*

a) *the second fundamental tensor $h_{\alpha\beta}$ is parallel, i.e.*

$$\nabla_\gamma h_{\alpha\beta} = 0,$$

b) *the quadratic form with the components $R_{\alpha\beta}$ of the Ricci tensor as coefficients is negatively definite, then M^m does not allow non-trivial infinitesimal affine deforma-*

tion for which the divergence of the tangential component of the deformation vector is equal to zero and the deformation vector is tangent to M^m at least at one point of M^m.

Proof: Let us suppose that M^m allows non-trivial infinitesimal affine deformations. Then ξ^α and ψ do not vanish at the same time and satisfy the equation (1.6).

The equation (1.6) in view of condition $\nabla_\gamma h_{\alpha\beta} = 0$ becomes

(2.19) $$\nabla_\gamma \nabla_\beta \zeta_\alpha + R'_{\varepsilon\gamma\beta\alpha} \zeta^\varepsilon = h_{\beta\alpha} \nabla_\gamma \psi + h_{\gamma\alpha} \nabla_\beta \psi - h_{\beta\gamma} \nabla_\alpha \psi.$$

From (2.19) we can get the following equations:

(2.20) $$\nabla^\beta \nabla_\beta \zeta_\alpha + R_{\varepsilon\alpha} \zeta^\varepsilon = 2h^\beta_\alpha \nabla_\beta \psi - h \nabla_\alpha \psi,$$

(2.21) $$\nabla_\alpha \zeta^\alpha = h\psi + C,$$

where C is a global constant.

Since the divergence of the vector ζ^α is equal to zero we have

(2.22) $$\nabla_\alpha \zeta^\alpha = 0.$$

From (2.21) by virtue of (2.22) and $\nabla_\gamma h_{\alpha\beta} = 0$ we obtain

(2.23) $$h \nabla_\alpha \psi = 0.$$

The hypersurface M^m is not minimal, i.e. $h \neq 0$. Then from (2.23) it follows that

(2.24) $$\psi = \bar{C},$$

where $\bar{C}$ is a constant.

The equality (2.19) in view of (2.24) becomes

(2.25) $$\nabla_\gamma \nabla_\beta \zeta_\alpha + R_{\varepsilon\gamma\beta\alpha} \zeta^\varepsilon = 0,$$

which shows that ζ^α is an affine Killing vector. Since M^m is compact and orientable, ζ^α is also a Killing vector:

(2.26) $$\nabla_\alpha \zeta_\beta + \nabla_\beta \zeta_\alpha = 0.$$

For a compact orientable submanifold M^m the following integral formula is valid

(2.27) $$\int_{M^m} \{R_{\alpha\beta} \zeta^\alpha \zeta^\beta + \nabla^\alpha \zeta^\beta \nabla_\beta \zeta_\alpha - (\nabla_\alpha \zeta^\alpha)^2\} \, dV = 0,$$

for any vector ζ^α in M^m [3].

From (2.26), (2.22) and (2.27) we have

(2.28) $$\int_{M^m} R_{\alpha\beta} \zeta^\alpha \zeta^\beta \, dV = \int_{M^m} \nabla^\alpha \zeta^\beta \nabla_\alpha \zeta_\beta \, dV.$$

This equality, considering condition b) of the theorem, is fulfilled only if ζ^α is identically equal to zero. The theorem is proved.

Corollary 1. *If a hypersurface M^m satisfies the conditions of the theorem, then M^m does not allow non-trivial infinitesimal affine deformations for which the tangential component of the deformation vector is a harmonic vector and the deformation vector is tangent to M^m at least at one point.*

Corollary 2. *A compact orientable hypersurface M^m of an orientable Riemannian manifold M^n does not allow non-trivial tangential infinitesimal affine deformation if the Ricci form $R_{\alpha\beta}$ is negatively definite.*

Theorem 5. *Let M^m be a non-minimal compact orientable hypersurface of an orientable Riemannian manifold M^n with negative (or equal to zero) constant scalar curvature. If M^m has a parallel second fundamental tensor ($\nabla_\gamma h_{\alpha\beta} = 0$), and the quadratic form with coefficients $h_\gamma h^\lambda_{\alpha\beta} - h^\varepsilon_{\beta\lambda} h^\lambda_{\alpha\varepsilon}$, is negatively definite, then M^m does not allow non-trivial infinitesimal affine deformation for which the divergence of the tangential component of the deformation vector is equal to zero and the deformation vector is tangent to M^m at least at one point.*

Proof: The Gauss equation of a submanifold of M^n is:

$$R_{\alpha\beta\gamma\delta} = R_{ijkh} B^i_\alpha B^j_\beta B^k_\gamma B^h_\delta + h_{\alpha\delta\lambda} h^\lambda_{\beta\gamma} - h_{\beta\delta\lambda} h^\lambda_{\alpha\gamma}. \tag{2.29}$$

The curvature tensor of a manifold M^n with constant scalar curvature K is:

$$R_{ijkh} = \frac{K}{n(n-1)} (g_{ih} g_{jk} - g_{ik} g_{jh}). \tag{2.30}$$

From (2.29), (2.30) and $g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta$ we obtain

$$R_{\beta\gamma} = \frac{m(m-1)}{n(n-1)} K g_{\beta\gamma} + h_\lambda h^\lambda_{\beta\gamma} - h^\alpha_{\beta\lambda} h^\lambda_{\alpha\gamma}. \tag{2.31}$$

Further the proof is analogous to that of Theorem 4.

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