Archivum Mathematicum

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Archivum Mathematicum, Vol. 18 (1982), No. 3, 145--149

Persistent URL: http://dml.cz/dmlcz/107136

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVIII: 145—150, 1982

DISTINGUISHING SUBSETS IN LATTICES

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(Received July 17, 1980)

Summary

The present paper investigates distinguishing (also called disjunctive) subsets in relative complemented lattices and in chains. For relative complemented·lattices it appears that each singleton is distinguishing. Hence, this fact holds for special cases—complemented modular lattices and for Boolean algebras as well. The structure of distinguishing subsets in chains is described.

Motivation

Distinguishing (disjunctive) subsets were studied in a connection with semigroup investigation at first (B. M. Schein, M. P. Schützenberger, E. J. Tully Jr., [6], [8], [10]). M. Novotný and H. J. Shyr have considered distinguishing subsets in monoids from the point of view of the formal language theory ([4], [8]). J. Zapletal, H. Jürgensen and G. Thierin have investigated distinguishing subsets for some special classes of semigroups ([11], [12], [13], [2]). I. Kopeček has generalized distinguishing subsets for universal algebras ([3]). This generalization enables to study more generally defined languages (see, for instance, [5]) in a connection with the notion of distinguishing subsets and to study distinguishing subsets in other classes of algebras. In [3], the problem of the existence of distinguishing subsets in connected monounary algebras is solved. The aim of this paper is to study distinguishing subsets in lattices.

1. INTRODUCTION

We shall use the following notation. If A is an algebra, F_A denotes the set of all operations of the algebra A. $M \subseteq A$ expresses that M is a subset of the algebra A (hence, we do not distinguish between A and its support). N_0 denotes the set of all non-negative integers.

1.1. Definition (see, for instance, [1], Chapter 6). An elementary translation on an algebra A is a function of the type

$$x \mapsto f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n),$$

where $f \in F_A$, $1 \leqq i \leqq n$, $n \in N_0$ and $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$.

A mapping $g: A \to A$ is said to be a translation on the algebra A, if either g is the identity mapping or g is composed of a finite number of elementary translations.

We shall recall the notion of distinguishing subset in universal algebra ([3]). This notion generalizes the special one introduced for semigroups.

1.2. Definition. Let A be an algebra and $L \subseteq A$. We shall say that L distinguishes A, if the following condition (*) holds.

(*): If $x, y \in A$ and $x \neq y$, then there exists a translation f such that exactly one of the elements $f(x)$, $f(y)$ belongs to L.

L is said to be a distinguishing subset of A, if L satisfies the condition (*).

Throughout this paper, a lattice A with a join operation $\vee$ and a meet operation $\wedge$ will be denoted as $A(\vee, \wedge)$. The induced ordering will be denoted as $\leqq$. Conversely, if each pair of a poset A has an infimum and a supremum, we shall consider A as a lattice, and the induced operations will be denoted as $\wedge$, $\vee$. Fundamental notions of the lattice theory are supposed to be known. They can be found, for instance, in [9].

2. DISTINGUISHING SUBSETS IN RELATIVELY COMPLEMENTED LATTICES

2.2. Theorem. *Each singleton of a relatively complemented lattice is its distinguishing subset.*

Proof. Let $A(\vee, \wedge)$ be a relatively complemented lattice and $c \in A$. We have to prove, that $\{c\}$ is a distinguishing subset of A. Let $a, b \in A$ and $a \neq b$.

a) Suppose that a and b are not comparable. Then $a \wedge a = a$ and $a \wedge b < a$ hold. Hence, the translation $t: x \mapsto x \wedge a$ maps the pair a, b onto the pair $a' = a$, $b' = a \wedge b < a$. Therefore, if to each pair of comparable elements a', b' there exists a translation t' such that exactly one of the elements $t'(a')$, $t'(b')$ equals c then $\{c\}$ is distinguishing. The existence of t' is proved in b).

b) Suppose that a is comparable with b and let $b < a$. Let r_b be a relative complement of b with respect to $b \wedge c$ and a. This means

$$r_b \wedge b = c \wedge b,$$

$$r_b \vee b = a.$$

This implies

(i) $$a \wedge r_b = r_b,$$

(ii) $$b \wedge r_b = c \wedge b \neq r_b,$$

(because $r_b = c \wedge b$ would imply $r_b \vee b = (c \wedge b) \vee b = b$ contradicting $b < a$). From $b < a$ it follows that $b \wedge r_b = a \wedge r_b$ and therefore $b \wedge r_b < a \wedge r_b$. By (ii) we have $b \wedge r_b \leqq c$. Hence, the translation t_1: $x \mapsto x \wedge r_b$ maps the pair a, b onto the pair $a' = a \wedge r_b$, $b' = b \wedge r_b$ so that $b' < a'$ and $b' \leqq c$.

Now we shall consider all the cases that can occur.

b1. c is not comparable with a'.

Then the translation t_2: $x \mapsto x \vee c$ maps the pair b', a' onto the pair $t_2(b') = b' \vee c = c$, $t_2(a') = a' \vee c = c$. Hence, the translations $t_2 t_1$ satisfies (*).

b2. $a' > c$.

Then the translation t_3: $x \mapsto x \vee c$ maps the pair a', b' onto the pair $t_3(a') = a' \vee c = a' \neq c$, $t_3(b') = b' \vee c = c$. Hence, the translation $t_3 t_1$ satisfies (*).

b3. $a' \leqq c$.

Let d be a relative complement of a' with respect to b' and c. Then $d \vee a' = c$ and $d \wedge a' = b'$. Hence, $d \geqq b'$ and therefore $b' \vee d = d$. Further, we have $d \neq c$ because of $d \wedge a' = b'$, $c \geqq a'$ and $a' > b'$. The translation t_4: $x \mapsto x \vee d$ maps the pair a', b' onto the pair $t_4(a') = a' \vee d = c$, $t_4(b') = b' \vee d \neq c$. Hence, the translation $t_4 t_1$ satisfies (*).

2.3. Corollary. *Each singleton of a complemented modular lattice is its distinguishing subset.*

Proof. The assertion follows from the fact that every complemented modular lattice is relatively complemented (see, for instance, [9]).

2.4. Corollary. *Each singleton of a Boolean algebra is its distinguishing subset.*

Proof. The assertion follows from the fact that every Boolean algebra is a relatively complemented lattice.

3. DISTINGUISHING SUBSETS IN CHAINS

A completely ordered nonempty set L is said to be a chain. We shall consider L as a lattice with the induced operations ($a \wedge b = \inf\{a, b\}$, $a \vee b = \sup\{a, b\}$). If $a, b \in L$ then $\langle a, b\rangle$ denotes the set $\{x \in L;\ a \leqq x \leqq b\}$. For $R \subseteq L$, $L - R$ denotes the complement of R in L.

3.1. Theorem. *Let L be a chain and $R \subseteq L$. Then the following assertions are equivalent.*

(1) R is a distinguishing subset of L.

(2) $\langle a, b\rangle \cap R \neq \emptyset$ and $\langle a, b\rangle \cap (L - R) \neq \emptyset$ for every $a, b \in L$, $a < b$.

Proof. (1) $\Rightarrow$ (2): Let R be a distinguishing subset of L and suppose that there are $a, b \in L$, $a < b$, such that $\langle a, b\rangle \cap (L - R) = \emptyset$. Hence, $\langle a, b\rangle \subseteq R$.

For every $v_1, v_2 \in \langle a, b\rangle$ satisfying $v_1 < v_2$ and every $w \in L$, only the following cases can occur.

a) $w \geqq v_2$.
Then

$$v_1 \vee w = v_2 \vee w = w$$

and

$$v_1 \wedge w = v_1, \qquad v_2 \wedge w = v_2.$$

b) $w \leqq v_1$.
Then

$$v_\cdot \vee w = v_1, \qquad v_2 \vee w = v_2$$

and

$$v_1 \wedge w = v_2 \wedge w = w.$$

c) $v_1 \leqq w \leqq v_2$.
Then

$$v_1 \vee w = w, \qquad v_2 \vee w = v_2$$

and

$$v_1 \wedge w = v_1, \qquad v_2 \wedge w = w.$$

Therefore, each elementary translation (i.e. a mapping of the form $x \mapsto x \vee s$ or $x \mapsto x \wedge s$ for any $s \in L$) maps a pair $v_1, v_2 \in \langle a, b\rangle$ either on the same element or into the interval $\langle a, b\rangle \subseteq R$. Clearly, this assertion holds for translations (instead of elementary translations) as well. This contradicts the condition (*). This contradiction follows from the assumption $\langle a, b\rangle \cap R = \emptyset$ quite analogously. Hence, (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (1): Let (2) be true and $a, b \in L$, $a < b$. If $a \in R$ and $b \in L - R$ (or conversely), we can put $f = \mathrm{id}_L$ in (*).

Suppose $a, b \in R$. Then there exists $c \in \langle a, b\rangle$ satisfying $c \in L - R$, $a \vee c = c \notin R$ and $b \vee c = b \in R$. Hence, we can put $f = (x \mapsto x \vee c)$ in (*). Analogously if $a, b \in L - R$.

3.2. Corollary. *Let L be an infinite chain. Then every distinguishing subset of L is infinite.*

3.3. Corollary. *The set of all rational (irrational) numbers is a distinguishing subset in the chain of all real numbers.*

3.4. Corollary. *Let $L = \{c_i\}_{i=1}^n$ be a finite chain with the ordering defined by: $c_i \leqq c_j$ iff $i \leqq j$. Then L has exactly two distinguishing subsets L_0, L_e, where:*

$L_0 = \{c_i \in L;\ i$ is an odd number$\}$,

$L_e = \{c_i \in L;\ i$ is an even number$\}$.

Proof. If $n = 1$, the assertion follows directly from the definition 1.?. If $n > 1$, the assertion follows easily by Theorem 3.1.

3.5. Corollary. *Every finite chain has exactly two distinguishing subsets.*

4. PROBLEMS

4.1. *Characterize distinguishing subsets in other classes of lattices.*

4.2. *Does there exist a distinguishing subset in every chain?*

4.3. *Does there exists a distinguishing subset in every lattice?*

4.4. *Compare distinguishing subsets with dense subsets in topological spaces defined by ordering.*

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