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UNIFORM NORMALITY OF TOPOLOGICAL GROUPS AND *l*-GROUPS

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1. Introduction

The topological space of a topological group is a completely regular space. The question about normality of that topological space was solved in the negative by A. A. Markov. He has proved that every completely regular topological space is a closed subspace of the topological space of a suitable topological group.

In this paper there are investigated some questions concerning a normality of topological groups and topological *l*-groups, namely, some kind of separability called uniform separability.

Now, we introduce some preliminary notes and definitions. A topological space (G, τ) is a non empty set G with a topology τ in the sense of Kuratowski (T_1-space). A closure of a set $P \subseteq G$ is denoted by $\bar{P}$. A topological group (G, Σ) has an additive group operation and a topology $\tau(\Sigma)$ defined by a complete system Σ of (open) neighbourhoods of zero. A topological *l*-group (G, Σ) (shortly *tl*-group) is a lattice-ordered group (*l*-group), G being a topological group and topological lattice in the topology $\tau(\Sigma)$ at the same time. N denotes the set of all positive integers. Further, we denote $A + B = \{a + b : a \in A, b \in B\}$, $A - B = \{a - b : a \in A, b \in B\}$, $A \vee B = \{a \vee b : a \in A, b \in B\}$, $A \wedge B = \{a \wedge b : a \in A, b \in B\}$ for a sum, a difference, a supremum, an infimum of every subsets A, B in a group or in a lattice, respectively.

1.1. Definition. Let (G, τ) be a topological space and $P, Q \subseteq G$. Then we say that *sets P, Q are separable* (*in the topology* τ) if there exist open sets A, B in G such that $A \supseteq \bar{P}$, $B \supseteq \bar{Q}$ and $A \cap B = \emptyset$.

1.2. Definition. Let (G, Σ) be a topological group and $P, Q \subseteq G$. Then we say that *sets P, Q are uniformly separable* (*in the topology* $\tau(\Sigma)$) if there exists a neighbourhood $U \in \Sigma$ such that $(\bar{P} + U) \cap (\bar{Q} + U) = \emptyset$.

1.3. Definition. A topological space (G, τ) is said to be *normal* if each pair of disjoint closed subsets in G is separable. Let (G, Σ) be a topological group.

A topological space $(G, \tau(\Sigma))$ is said to be *uniformly normal* if each pair of disjoint closed subsets in G is uniformly separable.

Remark. Uniform normality on a topological group is stronger than normality (see 3.7 Example).

Finally, we sum up the main results of this paper:

1. Let (G, Σ) be a topological group. Then the following assertions are equivalent:
   1. The topological space $(G, \tau(\Sigma))$ is uniformly normal.
   2. The sum of any two closed subsets in G is a closed subset in G.
   3. The difference of any two closed subsets in G is a closed subset in G.
2. A compact topological group is uniformly normal.
3. Let (G, Σ) be a *tl*-group and $P, Q \subseteq G$. If $\wedge \mid \bar{P} - \bar{Q} \mid \neq 0$, or $\wedge \mid \bar{P} - \bar{Q} \mid$ does not exist, then P, Q are uniformly separable sets.
4. The linearly ordered additive group of real numbers (or rational numbers) is not uniformly normal in the interval topology.
5. Let (G, Σ) be a linearly ordered *tl*-group with the interval topology. If the topological space $(G, \tau(\Sigma))$ is uniformly normal, then it holds:

1. G is totally non-archimedean, i.e., for every element $g \in G$, $g \neq 0$ there exists an element $h \in G$, $h \neq 0$ such that $\mid g \mid > n \mid h \mid$, for every $n \in N$.

2. If M is a closed subset in G and $\vee M$ $(\wedge M)$ exists, then the set M has the greatest (smallest) element.

3. $\tau(\Sigma) = \tau(\Sigma_1)$, where Σ_1 is the set of all non-zero convex subgroups in G.

4. $(G, \tau(\Sigma))$ is a totally disconnected topological space.

5. There exists no strongly decreasing (strongly increasing) sequence in G having an infimum (a supremum) in G.

2. Uniform separability in topological groups

2.1. Proposition. *If (G, Σ) is a topological group and $A \subseteq G$, $\bar{A} = A$, $g \in G$, $g \notin A$, then $\{g\}$ and A are uniformly separable sets.*

Proof. Consider the set $P = A - \{g\}$. Then $\bar{P} = P$, 0 non $\in P$ and there exists a neighbourhood $U \in \Sigma$ such that $\bar{U} \cap P = \emptyset$, because any topological group is a regular space. Now, if we take a neighbourhood $V \in \Sigma$ with the property $-V + V \subseteq U$, then $(V + \{g\}) \cap \{V + A\} = \emptyset$. Namely, if there exist elements $v_1, v_2 \in V$, $a \in A$ such that $v_1 + g = v_2 + a$, then $-v_2 + v_1 = a - g$ and $(-V + V) \cap (A - \{g\}) \subseteq \bar{U} \cap P = \emptyset$, a contradiction.

In the following part we investigate sums and differences of open or closed subsets in topological groups.

2.2. Proposition. 1. *If (G, Σ) is a topological group and Λ is a set of all open subsets in $\tau(\Sigma)$, then it holds: $A, B \in \Lambda \Rightarrow A + B \in \Lambda$, $-A \in \Lambda$.*

2. *If* (G, Σ) *is a tl-group, then it holds*:

$$A, B \in \Lambda \Rightarrow A \vee B \in \Lambda,\ A \wedge B \in \Lambda.$$

Proof. We prove only the implication: $A, B \in \Lambda \Rightarrow A \wedge B \in \Lambda$, in a *tl*-group (G, Σ): If $x \in A \wedge B$, then $x = a \wedge b$ for suitable elements $a \in A$, $b \in B$, and a neighbourhood $U \in \Sigma$ exists such that $a + U \subseteq A$, $b + U \subseteq B$. From this $x + U = (a \wedge b) + U \subseteq (a + U) \wedge (b + U) \subseteq A \wedge B$ follows, i.e., $A \wedge B \in \Lambda$.

2.3. Proposition. *Let* (G, Σ) *be a tl-group (a topological group) and* $\circ \in \{+, -, \vee, \wedge\}$ $(\circ \in \{+, -\})$ *an operation and let* $A, B \in G$, $g \in G$ *hold. Then it holds*:

1. $\overline{A \circ B} = \bar{A} \circ \bar{B}$, 2. $\overline{A \circ B} = \overline{\bar{A} \circ \bar{B}}$, 3. $\overline{A \circ \{g\}} = \bar{A} \circ \{g\}$.

Proof. 1. If $x \in \bar{A} \circ \bar{B}$ then $x = a \circ b$ for suitable elements $a \in \bar{A}$, $b \in \bar{B}$. If we choose an arbitrary neighbourhood $U \in \Sigma$ then from continuity of the operation $\circ$ there follows the existence of a neighbourhood $V \in \Sigma$ such that $(a \circ b) + U \supseteq (a + V) \circ (b + V)$. It means that there exist elements $v_1, v_2 \in V$ such that $a + v_1 \in A$, $b + v_2 \in B$ and thus $(a + v_1) \circ (b + v_2) \in A \circ B$ and $(a + v_1) \circ (b + v_2) = (a \circ b) + u = x + u$ for a suitable element $u \in U$. Finally, we have $x \in \overline{A \circ B}$.

2. $\overline{A \circ B} \subseteq \overline{\bar{A} \circ \bar{B}} \subseteq \overline{\overline{A \circ B}} = \overline{A \circ B} \Rightarrow \overline{A \circ B} = \overline{\bar{A} \circ \bar{B}}$.

3a. We have $\bar{A} + \overline{\{g\}} \subseteq \overline{A + \{g\}}$ by 1. and thus $\overline{A + \{g\}} = \overline{(A + \{g\}} - \overline{\{g\}}) + \overline{\{g\}} \subseteq \overline{A + \{g\}} - \overline{\{g\}} + \overline{\{g\}} = \bar{A} + \overline{\{g\}}$. We can prove similarly that $\bar{A} - \overline{\{g\}} = \overline{A - \{g\}}$.

3b. First, $\bar{M} \vee \bar{0} = \overline{M \vee 0}$, for every set $M \subseteq G$ (see [2], the proof of Prop. 4) and from this $\overline{A \vee \{g\}} = \overline{[(A - \{g\} \vee 0] + \{g\}} = \overline{(A - \{g\}} \vee \{0\}) + \overline{\{g\}} = (\overline{A - \{g\}} + \overline{\{g\}}) \vee \overline{\{g\}} = \overline{A - \{g\} + \{g\}} \vee \overline{\{g\}} = \bar{A} \vee \overline{\{g\}}$, by 1. We can prove similarly that $\bar{A} \wedge \overline{\{g\}} = \overline{A \wedge \{g\}}$.

2.4. Proposition. *If* (G, Σ) *is a topological group then* $-\bar{A} = \overline{-A}$ *for every* $A \subseteq G$.

Proof. If $x \in -\bar{A}$ then $x = -y$ for a suitable $y \in \bar{A}$, i.e., $(y + U) \cap A \neq \emptyset$ for every neighbourhood $U \in \Sigma$. It implies the existence of $u \in U$ and $a \in A$ such that $y + u = a$ and from this $x = -(a - u) = u - a$, $-u + x = -a$. It means $(-U + x) \cap (-A) \neq \emptyset$. Now, we have an arbitrary $V \in \Sigma$ and choose $U \in \Sigma$ such that $-U \subseteq V$ thus $(V + x) \cap (-A) \neq \emptyset$, i.e., $x \in \overline{-A}$. Thus $-\bar{A} \subseteq \overline{-A}$. The converse inclusion follows putting $-A$ instead of A in the preceding proof.

2.5. Proposition. *Let* (G, Σ) *be a topological group and let* $x \in G, A, B \subseteq G$ *hold. Then we have*:

$$x \in \overline{\bar{A} - \bar{B}} \setminus (\bar{A} - \bar{B}) \Leftrightarrow 0 \in \overline{\bar{A} - \bar{B}_1} \setminus (\bar{A} - \bar{B}_1), \quad \textit{where} \quad \bar{B}_1 = x + \bar{B}.$$

Proof. First, x non $\in \bar{A} - \bar{B} \Leftrightarrow 0$ non $\in \bar{A} - \bar{B} - x \Leftrightarrow 0$ non $\in \bar{A} - (x + \bar{B}) = \bar{A} - \bar{B}_1$. Now for every $U \in \Sigma$ a) to e) are equivalent: a) $x \in \overline{\bar{A} - \bar{B}}$, b) $(U + x) \cap (\bar{A} - \bar{B}) \neq \emptyset$, c) There exist elements $u \in U$, $a_1 \in \bar{A}$ and $b_1 \in \bar{B}$ such that $u + x = a_1 - b_1$ (or equivalently $u = a_1 - (x + b_1)$), d) $U \cap [\bar{A} - (x + \bar{B})] \neq \emptyset$, e) $0 \in \overline{\bar{A} - \bar{B}_1}$.

2.6. Proposition. *If (G, Σ) is a topological group then $\overline{\bar{A} - \bar{B}} = \cap \{\bar{A} - \bar{B} - U : U \in \Sigma\}$ for every $A, B \subseteq G$.*

Proof. If $x \in \overline{\bar{A} - \bar{B}}$ then $(x + U) \cap (\bar{A} - \bar{B}) \neq \emptyset$ for every $U \in \Sigma$. It means that elements $u \in U$, $a_1 \in \bar{A}$, $b_1 \in \bar{B}$ exist such that $x + u = a_1 + b_1$, i.e., $x = a_1 + b_1 - u \in \bar{A} - \bar{B} - U$. Finally $\overline{\bar{A} - \bar{B}} \subseteq \cap \{\bar{A} - \bar{B} - U : U \in \Sigma\}$.

If $x \in \cap \{\bar{A} - \bar{B} - U : U \in \Sigma\}$ then $x = a_1 - b_1 - u$, for suitable $a_1 \in \bar{A}$, $b_1 \in \bar{B}$, $u \in U$ thus $x + u = a_1 + b_1$ implies $(x + U) \cap (\bar{A} - \bar{B}) \neq \emptyset$ for every $U \in \Sigma$. It means that $x \in \overline{\bar{A} - \bar{B}}$ holds.

2.7. Proposition. *If (G, Σ) is a topological group and $A, B \subseteq G$ then $\bar{A} - \bar{B} \subseteq \overline{\bar{A} - \bar{B}} = \overline{A - B}$ holds.*

Proof. The facts $\bar{A} - \bar{B} \subseteq \overline{\bar{A} - \bar{B}}$ and $\overline{A - B} \subseteq \overline{\bar{A} - \bar{B}}$ are clear. Consider an arbitrary element $x \in \overline{\bar{A} - \bar{B}}$. Then for every $U \in \Sigma$ there exist neighbourhoods $V, U_0, U_1 \in \Sigma$ such that $U_1 - U_1 \subseteq U_0$, $U_0 - U_0 \subseteq V$, $-V \subseteq U$. Then $(x + U_1) \cap (\bar{A} - \bar{B}) \neq \emptyset$, i.e., $x + u = a_1 - b_1$ for suitable $u \in U_1$, $a_1 \in \bar{A}$, $b_1 \in \bar{B}$. Further $U_2 \in \Sigma$ exists such that $U_2 \subseteq U_1$ and $-x + U_2 + x \subseteq U_1$. We have $(U_2 + a_1) \cap A \neq \emptyset$, $(U_2 + b_1) \cap B \neq \emptyset$ and therefore $u_1, u_2 \in U_2$, $a \in A$, $b \in B$ exist such that $u_1 + a_1 = a$, $u_2 + b_1 = b$. This implies $x + u = (-u_1 + a) - (-u_2 + b) \Rightarrow x = -u_1 + (a - b) + u_2 - u \Rightarrow u_1 + x = (a - b) + u_2 - u$. We have $u_1 + x = x + u_3$ for suitable element $u_3 \in U_1$ because $u_1 + x \in U_2 + x \subseteq x + U_1$. Now $x + u_3 = u_1 + x = (a - b) + u_2 - u$, i.e., $x = (a - b) + u_2 - u - u_3 \in (a - b) + U_2 - U_1 - U_1 \subseteq (a - b) + (U_1 - U_1) - U_1 \subseteq (a - b) + U_0 - U_1 \subseteq (a - b) + U_0 - U_0 \subseteq (a - b) + V \subseteq (a - b) - U$. Finally, $x = (a - b) - W$, for a suitable element $w \in U$ there holds $x + w = a - b$ and $(x + U) \cap (A - B) \neq \emptyset$ for every $U \in \Sigma$. The inclusion $\overline{\bar{A} - \bar{B}} \subseteq \overline{A - B}$ is proved.

2.8. Proposition. *If (G, Σ) is a topological group and $A, B \subseteq G$, then it holds:*

$$\bar{A} - \bar{B} = \overline{A - B} \Leftrightarrow \bar{A} + \bar{B} = \overline{A + B}.$$

Proof. $\Rightarrow$: $\bar{A} + \bar{B} = \bar{A} - (-\bar{B}) = \bar{A} - \overline{(-B)} = \overline{A - (-B)} = \overline{A + B}$. $\Leftarrow$: $\overline{A - B} = \overline{A + (-B)} = \bar{A} + \overline{(-B)} = \bar{A} - \bar{B}$; see 2.4.

2.9. Theorem. *Let (G, Σ) be a topological group and $A, B \subseteq G$. Then the following assertions are equivalent:*

1. $\overline{\bar{A} - \bar{B}} = \bar{A} - \bar{B}$.

2. $\bar{A} \cap \bar{B} = \emptyset \Rightarrow$ *there exists* $V \in \Sigma$ *such that*

$$(V + \bar{A}) \cap (V + \bar{B}) = \emptyset.$$

3. $\bar{A} \cap \bar{B} = \emptyset \Rightarrow 0 \text{ non} \in \overline{\bar{A} - \bar{B}}$.

Proof. 1 $\Rightarrow$ 2: We have: $\bar{A} \cap \bar{B} = \emptyset \Rightarrow 0 \text{ non} \in \bar{A} - \bar{B} \Rightarrow 0 \text{ non} \in \cap\{\bar{A} - \bar{B} - U : U \in \Sigma\}$ (see 2.6) $\Rightarrow$ there exists $U \in \Sigma$ such that $0 \text{ non} \in \bar{A} - \bar{B} - U \Rightarrow \bar{A} \cap (U + \bar{B}) = \emptyset$ for this U. Then there exists neighbourhood $V \in \Sigma$ such that $-V + V \subseteq U$ and $(V + \bar{A}) \cap (V + \bar{B}) = \emptyset$. Namely, the existence of elements $v_1, v_2 \in V$, $a \in \bar{A}$, $b \in \bar{B}$ such that $v_1 + a = v_2 + b$ implies $a = -v_1 + v_2 + b \in \bar{A} \cap (-V + V + \bar{B}) \subseteq \bar{A} \cap (U + \bar{B})$, a contradiction.

2 $\Rightarrow$ 1: If $x \in \overline{\bar{A} - \bar{B}} \setminus (\bar{A} - \bar{B})$ then $x \in \bar{A} - \bar{B} - U$ for every $U \in \Sigma$ (see 2.6). Further, if we denote $\bar{B}_1 = x + \bar{B}$ then it holds $0 \text{ non} \in \bar{A} - \bar{B}_1$, i.e., $\bar{A} \cap \bar{B}_1 = \emptyset$ (see 2.6). With regard to the assumption there exists a neighbourhood $V \in \Sigma$ exists such that $(V + \bar{A}) \cap (V + \bar{B}_1) = \emptyset$. If we choose $U \in \Sigma$ such that $U \subseteq V$ and $x + U - x \subseteq V$ then $x = a - b - u$ for suitable $u \in U$, $a \in \bar{A}$ and $b \in \bar{B}$. From this $a = x + u + b \in x + U + \bar{B} \subseteq V + x + \bar{B}$ thus $\emptyset \neq \bar{A} \cap (V + x + \bar{B}) \subseteq (V + \bar{A}) \cap (V + \bar{B}_1)$, a contradiction.

2 $\Rightarrow$ 3: $\bar{A} \cap \bar{B} = \emptyset \Rightarrow 0 \text{ non} \in \bar{A} - \bar{B} = \overline{\bar{A} - \bar{B}}$.

3 $\Rightarrow$ 2: If $\bar{A} \cap \bar{B} = \emptyset$ and $(V + \bar{A}) \cap (V + \bar{B}) \neq \emptyset$ holds for every $V \in \Sigma$ then $v_1 + a = v_2 + b$ for suitable elements $a \in \bar{A}$, $b \in \bar{B}$, $v_1, v_2 \in V$, i.e., $a - b = -v_1 + v_2 \in -V + V$. It means that for every $U \in \Sigma$ and a suitable neighbourhood $V \in \Sigma$ such that $-V + V \subseteq U$ we have $a - b \in U$ and $U \cap (\bar{A} - \bar{B}) \neq \emptyset$, which contradicts $0 \text{ non} \in \overline{\bar{A} - \bar{B}}$.

2.10. Corollary. *If* (G, Σ) *is a topological group, then the following assertions are equivalent*:

1. *The topological space* $(G, \tau(\Sigma))$ *is uniformly normal.*
2. *The sum of two closed sets in G is a closed set in G.*
3. *The difference of two closed sets in G is a closed set in G.*

Proof follows from 2.8 and 2.9.

2.11. Corollary. *If* (G, Σ) *is a uniformly normal topological group and H is a closed normal subgroup in G, then the factor group* $(G/H, \Sigma_H)$ *is uniformly normal.*

Proof. If a topological factor group $(G/H, \Sigma_H)$ is not uniformly normal, where $\Sigma_H = \{(U + H)/H : U \in \Sigma\}$, then there exist sets $A, B \subseteq G/H$ such that $\bar{A} = A$, $\bar{B} = B$, $A \cap B = \emptyset$ and A, B are not uniformly separable sets in $\tau(\Sigma_H)$. It holds $0_H \in \overline{A - B}$ according to Theorem 2.9 and it means that $U_H \cap (A - B) \neq \emptyset$ for every $U_H \in \Sigma_H$. Further, there exists a neighbourhood $U \in \Sigma$ and closed sets A_0, B_0 in (G, Σ) such that $U_H = (U + H)/H$, $A = A_0/H$, $B = B_0/H$. It follows that for every $U \in \Sigma$ there exists an element $u \in U$ such that $u + H \subseteq A_0 - B_0$, i.e., there exist elements $a \in A_0$, $b \in B_0$ such that $u = a - b \in A_0 - B_0$. Finally $0 \in \overline{A_0 - B_0}$, $\bar{A}_0 = A_0$, $\bar{B}_0 = B_0$, $A_0 \cap B_0 = \emptyset$ which, according to Theorem 2.9., means that sets A_0, B_0 are not uniformly separable in (G, Σ).

2.12. Theorem. *Every compact topological group is a uniformly normal space.*

Proof. Let (G, Σ) be a compact topological group. Suppose that there exist closed sets P, Q in G such that $P \cap Q = \emptyset$ and $0 \in \overline{P - Q}$. It means that $U \cap \cap (P - Q) \neq \emptyset$ for every $U \in \Sigma$ and thus there exist elements $u \in U$, $p \in P$, $q \in Q$ such that $u = p - q$. From this $p = u + q$, i.e., $P \cap \overline{(U + Q)} \neq \emptyset$. We consider the system $\{P \cap \overline{(U + Q)}: U \in \Sigma\}$ and we prove that it is a collection of closed sets satisfying the finite intersection condition. Namely, an arbitrary finite system $\{P \cap \overline{(U_i + Q)}: U_i \in \Sigma,\ i = 1, 2, \ldots, k\}$ has the property $\emptyset \neq P \cap \overline{(V + Q)} \subseteq \subseteq \cap\{P \cap \overline{(U_i + Q)}: U_i \in \Sigma,\ i = 1, 2, \ldots, k\}$, where $V \in \Sigma$, $V \subseteq \cap\{U_i: i = = 1, 2, \ldots, k\}$. It follows $\cap\{P \cap \overline{(U + Q)}: U \in \Sigma\} = P \cap \cap\{\overline{U + Q}: U \in \Sigma\}$. Therefore $x \in P$ and $x \in \overline{U + Q}$ for every $U \in \Sigma$. If $V \in \Sigma$ such that $-V + V \subseteq U$ then $x \in \overline{V + Q}$, i.e., $(V + x) \cap (V + Q) \neq \emptyset$ which implies the existence of elements $x_1, v_2 \in V$, $q \in Q$ such that $v_1 + x = v_2 + q$. We have $x = -v_1 + v_2 + + q \in (-V + V) + Q \subseteq U + Q$ for every $U \in \Sigma$. Now, we choose arbitrary neighbourhoods $U \in \Sigma$ and $V \in \Sigma$ such that $-V \subseteq U$. Then elements $v \in V$, $q \in Q$ exist such that $x = v + q$, i.e., $q = -v + x \in (-V + x) \cap Q \subseteq (U + x) \cap Q$. It means $x \in Q$, which contradicts $P \cap Q = \emptyset$.

3. Some results on topological l-groups

Now we attempt to include into investigating uniform separability of closed sets in *tl*-groups also lattice operations and the lattice order.

3.1. Proposition. *Let* (G, Σ) *be a tl-group and* $P, Q \subseteq G$. *If* $\wedge \mid \bar{P} - \bar{Q} \mid \neq 0$, *or* $\wedge \mid \bar{P} - \bar{Q} \mid$ *does not exist then* $\bar{P}, \bar{Q}$ *are uniformly separable sets.*

Proof. According to Theorem 2.9 it is sufficient to prove $0 \text{ non} \in \overline{\bar{P} - \bar{Q}}$. We have $\bar{P} \cap \bar{Q} = \emptyset \Leftrightarrow 0 \text{ non} \in \mid \bar{P} - \bar{Q} \mid$. Now, if $\wedge \mid \bar{P} - \bar{Q} \mid \neq 0$ then $\wedge \mid \bar{P} - \bar{Q} \mid = = m > 0$. If $\wedge \mid \bar{P} - \bar{Q} \mid$ does not exist then $g \in G$ exists such that $\mid p - q \mid \geqq g$, for every $p \in \bar{P}$, $q \in \bar{Q}$, and $g > 0$ or $g \parallel 0$. In the case $g \parallel 0$ we consider the element $g_1 = g \vee 0$ and then $\mid p - q \mid \geqq g \vee 0 = g_1 > 0$ for every $p \in \bar{P}$, $q \in \bar{Q}$. In both cases there exists a positive (non zero) lower bound m of $\mid \bar{P} - \bar{Q} \mid$. If $0 \in \overline{\bar{P} - \bar{Q}}$ then by means of contradiction then $U \cap (\bar{P} - \bar{Q}) \neq \emptyset$ for every $U \in \Sigma$. We choose a neighbourhood $V \in \Sigma$ such that $V \subseteq U$, $V \vee -V \subseteq U$. Then $v_0 = p - q$ for suitable elements $v_0 \in V, p \in \bar{P}, q \in \bar{Q}$ and from this $\mid p - q \mid = (p - q) \vee (q - p) = = v_0 \vee -v_0 \in V \vee -V \subseteq U$. If we choose $U \in \Sigma$ such that $m > \mid u \mid$ or $m \parallel \mid u \mid$, for every $u \in U$ (see [4], 2.2) then we have a contradiction with $\mid v_0 \mid = \mid p - q \mid \geqq \geqq m$ and $v_0 \in U$.

3.2. Lemma. *Let* (G, Σ) *be a tl-group and* $P, Q \subseteq G$. *Then it holds*:

$$0 \in \overline{P - Q} \Rightarrow 0 \in \overline{\mid P - Q \mid} \Rightarrow \wedge \mid P - Q \mid = 0.$$

If $\tau(\Sigma)$ *is a locally convex topology then* $0 \in \overline{|P - Q|} \Rightarrow 0 \in \overline{P - Q}$ *holds.*

Proof. 1. If $0 \in \overline{P - Q}$ then for every $U \in \Sigma$ there exists $V \in \Sigma$ and elements $v \in V$, $p \in P$, $q \in Q$ such that $V \vee -V \subseteq U$, $V \subseteq U$ and $v = p - q$. It implies $-v = q - p$, i.e., $v \vee - v = (p - q) \vee (q - p) = |p - q| \in (V \vee -V) \cap |P - Q| \subseteq \subseteq U \cap |P - Q|$. Finally $0 \in \overline{|P - Q|}$.

2. Now, suppose $0 \in \overline{|P - Q|}$ and assume (by the way of contradiction) that $\wedge |P - Q| = 0$ is not true. As above (see the proof of Prop. 3.1), there exists a lower bound m of the set $|P - Q|$ with $m > 0$. With regard to [4], 2.2 there exists a neighbourhood $U \in \Sigma$ such that $|u| < m$ or $|u| \parallel m$, for every $u \in U$. The fact $0 \in \overline{|P - Q|}$ implies the existence of elements $p_1 \in P$, $q_1 \in Q$, $u_1 \in U$ such that $u_1 = |p_1 - q_1|$. But $|p_1 - q_1| = u_1 < m$ or $|p_1 - q_1| = u_1 \parallel m$, a contradiction.

3. Now, if $\tau(\Sigma)$ is a locally convex topology and $0 \in \overline{|P - Q|}$ then for every $U \in \Sigma$ there exists a convex neighbourhood $V \in \Sigma$ such that $\pm V \subseteq U$ and $V \cap \cap |P - Q| \neq \emptyset$. There exist elements $v \in V, p \in P, q \in Q$ such that $v = |p - q| \geqq \geqq p - q$ and $-v = -|p - q| = (q - p) \wedge (p - q) \leqq p - q$. Finally, $-v \leqq \leqq p - q \leqq v$, i.e., $p - q \in V \subseteq U$, $U \cap (P - Q) \neq \emptyset$ and $0 \in \overline{P - Q}$.

3.3. Corollary. *If* (G, Σ) *is a tl-group and* P, Q *are disjoint closed subsets in* G *which are not uniformly separable, then it holds*:

1. $0 \in \overline{P - Q}$, 2. $0 \in \overline{|P - Q|}$, 3. $\wedge |P - Q| = 0$.

Proof. Follows from Theorem 2.9 and Lemma 3.2.

Now, let us investigate linearly ordered *tl*-groups with the interval topology in connection with uniform separability. It is known (for example see [1]) that this topological space is normal.

3.4. Theorem. *Let* (G, Σ) *be a linearly ordered tl-group. Then sets* P, Q *in* G *are uniformly separable if and only if there exists an element* $m \in G$ *such that* $m > 0$ *and* $|p - q| \geqq m$, *for every* $p \in \bar{P}$, $q \in \bar{Q}$.

Proof. $\Rightarrow$: If there exists no element $m \in G$ such that $m > 0$ and $|p - q| \geqq m$ for every $p \in \bar{P}$, $q \in \bar{Q}$, then $\wedge \{|p - q| : p \in \bar{P}, q \in \bar{Q}\} = 0$. It means that for every $U \in \Sigma$ and every $m \in G$, $m > 0$ there exist elements $p \in \bar{P}$, $q \in \bar{Q}$ such that $|p - q| < < m$, i.e., $(\bar{P} - \bar{Q}) \cap (-m, m) \neq \emptyset$. Therefore there exists an element $m \in G$ such that $m > 0$ and $U \supseteq (-m, m)$. We have $(\bar{P} - \bar{Q}) \cap U \neq \emptyset$, i.e., $0 \in \overline{\bar{P} - \bar{Q}}$, a contradiction.

$\Leftarrow$: We have $|\bar{P} - \bar{Q}| \cap (-m, m) = \emptyset$. It follows that 0 non $\in \overline{\bar{P} - \bar{Q}}$ and thus P, Q are uniformly separable (see Theorem 2.9).

3.5. Definition. We say that an *l*-group G is *dense* if for every $g, h \in G$ such that $h > g$ there exists an element $k \in G$ such that $h > k > g$.

3.6. Lemma. *A linearly ordered tl-group with the non-discrete interval topology is dense.*

Proof. If elements $x, y \in G$ exist such that there exists no element $z \in G$ with the property $x > z > y$, then the open intervals (y, x) and $(0, m)$ are empty sets, where $m = x - y$. Further $(m, 2m) = \emptyset$ and thus $(0,2m) = \{m\}$. It follows that $\tau(\Sigma)$ is the discrete topology, a contradiction.

3.7. Example. Let R be a linearly ordered additive group of real numbers with the interval topology and $A = N$, $B = \cup\{\langle n - 1 + 1/n, n - 1/n\rangle : n \in N, n \geqq 2\}$, where $\langle a, b\rangle = \{g \in R: a \leqq g \leqq b\}$. Then A, B are disjoint closed subsets in R which are not uniformly separable (see Theorem 3.4).

Remark. It can be proved similarly that a linearly ordered additive group Q of rational numbers is not a uniformly normal space.

3.8. Theorem. *If (G, Σ) is a linearly ordered tl-group with the non-discrete topology and (G, Σ) is a uniformly normal space, then it holds:*

1. *G is totally non-archimedean, i.e., for every element $g \in G$, $g \neq 0$ there exists an element $h \in G$, $h \neq 0$ such that $| g | > n | h |$, for every $n \in N$.*
2. *If M is a closed subset of G and $\vee$ M($\wedge$ M) exists, then* M *has the greatest (the smallest) element.*
3. *$\tau(\Sigma) = \tau(\Sigma_1)$, where Σ_1 is the set of all convex subgroups P of G such that $P \neq \{0\}$.*
4. *$(G, \tau(\Sigma))$ is a totally disconnected topological space.*
5. *There exists no strongly decreasing (increasing) sequence in G which has in G an infimum (a supremum).*

Proof. 1. If there exists an element $g \in G$ such that $g > 0$ and $nh > g$, for every $h \in G$, $h > 0$ and a suitable number $n \in N$, then the convex subgroup $\langle g\rangle$ generated by g in G is archimedean. Namely, $\langle g\rangle = \{x \in G: 0 \leqq | x | \leqq ng \text{ for some } n \in N\}$. If $a \in \langle g\rangle^+$ is an archimedean element then $0 \leqq a \leqq mg$, for a suitable number $m \in N$. Further, for every $h \in \langle g\rangle^+$ there exists $n \in N$ such that $nh > g$, i.e., $mnh > > mg \geqq a \geqq 0$. It means that a is an archimedean element in $\langle g\rangle$ and $\langle g\rangle$ is a linearly ordered archimedean group. $\langle g\rangle$ is 1-isomorphic with a subgroup of R and because $\langle g\rangle$ is dense we have that $\langle g\rangle$ is 1-isomorphic with a dense subgroup of R which contains the additive group Q of rational numbers. Further, $\overline{\langle g\rangle}$ is 1-isomorphic with R. Finally, there exists a closed subgroup of G which is 1-isomorphic with R, and which with regard to Example 3.7, contains sets which are not uniformly separable, a contradiction. G has no archimedean element and thus G is totally non-archimedean.

2. If P is a closed subset in G and $\vee P$ non $\in P$ then P and $Q = \{g \in G: g \geqq \vee P\}$ are disjoint closed sets. With regard to Theorem 3.4 there exists an element $m \in G$ such that $m > 0$ and $\vee P - p > m$, for every $p \in P$, i.e., $\vee P > m + p$, for every $p \in P$, a contradiction. The second part for $\wedge P$ can be proved similarly.

3. If $V \in \Sigma_1$, then V is a convex subgroup of G, $V \neq \{0\}$, there exists an element $v \in V$, $v \neq 0$ such that $(-v, v) \subseteq V$. We have $\tau(\Sigma) \leqq \tau(\Sigma_1)$.

If $U \in \Sigma$ then there exists $u \in U$ such that $(-u, u) \subseteq U$. If for every $g \in G$, $0 \neq g \in (-u, u)$ there exists $n \in N$ such that $ng > u$ then we can prove similarly as in the first part that the convex subgroup $\langle u \rangle$ in G is archimedean, a contradiction. It means that there exists $g \in G$ such that $g > 0$, and a convex subgroup $\langle g \rangle$ has a property $\langle g \rangle \subseteq (-u, u)$, i.e., $\tau(\Sigma_1) \leqq \tau(\Sigma)$.

4. It follows immediately from 3.

5. Let $\{x_n\}$ be a descreasing sequence in G and let $\wedge \{x_n : n \in N\}$ exist. Then $\{y_n\}$, where $y_n = x_n - \wedge \{x_n : n \in N\}$ for $n \in N$, has the infimum 0. Let us denote $A = \cup \{\langle -y_{4n-3}, -y_{4n-2} \rangle : n \in N\}$, $B = \cup \{\langle -y_{4n-1}, -y_{4n} \rangle : n \in N\}$, and prove that A, B are disjoint closed subsets in G which are not uniformly separable.

a) If $a \in \bar{A} \setminus A$, then there exists $n \in N$ such that $-y_{4n-2} < a < -y_{4n+1-3}$ or $-y_1 > a$. Then $A \cap [a + (-g, g)] = \emptyset$ and consequently a non $\in \bar{A}$, a contradiction. Thus A (and similarly B) is closed.

b) $A \cap B = \emptyset$ is clear. We shall prove that A, B are not uniformly separable. For every $m \in G$, $m > 0$ there exists an element $s \in G$ such that $0 < 2s < m$. Furthermore, for some $n \in N$ we have $y_{4n} < s$, whence for every $a \in \langle -y_{4n-3}, -y_{4n-2} \rangle$ $(\subseteq A)$ and every $b \in \langle -y_{4n-1}, -y_{4n} \rangle$ $(\subseteq B)$, we have $0 < b - a \leqq -y_{4n} + y_{4n-3} < s + s = 2s < m$. By Theorem 3.4 A and B are not uniformly separable.

REFERENCES

[1] Fuchs, L.: *Partially ordered algebraic systems*, Pergamon Press 1963.

[2] Madell, R. L.: *Embeddings of topological latticeordered groups*, TAMS, 146 (1969), 447—455.

[3] Pontrjagin, L. S.: *Nepreryvnyje gruppy* (russian), Moskva 1973.

[4] Šmarda, B.: *Some types of topological l-groups*, Scripta Fac. Sci. Nat. UJEP Brunensis, 507 (1969), 341—352.

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