Milan Sekanina; Anna Sekaninová

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ARBITRARILY TRACEBLE EULERIAN GRAPH HAS THE HAMILTONIAN SQUARE

MILAN SEKANINA, ANNA SEKANINOVÁ, Brno

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We shall deal with finite undirected graphs $G = (V, E)$ without loops and multiple edges. $V_k(G)$ is the set of vertices of degree k in G. If G is connected, then d_G denotes the usual metric on V and n being a positive integer $G^n = (V, E^n)$ is the n-power (called the square for $n = 2$) of G, i.e. $(x, y) \in E^n$ iff $1 \leqq d(x, y) \leqq n$. It is known ([8], [4]) that the 3-power is always Hamiltonian. Several papers [5], [1], [2], [3] et al. were devoted to the question on hamiltonicity of the square of a connected graph. In [2] an important role of Eulerian graphs has been discovered.

In this note, we are interested in so called Eulerian graphs arbitrarily traceble from a vertex, which were introduced and studied by 0. Ore in [6] (we admit no loops and no multiple edges, but this is not essential, in fact). These are such Eulerian graphs, in which an Eulerian circle can be drawn starting from a suitable vertex and obeying one rule only: to draw every edge only once. In [6] (see also [7], chapter IV, 4.5.6) it is proved that every such arbitrarily traceble graph is given by the following construction: we take a forest $G_1 = (V_1, E_1)$ without one vertex components and v non $\in V_1$. We add v to G_1 and we connect v by an edge to each vertex from V_1 having an uneven degree in G_1. So we get the Eulerian graph G arbitrarily traceble from the vertex v. We shall prove that such a graph has the Hamiltonian square. This assertion will be a corollary of an auxiliary statement on forests, generalizing the notion of the square for trees, which can be of some interest by itself.

Proposition. Let $G = (V, E)$ be a forest. There exists such an ordering of V in a sequence $v_1, \ldots, v_p$ $(p = |V|)$, for which

1. $v_1, v_p \in V_1(G)$,
2. for all i $d_G(v_i, v_{i+1}) \leqq 2$ or $v_i, v_{i+1} \in V_1(G)$.

Proof (by induction on p). If $p = 1$, clear. Let $p > 1$. If G is no tree, we can use induction for connected components of G. So let G be a tree.

a) Let there be two vertices $v, w \in V$, which are neighbors and both of them of degrees at least 3. Delete the edge (v, w) from G. After that G decomposes in G_1

and G_2 for which $V_1(G) = V_1(G_1) \cup V_1(G_2)$. We can apply the induction assumption to G_1 and G_2.

b) If the assumption in a) is not valid in G (i.e. v, w do not exist) we take some maximal (further non-prolongable) way in G with vertices $x_1, \ldots, x_k$. So $d_G(x_i, x_{i+1}) = 1$ and $x_1, x_k \in V_1(G)$. From the sequence $x_1, \ldots, x_k$ select the vertices belonging to $V_1(G) \cup V_2(G) \cup V_3(G)$. We get the sequence $\mu : x_1, x_2, \ldots, x_{j_1-1}, x_{j_1+1}, \ldots, x_{j_2-1}, x_{j_2+1}, \ldots, x_{j_n-1}, x_{j_n+1}, \ldots, x_k$. Our assumption b) implies $j_1 + 1 < j_2 < \ldots$ Let x be a vertex from the sequence μ belonging to $V_3(G)$. Let G_x be the part of the branch in G starting in x contained in $[G - \{x_1, \ldots, x_k\}] \cup \{x\}$ up to the first vertex (if it exists) of degree at least 3 in G (this vertex not including). I.e. G_x is a way $x, x^1, x^2, \ldots, x^r$ in G which is „longest possible" such that $x^1, x^2, \ldots, x^{r-1}$ are of degree 2 in G, the degree of x^r is 2 or 1. Take now the sequence of vertices $x_1, \ldots, x_{j_1-1}$ and let $x', x'', \ldots, x^{(m)}$ be from $V_3(G)$ among them. Put H_1 to be the subgraph in G with the vertices $x_1, \ldots, x_{j_1-1}$ and those of $G_{x'}, G_{x''}, \ldots, G_{x^{(m)}}$.

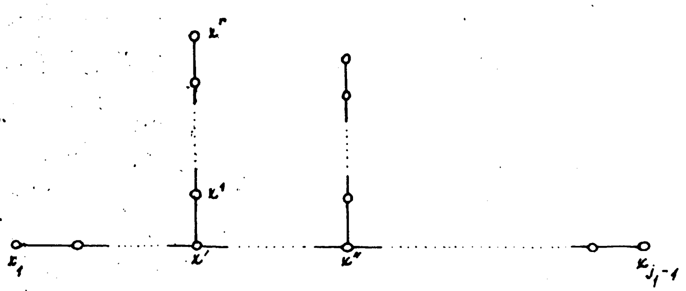

Fig. 4.

H_1 is a graph of the type (we put $x = x'$)

Similarly $H_2, \ldots, H_{n+1}$ are defined. By [5] (and it is easily seen here) there exists an ordering of the set of vertices in H_1 in such a sequence that the first vertex is x_1, the last vertex x_{j_1-1} and the neighbors in this sequence have in H_1 (i.e. in G) the distance at most 2. Let us denote this sequence by π_1. We get a sequence of such sequences $\pi_1, \pi_2, \ldots, \pi_{n+1}$. The end of π_{n+1} is the vertex x_k and the end of π_i has the distance 2 from the starting vertex of π_{i+1}. Therefore the sequence $\pi = (\pi_1, \pi_2, \ldots, \pi_{n+1})$ obtained by juxtaposition of the sequences $\pi_1, \pi_2, \ldots, \pi_{n+1}$ has the property 1. and 2. from the Proposition. We now delete the vertices participating in the sequence π and the edges incident to them from G. We get the forest W such that $V_1(W) \subset V_1(G)$. By the induction assumption we can order the set of vertices of W in a sequence ϱ fulfilling 1. and 2. from the Proposition. Then the sequence $\sigma = (\pi, \varrho)$ is a required sequence.

Corollary 1. Let $G = (V, E)$ be a graph with a vertex v having the following property:

$$H = G - \{v\} \text{ is a forest and } u \in V_1(H) \Rightarrow \text{the edge } (u, v)$$

is in E. Then G^2 is Hamiltonian.

Corollary 2. Let G be an Eulerian graph arbitrarily traceble from a vertex. Then G^2 is Hamiltonian.

Proof is an immediate consequence of Corollary 1. and Ore's construction of such graphs.

REFERENCES

[1] Fleischner, H.: *The square of every nonseparable graph is Hamiltonian,* Bull. Am. Math. Soc., 77 (1971), 1052—1054.

[2] Fleischner, H., Hobbs, A. M.: *A necessary condition, for the square of a graph to be Hamiltonian* J. of Comb. Theory, 19, 1975, 97—118.

[3] Hobbs, A. M.: *Hamiltonian square of cacti,* J. of Comb. Theory, Series B, 26 (1979), 50—65.

[4] Karaganis, J. J.: *On the cube a graph,* Can. Math. Bull., 11 (1968), 295—296.

[5] Neuman, F.: *On a certain ordering of the vertices of a tree,* Časopis pěst. mat., 89 (1964) 323—339.

[6] Ore, O.: *A problem regarding the tracing of graphs,* Elem. Math., 6 (1951), 49—53.

[7] Ore, O.: *Theory of graphs,* A. M. S., Providence, 1962.

[8] Sekanina, M.: *On an ordering of the set of vertices of a connected graph,* Publ. Fac. Sc. Univ. Brno, No 412, 1960, 137—142.

M. Sekanina, A. Sekaninová
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia