Archivum Mathematicum

Pavol Marušiak

Some results on the oscillatory and asymptotic behavior of solutions of nonlinear delay differential inequalities

Archivum Mathematicum, Vol. 18 (1982), No. 2, 77--87

Persistent URL: http://dml.cz/dmlcz/107126

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVIII: 77—88, 1982

SOME RESULTS ON THE OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR DELAY DIFFERENTIAL INEQUALITIES

PAVOL MARUŠIAK, Žilina

(Received December 5, 1980)

We consider the following nonlinear delay differential inequality

(r) $\{(r_{n-1}(t)\,(\dots(r_2(t)\,(r_1(t)\,y'(t))')'\dots)' + p(t)\,f(y(t), y[h(t)])\}\operatorname{sgn} y[h(t)] \leqq 0,$

where $n \geqq 2$.

The following conditions are always assumed:

(i) $r_i \in C[\langle 0, \infty), (0, \infty)]$, $i = 1, 2, \dots, n-1$,

(ii) $h \in C[\langle 0, \infty), R]$, $h(t) \leqq t$ for $t \geqq 0$ and $\lim_{t\to\infty} h(t) = \infty$,

(iii) $p \in C[\langle 0, \infty), \langle 0, \infty)]$ and p is not identically zero in any neighborhood $O(\infty)$,

(iv) $f \in C[R^2, R]$, $yf(x, y) > 0$ for $xy > 0$ and nondecreasing in $x(> 0)$, $y(> 0)$.

We introduce the notation:

(D_1) $D^0(y) = y$, $D^1(y; r_1) = r_1 y'$, $D^i(y; r_1, \dots, r_i) = r_i(D^{i-1}(y; r_1, \dots, r_{i-1}))'$, $i = 2, 3, \dots, n$, with $r_n = 1$.

Moreover, if $D^i(y; r_1, \dots, r_i)$ is defined as a continuous function on $\langle T, \infty)$, then the function y is said to be i-times continuously r-differentiable on $\langle T, \infty)$.

Then in view of (D_1) we can rewrite the inequality (r) as follows:

$$\{D^n(y; r_1, \dots, r_{n-1}, 1)\,(t) + p(t)\,f(y(t), y[h(t)])\}\operatorname{sgn} y[h(t)] \leqq 0;$$

(D_2) $\bar{r}_i(t) = \max\{r_i(s)\colon t/2 \leqq s \leqq t\}$, $i = 1, 2, \dots, n-1$, $\tilde{r}_j(t) = \max\{\bar{r}_j(s)\colon t/2^{n-j-1} \leqq s \leqq t\}$, where $j \in \{1, \dots, n-1\}$;

$$R_j^i(t) = \tilde{r}_j(t)\,\tilde{r}_{j-1}(t)\dots\tilde{r}_{i+1}(t), j = i+1, \dots, n-1,\ R_j^0(t) = R_j(t);$$

$$J_i(t, s, r_i, \dots, r_{n-1}) = \int_s^t \frac{1}{r_i(s_i/2^{n-i-1})} \int_s^{s_i} \dots \frac{1}{r_{n-2}(s_{n-2}/2)} \cdot$$

$$\int_s^{s_{n-2}} \frac{ds_{n-1}}{r_{n-1}(s_{n-1})}\, ds_{n-2} \dots ds_i, \qquad i = 1, 2, \dots, n-1,$$

$$I_0 = 1,$$

$$I_k(t, t_0, r_{i_k}, \dots, r_{i_1}) = \int_{t_0}^{t} \frac{1}{r_{i_k}(s)} I_{k-1}(s, t_0, r_{i_{k-1}}, \dots, r_{i_1}) \, ds,$$

$$i_k \in \{1, 2, \dots, n-1\}, k \in \{1, 2, \dots, n-1\},$$

(D_3) $\gamma(t) = \sup \{s \geqq 0; h(s) < t\}$ for $t \geqq 0$.

Denote by W the set of all solutions $y(t)$ of (r) which exist on a ray $\langle t_0, \infty) \subset \subset \langle 0, \infty)$ and satisfy

$$\sup \{| y(s) | : s \geqq t\} > 0$$

for every $t \geqq t_0$.

A solution $y(t) \in W$ is called oscillatory if it has arbitrarily large zeros. Otherwise the solution $y(t) \in W$ is called nonoscillatory.

Definition 1. We shall say that the inequality (r) has the property A, if every solution $y(t) \in W$ is oscillatory for n even, while for n odd is either oscillatory or $| D^i(y; r_1, \dots, r_i)(t) | \downarrow 0$ as $t \uparrow \infty$ $(i = 0, 1, \dots, n-1)$.

Definition 2. We shall say that the inequality (r) has the property A_0, if every solution $y(t) \in W$ is either oscillatory or $| D^i(y; r_1, \dots, r_i)(t) | \downarrow 0$ as $t \uparrow \infty$ $(i = 0, 1, \dots, n-2)$.

In this paper we shall prove sufficient conditions for the inequality (r) to have either the property A or A_0. The oscillatory properties of solutions of functional differential equations of n-th order, involving general differential operators of the form (D_1) are studied for example in [3, 4, 6–9].

To obtain our results, we shall need the following lemmas which are extensions of two lemmas due to Kiguradze [1], [2].

Lemma 1. Let $r_i: \langle T_0, \infty) \to (0, \infty)$, $i = 1, \dots, k$ be continuous functions and

(1) $$\int^{\infty} \frac{dt}{r_i(t)} = \infty, \qquad i = 1, 2, \dots, k-1.$$

Let $u(\neq 0)$ be k-times continuous r-differentiable function on $\langle T_0, \infty)$. If

(2) $$\delta u(t) D^k(u; r_1, \dots, r_k)(t) \leqq 0, (\delta = \pm 1) \qquad \text{for } t \geqq T_0,$$

and not identically zero in any neighborhood $O(\infty)$, then there exists an integer $l \in \{0, 1, \dots, k\}$, with $k + l$ odd (even) if $\delta = 1 (\delta = -1)$ and a $t_0 > T_0$ such that

(3) $u(t)D^i(u; \bar{r}_1, \dots, r_i)(t) > 0$ on $\langle t_0, \infty)$ for $i = 0, 1, \dots, l$,

(4) $(-1)^{l+i} u(t) D^i(u; r_1, \dots, r_i)(t) > 0$ on $\langle t_0, \infty)$ for $i = l+1, \dots, k-1$,

This Lemma generalizes the well-known lemma of Kiguradze [1] and can be proved similarly.

Lemma 2. Let $r_i: \langle T_0, \infty) \to (0, \infty)$, $i = 1, \dots, n-1$ be continuous functions and

(5) $$\int^{\infty} \frac{dt}{r_i(t)} = \infty, \qquad i = 1, 2, \ldots, n - 2.$$

Let $u(\neq 0)$ be an $n - 1$-times continuously r-differentiable function on the interval $\langle T_0, \infty)$. If for every $t \geqq T_0$,

(6) $$u(t)\, D^{n-1}(u; r_1, \ldots, r_{n-1})(t) > 0,$$

(7) $$u(t)\, D^n(u; r_1, \ldots, r_{n-1}, 1)(t) \leqq 0$$

and not identically zero on any neighborhood $O(\infty)$, then there exist $t_0 \geqq T_0$ and an integer $l \in \{0, 1, \ldots, n - 1\}$, $n + l$ odd, such that (3),

(4′) $$(-1)^{l+i}\, u(t)\, D^i(u; r_1, \ldots, r_i)(t) > 0 \quad \text{on} \quad \langle t_0, \infty) \quad \text{for } i = l + 1, \ldots, n - 1$$

hold, and

(8) $$|D^i(u; r_1, \ldots, r_i)(t/2^{n-i-1})| \geqq$$
$$\geqq a_i t^{n-i-1} \frac{|D^{n-1}(u; r_1, \ldots, r_{n-1})(t)|}{\bar{r}_{n-1}(t)\, \bar{r}_{n-2}(t/2) \ldots \bar{r}_{i+1}(t/2^{n-i-2})} \quad \text{for } t \geqq 2^{n-i-1} t_0,$$

where

$$a_i = \frac{2^{-(n-i-1)^3/2}}{(n - i - 1)!}, \qquad i = l, l + 1, \ldots, n - 1.$$

(9) $$|D^i(u; r_1, \ldots, r_i)(t)| \geqq \left(\frac{t}{2}\right)^{l-i} \frac{|D^l(u; r_1, \ldots, r_l)(t)}{(l - i)!\, \bar{r}_{i+1}(t) \ldots \bar{r}_l(t)}$$
$$\text{for } t \geqq 2t_0, \qquad i = 0, 1, \ldots, l - 1,$$

(10) $$|D^i(u; r_1, \ldots, r_i)(t)| \geqq A \frac{t^{n-i-1}\, |D^{n-1}(u; r_1, \ldots, r_{n-1})(t)|}{\bar{r}_{n-1}(t)\, \bar{r}_{n-2}(t/2) \ldots \bar{r}_l(t/2^{n-l-1}) \ldots \bar{r}_{i+1}(t/2^{n-l-1})}$$

$$\text{for } t \geqq 2^{n-1} t_0, \qquad \text{where} \qquad A = \frac{2^{-(n-1)(n^2+1)}}{[(n - 1)!]^2}, \qquad i = 0, 1, \ldots, l.$$

Proof. By Lemma 1 in view of (6), there exist $t_0 \geqq T_0$ and an integer l, $l \in \{0, 1, \ldots, n - 1\}$, with $l + n - 1$ even such that (3) and (4′) hold.

Without loos of generality, we assume that $u(t) > 0$ for every $t \geqq t_0$. Next by virtue of (4′) and (7) we obtain

$$-D^{n-2}(u; r_1, \ldots, r_{n-2})(t/2) \geqq \int_{t/2}^{t} \frac{D^{n-1}(u; r_1, \ldots, r_{n-2})(s)\, ds}{r_{n-1}(s)} \geqq$$
$$\geqq D^{n-1}(u; r_1, \ldots, r_{n-1})(t)\, J_{n-1}(t, t/2, r_{n-1}), \qquad t \geqq 2t_0,$$

$$D^{n-3}(u; r_1, \ldots, r_{n-3})(t/4) \geqq -\int_{t/2}^{t} \frac{D^{n-2}(u; r_1, \ldots, r_{n-2})(s/2)\, ds}{2r_{n-2}(s/2)} \geqq$$
$$\geqq \frac{1}{2} D^{n-1}(u; r_1, \ldots, r_{n-1})(t)\, J_{n-2}(t, t/2, r_{n-2}, r_{n-1}), \qquad t \geqq 4t_0,$$

...

$$(-1)^{n-i-1} D^i(u; r_1, \ldots, r_i)(t/2^{n-i-1}) \geqq$$
$$\geqq \frac{(-1)^{n-i}}{2^{n-i-2}} \int_{t/2}^{t} \frac{D^{i+1}(u; r_1, \ldots, r_{i+1})(s/2^{n-i-2})\,ds}{r_{i+1}(s/2^{n-i-2})} \geqq$$
$$\geqq \frac{1}{2^{(n-i-1)^2/2}} D^{n-1}(u; r_1, \ldots, r_{n-1})(t)\, J_{i+1}(t, t/2, r_{i+1}, \ldots, r_{n-1}),$$
$$t \geqq 2^{n-i-1} t_0,\ i \geqq l.$$

From the last inequalities we get

(11) $$(-1)^{n-i-1} D^i(u; r_1, \ldots, r_i)(t/2^{n-i-1}) \geqq \frac{D^{n-1}(u; r_1, \ldots, r_{n-1})(t)}{2^{(n-i-1)^2/2}\bar{r}_{i+1}(t/2^{n-i-2})} \times$$
$$\times \int_{t/2}^{t} \frac{t - s_{i+2}}{r_{i+2}(s_{i+2}/2^{n-i-3})} J_{i+3}(s_{i+2}, t/2, r_{i+3}, \ldots, r_{n-1})\, ds_{i+2} \geqq \ldots \geqq$$
$$\geqq \frac{D^{n-1}(u; r_1, \ldots, r_{n-1})(t)}{2^{(n-i-1)^2/2}\bar{r}_{i+1}(t/2^{n-i-2}) \ldots \bar{r}_{n-1}(t)} \int_{t/2}^{t} \frac{(t-s)^{n-i-2}}{(n-i-2)!}\, ds,$$
$$\text{for } t \geqq 2^{n-i-1} t_0,\ i = l, \ldots, n-1.$$

The inequality (8) follows from (11).

Next, in view of (3) and (4′) we have

$$D^{l-1}(u; r_1, \ldots, r_{l-1})(t) \geqq \int_{t_0}^{t} \frac{D^l(u; r_1, \ldots, r_l)(s)}{r_l(s)}\, ds \geqq$$
$$\geqq D^l(u; r_1, \ldots, r_l)(t)\, I_1(t, t_0, r_l),$$
$$D^{l-2}(u; r_1, \ldots, r_{l-2})(t) \geqq \int_{t_0}^{t} \frac{D^{l-1}(u; r_1, \ldots, r_{l-1})(s)}{r_{l-1}(s)}\, ds \geqq$$
$$\geqq D^l(u; r_1, \ldots, r_l)(t)\, I_2(t, t_0, r_{l-1}, r_l),$$
$$\ldots$$
$$D^i(u; r_1, \ldots, r_i)(t) \geqq \int_{t_0}^{t} \frac{D^{i+1}(u; r_1, \ldots, r_{i+1})(s)}{r_{i+1}(s)}\, ds \geqq$$
$$\geqq D^l(u; r_1, \ldots, r_l)(t)\, I_{l-i}(t, t_0, r_{i+1}, \ldots, r_l),$$
$$\text{for } i = 0, 1, \ldots, l-1,\ t \geqq 2t_0.$$

The last inequality implies (9).

If we put $t/2^{n-l-1}$ in place of t in (9) and use the monotonicity of the function $D^i(u; r_1, \ldots, r_i)(t)$, we obtain

(12) $$D^i(u; r_1, \ldots, r_i)(t) \geqq D^i(u; r_1, \ldots, r_i)(t/2^{n-l-1}) \geqq$$
$$\geqq \frac{t^{l-i}}{(2^{n-l})^{l-i}} \frac{D^l(u; r_1, \ldots, r_l)(t/2^{n-l-1})}{(l-i)!\, \bar{r}_{i+1}(t/2^{n-l-1}) \ldots \bar{r}_l(t/2^{n-l-1})}.$$

Combining (12) with (8) for $i = l$, we get (10).

Remark. If we use (D_2), the inequality (10) can be rewritten to the following form:

$$(10') \qquad |D^i(u; r_1, \dots, r_i)(t)| \geqq A \frac{|D^{n-1}(u; r_1, \dots, r_{n-1})(t)|}{R^i_{n-1}(t)} t^{n-i-1},$$

$$\text{for } t \geqq 2^{n-1} t_0,\ i = 0, 1, \dots, l.$$

Further, we assume that

$$(13) \qquad \int^{\infty} \frac{dt}{r_i(t)} = \infty, \qquad i = 1, 2, \dots, n-2$$

holds.

Lemma 3. Let (i) – (iv), (13) hold.

a) If

$$(14) \qquad \int^{\infty} \frac{dt}{r_{n-1}(t)} = \infty,$$

then conditions (6) and (7) are satisfied for every nonoscillatory solution $y(t) \in W$ of (r).

b) If for every $T \geqq t_0$

$$(15) \qquad \int_T^{\infty} \frac{\int_T^t p(s)\,ds}{r_{n-1}(t)} dt = \infty,$$

then conditions (6) and (7) hold for every nonoscillatory solution $y(t)$ of (r) with $\lim_{t\to\infty} y(t) \neq 0$.

Proof. Let $y(t)$ be a nonoscillatory solution of (r). Without loss of generality we suppose that $y(t) > 0$ for every $t \geqq t_0$, since the substitution $y = -u$ transforms (r) into an equation of the same form subject to similar assumptions.

Next by (ii) there exists a $t_1 \geqq t_0$ such that $y[h(t)] > 0$ for every $t \geqq t_1$. Thus from (r), in view of (iv) we have

$$(16) \qquad (D^{n-1}(y; r_1, \dots, r_{n-1})(t))' = -p(t) f(y(t), y[h(t)]) \leqq 0, \qquad t \geqq t_1.$$

Moreover, since $p(t)$ is not identically zero in any neighborhood $O(\infty)$, the same holds for $(D^{n-1}(y; r_1, \dots, r_{n-1})(t))'$ and consenquently either $D^{n-1}(y; r_1, \dots, r_{n-1})(t) > 0$, or $D^{n-1}(y; r_1, \dots, r_{n-1})(t) < 0$ for all large (t).

We shall prove that the last assumption cannot hold in both cases, provided that in case b) we have $\lim y(t) \neq 0$ as $t - \infty$.

a) We assume that for some $t_2 \geqq t_1$ we have

$$D^{n-1}(y; r_1, \dots, r_{n-1})(t_2) = K < 0.$$

The inequality (16) yields

$$D^{n-1}(y; r_1, \dots, r_{n-1})(t) \leqq D^{n-1}(y; r_1, \dots, r_{n-1})(t_2) = K < 0, \qquad t \geqq t_2,$$

and consenquently

$$(D^{n-2}(y; r_1, \ldots, r_{n-2})(t))' \leqq \frac{r_{n-1}(t_2) K}{r_{n-1}(t)} = \frac{K_1}{r_{n-1}(t)}.$$

Integrating the last inequality from t_2 to t ($t_2 \leqq t$), we obtain

$$D^{n-2}(y; r_1, \ldots, r_{n-2})(t) \leqq D^{n-2}(y; r_1, \ldots, r_{n-2})(t_2) + K_1 \int_{t_2}^{t} \frac{ds}{r_{n-1}(s)}.$$

Then, in view of (14), we have

$$\lim_{t \to \infty} D^{n-2}(y; r_1, \ldots, r_{n-2})(t) = -\infty,$$

which contradicts the positivity of y. This contradiction proves the case a).

To prove b) we remark that the assumption $\lim_{t \to \infty} y(t) > 0$ implies the existence of a constant $L > 0$ and $t_3 \geqq t_2$ such that $y(t) \geqq L$, $y[h(t)] \geqq L$ for every $t \geqq t_3$. Then by (iv) we have

$$f(y(t), y[h(t)]) \geqq f(L, L) = M > 0 \qquad \text{for } t \geqq t_3.$$

This, by (r), leads to the inequality

$$(D^{n-1}(y; r_1, \ldots, r_{n-1})(t))' \leqq Mp(t) \qquad \text{for } t \geqq t_3.$$

Integrating the last inequality from T ($T \geqq t_3$) to t, we get

$$D^{n-1}(y; r_1, \ldots, r_{n-1})(t) \leqq -M \int_T^t p(s)\, ds, \qquad t \geqq T,$$

and consenquently

$$(D^{n-2}(y; r_1, \ldots, r_{n-2})(t))' \leqq \frac{1}{r_{n-1}(t)} \int_T^t p(s)\, ds.$$

Integrating again from T to t ($\geqq T$), with regard to (15) we have

$$\lim_{t \to \infty} D^{n-2}(y; r_1, \ldots, r_{n-2})(t) = -\infty,$$

which contradicts the positivity of y.

The proof of Lemma 3* is complete.

Theorem 1. Suppose that (i), (iii), (iv), (13) are satisfied and, in addition, suppose that

(v) $h \in C^1[\langle 0, \infty), R]$, $h(t) \leqq t$, $h'(t) \geqq 0$ for $t \geqq 0$, $\lim_{t \to \infty} h(t) = \infty$,

(17) $$\int^{\infty} \frac{dt}{f(t, t)} < \infty.$$

If

(18) $$\int_T^{\infty} p(t) \int_T^{h(t)} \frac{s^{n-2}\, ds}{R_{n-1}(s)}\, dt = \infty,$$

for every $T : \gamma(T) \geqq t_0$, then

α) under the condition (14) the inequality (r) has the property A.

β) under the condition (15) the inequality (r) has the property A_0.

Proof. Let y be a nonoscillatory solution of (r) with $\lim_{t\to\infty} y(t) \neq 0$. Without loss of generality we assume that $y[h(t)] > 0$ for $t \geqq t_1 \geqq t_0$. From this, by (r) and (i), (iii) it follows that

$$(D^{n-1}(y; r_1, \ldots, r_{n-1})(t))' \leqq 0 \qquad \text{for } t \geqq t_1 .$$

This last function is not identically zero in any neighborhood $O(\infty)$. Now, under the one of the conditions (14), (15) we have by Lemma 3

$$D^{n-1}(y; r_1, \ldots, r_{n-1})(t) > 0 \qquad \text{for } t \geqq t_2 \geqq t_1 .$$

By Lemma 1 there exists a $t_3 \geqq t_2$ such that either

$$D^1(y; r_1)(t) > 0, \qquad \text{or} \qquad D^1(y; r_1)(t) < 0 \qquad \text{for } t \geqq t_3 .$$

Further we shall use the analogous method as in the proof of Theorem 1 in [10].

Case 1. Let $D^1(y; r_1)(t) > 0$ on $\langle t_3, \infty)$. Let z be the function defined by the formula

$$(19) \quad z(t) = -D^{n-1}(y; r_1, \ldots, r_{n-1})(t) \int_{t_3}^{t} \frac{[h(s)]^{n-2} h'(s)\, ds}{R_{n-1}[h(s)]\, f(y(s), y[h(s)])}, \qquad t \geqq t_3 .$$

We obviously have

$$(20) \qquad z(t) \leqq 0 \qquad \text{on } \langle t_3, \infty).$$

From (19), in view of (r) we get

$$z'(t) \geqq p(t) f(y(t), y[h(t)]) \int_{t_3}^{t} \frac{[h(s)]^{n-2} h'(s)}{R_{n-1}[h(s)]\, f(y(s), y[h(s)])}\, ds - \\ - \frac{D^{n-1}(y; r_1, \ldots, r_{n-1})(t)\, [h(t)]^{n-2} h'(t)}{R_{n-1}[h(t)]\, f(y(t), y[h(t)])} .$$

Since the function f, y are nondecreasing and $D^{n-1}(y; r_1, \ldots, r_{n-1})$ is nonincreasing, we have

$$z'(t) \geqq p(t) \int_{t_3}^{t} \frac{[h(s)]^{n-2} h'(s)}{R_{n-1}[h(s)]}\, ds - \\ - \frac{D^{n-1}(y; r_1, \ldots, r_{n-1})(h(t))\, [h(t)]^{n-2} h'(t)}{R_{n-1}[h(t)]\, f(y[h(t)], y[h(t)])} \qquad \text{for } t \geqq t_3 .$$

Thus applying (10′) for $i = 1$, $u = y$, $h(t)$ in place of t and using $r_1[h(t)] \leqq \leqq R_1[h(t)]$, we obtain

$$z'(t) \geqq p(t) \int_{t_3}^{t} \frac{[h(s)]^{n-2} h'(s)}{R_{n-1}[h(s)]}\, ds - \frac{1}{A} \frac{D^1(y; r_1)(h(t))\, h'(t)}{R_1[h(t)]\, f(y[h(t)], y[h(t)])} \geqq$$

$$\geqq p(t) \int_{h(t_3)}^{h(t)} \frac{x^{n-2}}{R_{n-1}(x)} \mathrm{d}x - \frac{1}{A} \frac{y'[h(t)]\, h'(t)}{f(y[h(t)], y[h(t)])} \qquad \text{for } t \geqq t_4 \geqq t_3 .$$

Integrating the last inequality from t_4 to t ($\geqq t_4$) and taking into account (17) and (18), we obtain $\lim_{t \to \infty} z(t) = \infty$, which contradicts (20).

Case 2. Let $D^1(y; r_1)(t) < 0$ on $\langle t_3, \infty)$. Let w be the function defined by the formula

$$(21) \qquad w(t) = -D^{n-1}(y; r_1, \ldots, r_{n-1})(t) \int_{t_3}^{t} \frac{[h(s)]^{n-2} h'(s)\, \mathrm{d}s}{R_{n-1}[h(s)]}, \qquad t \geqq t_3 .$$

We obviously have

$$(22) \qquad w(t) \leqq 0 \qquad \text{on } \langle t_3, \infty).$$

From (21), in view of (r) and the monotonicity of $D^{n-1}(y; r_1, \ldots, r_{n-1})$, we have

$$(23) \qquad w'(t) = p(t)\, f(y(t), y[h(t)]) \int_{t_3}^{t} \frac{[h(s)]^{n-2} h'(s)}{R_{n-1}[h(s)]} \mathrm{d}s - \frac{D^{n-1}(y; r_1, \ldots, y_{n-1})(h(t))\, [h(t)]^{n-2} h'(t)}{R_{n-1}[h(t)]} .$$

Moreover, since $y[h(t)] > 0$ for $t \geqq t_3$, there exists a positive constant C such that

$$f(y(t), y[h(t)]) \geqq C \qquad \text{for } t \geqq t_3 .$$

From (23), by applying (8) with $l = 0$, $i = 1$, $h(t)$ in place of t and using $r_1[h(t)/2^{n-2}] \leqq R_1[h(t)]$, we get

$$w'(t) \geqq C p(t) \int_{h(t_3)}^{h(t)} \frac{x^{n-2}\, \mathrm{d}x}{R_{n-1}(x)} + \frac{y'[h(t)/2^{n-2}]\, h'(t)}{a_1} \qquad \text{for } t \geqq t_4 .$$

Integrating the last inequality from t_4 to t ($\geqq t_4$), by (18) and the fact that the solution y is bounded, we obtain $\lim_{t \to \infty} w(t) = \infty$, which contradicts (22).

We have just proved that for every nonoscillatory solution y of (r) $\lim_{t \to \infty} y(t) = 0$ and $y(t)\, y'(t) < 0$ for all large t. If condition (14) is satisfied, then by Lemma 3 and Lemma 2 n must be odd.

Moreover, as it is easy to see, $\lim_{t \to \infty} y(t) = 0$ implies that

in α) $| D^i(y; r_1, \ldots, r_i)(t) | \downarrow 0$ as $t \uparrow \infty$ for $i = 0, 1, \ldots, n - 1$, and

in β) $| D^i(y; r_1, \ldots, r_i)(t) | \downarrow 0$ as $t \uparrow \infty$ for $i = 0, 1, \ldots, n - 2$.

Remark. If the functions r_i ($i = 1, 2, \ldots, n - 1$) are nondecreasing, then the condition (18) can be replaced by

$$\int_T^{\infty} p(t) \int_T^{h(t)} \frac{s^{n-2}}{r_1(s) \ldots r_{n-1}(s)} \mathrm{d}s = \infty .$$

Theorem 2. Let the conditions (i)–(iv) (13) be satisfied. Let

$$|f(g(t)\,u, g(t)\,v)| \geqq G(g(t))\,|f(u, v)| \quad \text{for } u\,.\,v > 0, \tag{24}$$

where $g \in C[(0, \infty), (0, K)]$, $G \in C[(0, K), (0, \infty)]$ and

$$\int_0^K \frac{ds}{G(s)} < \infty.$$

If

$$\int^{\infty} p(t) \left| f\left(\pm \frac{t^{n-1}}{R_{n-1}(t)}, \pm \frac{[h(t)]^{n-1}}{R_{n-1}[h(t)]} \right) \right| dt = \infty, \tag{25}$$

then α) under the condition (14) the inequality (r) has the property A.

β) under the condition (15) the inequality (r) has the property A_0.

Proof. Let $y(t)$ be a nonoscillatory solution of (r) with $\lim_{t\to\infty} y(t) \neq 0$. We assume, without loss of generality that

$$\lim_{t\to\infty} y(t) > 0. \tag{26}$$

Then, in view of (ii), we can choose t_1 such that $y[h(t)] > 0$ for every $t \geqq t_1$. Similarly as in the proof of Theorem 1 we have $D^{n-1}(y; r_1, \ldots, r_{n-1})(t) > 0$ for $t \geqq t_2 \geqq t_1$. Then by using Lemma 2 for $u = y$, from (8) with $i = l = 0$ and from (10′) with $i = 0 < l$, we get

$$y(t/2^{n-1}) \geqq a_0 t^{n-1} \frac{D^{n-1}(y; r_1, \ldots, r_{n-1})(t)}{R_{n-1}(t)}, \qquad t \geqq 2^{n-1} t_0 = t_3 \tag{27}$$

and

$$y(t) \geqq A t^{n-1} \frac{D^{n-1}(y; r_1, \ldots, r_{n-1})(t)}{R_{n-1}(t)}, \qquad t \geqq t_3. \tag{28}$$

Integrating (r) from t ($\geqq t_1$) to ∞, we obtain

$$\infty > D^{n-1}(y; r_1, \ldots, r_{n-1})(t) \geqq \Phi(t), \qquad t \geqq t_1, \tag{29}$$

where $\Phi(t) = \int_t^{\infty} p(s)\, f(y(s), y[h(s)])\, ds$.

Then, with regard to the monotonicity of $D^{n-1}(y; r_1, \ldots, r_{n-1})$, we have

$$D^{n-1}(y; r_1, \ldots, r_{n-1})(h(t)) \geqq \Phi(t) \qquad \text{for every } t \geqq \bar{t}_1 = \gamma(t_1). \tag{29′}$$

I. Let $l \in \{1, 2, \ldots, n-1\}$. Then (28) and

$$y[h(t)] \geqq A[h(t)]^{n-1} \frac{D^{n-1}(y; r_1, \ldots, r_{n-1})(h(t))}{R_{n-1}[h(t)]}, \qquad t \geqq t_4 = \gamma(t_3) \tag{28′}$$

hold.

From (28) or (28′), in view of (29) or (29′), we get,

$$(30) \qquad y(t) \geqq A \frac{t^{n-1}\Phi(t)}{R_{n-1}(t)} \qquad \text{for } t \geqq T = \max\{\bar{t}_1, t_4\},$$

or

$$(30') \qquad y[h(t)] \geqq A \frac{[h(t)]^{n-1}\Phi(t)}{R_{n-1}[h(t)]} \qquad \text{for } t \geqq T, \text{ respectively.}$$

In view of the monotonicity of the function f, (30), (30′) and (24) we have

$$(31) \qquad f(y(t), y[h(t)]) \geqq f\left(A \frac{t^{n-1}\Phi(t)}{R_{n-1}(t)}, A \frac{[h(t)]^{n-1}\Phi(t)}{R_{n-1}[h(t)]}\right) \geqq$$

$$\geqq G(A\Phi(t)) f\left(\frac{t^{n-1}}{R_{n-1}(t)}, \frac{[h(t)]^{n-1}}{R_{n-1}[h(t)]}\right).$$

Multiplying both sides of (31) by $p(t)/G(A\Phi(t))$ and then integrating from T to t, we obtain

$$\int_T^t p(s) f\left(\frac{s^{n-1}}{R_{n-1}(s)}, \frac{[h(s)]^{n-1}}{R_{n-1}[h(s)]}\right) ds \leqq \int_T^t \frac{p(s)}{G(A\Phi(s))} f(y(s), y[h(s)]) \, ds =$$

$$= -\int_T^t \frac{\Phi'(s)}{G(A\Phi(s))} ds = \frac{1}{A} \int_{G(A\Phi(t))}^{G(A\Phi(T))} \frac{du}{G(u)} \leqq \frac{1}{A} \int_0^K \frac{du}{G(u)} < \infty,$$

which contradicts (25).

II. Let $l = 0$. Then (27) implies with regard to (26)

$$y(t) \gtreqqless \frac{y(t)}{y(t/2^{n-1})} y(t/2^{n-1}) \geqq M_0 t^{n-1} \frac{D^{n-1}(y; r_1, \ldots, r_{n-1})(t)}{R_{n-1}(t)},$$

where $M_0 = \inf\limits_{t \geqq t_0} \left\{\frac{y(t)}{y(t/2^{n-1})}\right\} a_0$.

Further, using the analogous method as in the case I, we get a contradiction with (25).

If (14) holds and $l \geqq 1$, then in view of (3), (26) is fulfilled. In all other cases (i.e. either (14) holds and $l = 0$, or (15) holds and $l \geqq 0$) we have to assume that (26) is satisfied. But, as shown above, this leads to a contradiction with (25). Then $\lim\limits_{t \to \infty} y(t) = 0$ for every nonoscillatory solution $y(t) \in W$. Hence it follows that in

α) $| D^i(y; r_1, \ldots, r_{n-1})(t) | \downarrow 0$ as $t \uparrow \infty$, $i = 0, 1, \ldots, n-1$, and in

β) $| D^i(y; r_1, \ldots, r_{n-1})(t) | \downarrow 0$ as $t \uparrow \infty$, $i = 0, 1, \ldots, n-2$.

The proof of Theorem 2 is complete.

Theorem 2 is extension of Theorem 1 in [5].

REFERENCES

[1] Кигурадзе И. Т.: *О колеблемости рещекий уравнения* $\frac{d^m u}{dt^m} + a(t) u^n \operatorname{sgn} u = 0$. Матем. сборник 65 (1964), 172—187.

[2] Кигурадзе И. Т.: *К вопросу колеблемости решений нелинейных дифференциальных уравнений*, Диф. уравнения 1, Ñ 8 (1965) 995—1006.

[3] Kitamura Y., Kusano T.: *Nonlinear oscillation of higher — order functional differential equations with deviating arguments.* J. Math. Anal. Appl. 77 (1980) 100—119.

[4] Kusano T., Naito M.: *Comparison theorems for functional differential equations with deviating arguments.* J. Math. Soc. Japan (To appear).

[5] Marušiak P.: *On oscillation of solutions of differential inequalities with retarded argument.* Čas. pěst. mat. 104 (1979), 281—294.

[6] Marušiak P., Werbowski J.: *Oscillation and asymptotic behavior of solution of differential equations generated by delays.* Fasc. Math. (To appear).

[7] Philos Ch. G.: *Some results on the oscillatory and asymptotic behavior of the solutions of differential equations with deviating arguments.* Arch. Math. 1. Scripta Fac. Sci. Nat. UJEP Brunenssis, XV. (1979) 19—36.

[8] Philos Ch. G.: *Oscillatory and asymptotic behavior of the bounded solutions of differential equations with deviating arguments.* Hiroshima Math. J. 8 (1978), 31—48.

[9] Philos Ch. G., Staikos V. A.: *Basic comparison results for the oscillatory and asymptotic behavior of differential equations with deviating arguments.* Technical Deport. Univ. of Ioannina No 12 (1978), 1—29.

[10] Sficas Y. G.: *On the oscillatory and asymptotic behavior of damped differential equations with retarded argument.* Technical Report Univ. of Ioannina, No 41 (1975) 1—25.

P. Marušiak
Marxa—Engelsa 25
010 88 Žilina
Czechoslovakia