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ON A "LIAPUNOV LIKE" FUNCTION FOR AN EQUATION $\dot{z} = f(t, z)$ WITH A COMPLEX—VALUED FUNCTION f

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1. Introduction

In earlier papers [2], [3], [4], [5] and [6], the author studied the asymptotic behaviour of the solutions of an equation

$$\dot{z} = G(t, z)\,[h(z) + g(t, z)], \tag{1}$$

where G is a real-valued function and h, g are complex-valued functions of a real variable t and a complex variable z. The function h is supposed to be holomorphic in a simply connected region Ω containing zero and the right hand side of (1) is assumed to be "close" to $h(z)$. It is shown that the asymptotic properties of the solutions of (1) are similar to those of

$$\dot{z} = h(z). \tag{2}$$

The technique of the proofs of the majority of these results is based on the Liapunov function method. On the assumption $h'(0) \neq 0$ and $h(z) = 0 \Leftrightarrow z = 0$, a suitable Liapunov function $W(z)$ is defined in the following manner:

$$W(z) = |z|\;|\exp[\int_0^z r(z^*)\,dz^*]|,$$

where

$$r(z) = \begin{cases} \dfrac{zh'(0) - h(z)}{zh(z)} & \text{whenever } z \in \Omega,\ z \neq 0, \\ -\dfrac{h''(0)}{2h'(0)} & \text{whenever } z = 0. \end{cases}$$

The purpose of the present paper is to give the definition and to describe some basic properties of a "Liapunov-like" function $W(z)$ which is convenient for the investigation of the asymptotic behaviour of the solutions of (1) in the case

$h(z) = 0 \Leftrightarrow z = 0$, $h^{(n)}(0) \neq 0$, $h^{(j)}(0) = 0$ for $j = 1, \ldots, n-1$, where $n \geqq 2$ is an integer. Notice that $W(z)$ does not satisfy all the conditions usually required for Liapunov functions. Namely, $W(z)$ is defined only for $z \in \Omega - \{0\}$ and there is no continuous extension of $W(z)$ to Ω.

Some results dealing with the asymptotic behaviour of the solutions of (1) will be published in next author's papers.

Throughout the paper we use the following notation:

C	– Set of all complex numbers
$\bar{b}$	– Conjugate of a complex number b
$\mathrm{Re}\, b$	– Real part of a complex number b
$\mathrm{Im}\, b$	– Imaginary part of a complex number b
$\mathrm{Arg}\, z$	– Principial value of the multivalued function $\arg z$
$\mathrm{Bd}\, \Gamma$	– Boundary of a set $\Gamma \subset C$
$\mathrm{Cl}\, \Gamma$	– Closure of a set $\Gamma \subset C$
$\mathrm{Int}\, \Gamma$	– Interior of a Jordan curve $z = z(t)$, $t \in [\alpha, \beta]$ whose points z form a set $\Gamma \subset C$; Γ will be called the geometric image of the Jordan curve $z = z(t)$, $t \in [\alpha, \beta]$
Ω	– Simply connected region in C such that $0 \in \Omega$
$\mathscr{H}(\Omega)$	– Class of all complex-valued functions defined and holomorphic in the region Ω
$\mathrm{Ind}_f(0)$	– Index of the point $z = 0$ with respect to the equation $\dot{z} = f(z)$.

2. Definitions and properties of $W(z)$ and $\hat{K}(\lambda)$

Let $n \geqq 2$ be an integer. Suppose $h(z) \in \mathscr{H}(\Omega)$, $h(z) = 0 \Leftrightarrow z = 0$, $h^{(j)}(0) = 0$ for $j = 1, \ldots, n-1$ and $h^{(n)}(0) \neq 0$. Define

$$a_1 = 1,$$

$$a_l = -\frac{n!}{h^{(n)}(0)} \sum_{j=1}^{l-1} a_j \frac{h^{(n+l-j)}(0)}{(n+l-j)!}, \qquad l = 2, 3, \ldots, n+1,$$

$$\Theta = 1 + (|a_n|^2 - 1) \operatorname{sgn} |a_n|,$$

$$k = \overline{[h^{(n)}(0)} + (\bar{a}_n - \overline{h^{(n)}(0)}) \operatorname{sgn} |a_n|]/(\Theta n!),$$

$$r(z) = \begin{cases} \dfrac{z^n h^{(n)}(0) - n!\, h(z) \sum_{j=1}^{n} a_j z^{j-1}}{n!\, h(z)\, z^n} & \text{for } z \in \Omega,\ z \neq 0, \\ -a_{n+1} & \text{for } z = 0. \end{cases}$$

Lemma 1. *The function $r(z)$ is holomorphic in Ω.*

Proof. $r(z)$ is holomorphic in $\Omega - \{0\}$. Using repeatedly L'Hospital's rule, we obtain

$$\lim_{z\to 0} \frac{z^n h^{(n)}(0) - n!\, h(z) \sum_{j=1}^{n} a_j z^{j-1}}{n!\, h(z)\, z^n} =$$

$$= \lim_{z\to 0} \frac{n! \sum_{j=1}^{n} a_j \frac{(2n)!}{(2n-j+1)!} h^{(2n-j+1)}(0)}{(2n)!\, h^{(n)}(0)} =$$

$$= \frac{-h^{(n)}(0)\,(2n)!\, a_{n+1}}{(2n)!\, h^{(n)}(0)} = -a_{n+1}.$$

Thus the singularity, at the point $z = 0$, is removable and $r(z)$ is holomorphic in Ω.

Put

$$w(z) = z^{|a_n|^2 \Theta^{-1}} \exp\left[-k^* \sum_{j=1}^{n-1} \frac{a_j}{(n-j)\, z^{n-j}}\right] \exp\left[k^* \int_0^z r(z^*)\, dz^*\right],$$

where $k^* = n!k$, and define

$$W(z) = |w(z)| \qquad \text{for } z \in \Omega,\ z \neq 0.$$

Lemma 2. *$W(z)$ is a first integral for an equation*

(3) $$\dot{z} = ik\overline{h^{(n)}(0)}\, h(z)$$

on the set $\Omega - \{0\}$. Moreover,

$$\left[\frac{\partial W(z)}{\partial \operatorname{Re} z}\right]^2 + \left[\frac{\partial W(z)}{\partial \operatorname{Im} z}\right]^2 \neq 0$$

for $z \in \Omega - \{0\}$.

Proof. For $z \in \Omega - \{0\}$ we obtain

$$\left[\frac{\partial W(z)}{\partial \operatorname{Re} z}\right]^2 + \left[\frac{\partial W(z)}{\partial \operatorname{Im} z}\right]^2 = |w'(z)|^2 =$$

$$= W^2(z)\, \Big| |a_n|^2 \Theta^{-1} z^{-1} + k^*\Big[\sum_{j=1}^{n-1} a_j z^{j-n-1} + r(z)\Big]\Big|^2 =$$

$$= W^2(z)\, |k|^2\, |h^{(n)}(0)|^2\, |h(z)|^{-2}.$$

Hence

$$\left[\frac{\partial W(z)}{\partial \operatorname{Re} z}\right]^2 + \left[\frac{\partial W(z)}{\partial \operatorname{Im} z}\right]^2 \neq 0 \qquad \text{for } z \in \Omega - \{0\}.$$

Further, if $z(t)$ is any differentiable function, then

$$\frac{d}{dt} W^2(z) = \frac{d}{dt}\left[w(z)\, \overline{w(z)}\right] = 2 \operatorname{Re}\left[w'(z)\, \overline{w(z)}\, \dot{z}\right] =$$

$$= 2W^2(z)\,\mathrm{Re}\,\{k^*[\sum_{j=1}^{n} a_j z^{j-n-1} + r(z)]\,\dot{z}\} = 2W^2(z)\,\mathrm{Re}\,\{kh^{(n)}(0)\,h^{-1}(z)\,\dot{z}\}$$

for all t for which $z = z(t) \in \Omega - \{0\}$. Therefore, if $z(t)$ is any solution of (3), then

$$\dot{W}(z(t)) = W(z(t))\,\mathrm{Re}\,\{kh^{(n)}(0)\,h^{-1}(z)\,\dot{z}(t)\} = \\ = W(z(t))\,|\,k\,|^2\,|\,h^{(n)}(0)\,|^2\,\mathrm{Re}\,i = 0$$

for all t such that $z(t) \neq 0$. The proof is complete.

Lemma 3. 1° *φ_μ is a characteristic direction for* (3) *if and only if* $\varphi_\mu = (n-1)^{-1}[\mu\pi - \mathrm{Arg}\,(i\bar{k})]$, *where μ is an integer.*

2° *There are positive numbers ϑ, δ, η ($\eta < 2$, $\vartheta < 2\arcsin(\eta/2)$) such that if μ is any integer and if a solution $z(t)$ of* (3) *satisfies*

$$z(t_1) \in \Omega_\mu = \left\{z \in \Omega\colon 0 < |\,z\,| < \delta, \left|\frac{z}{|\,z\,|} - e^{i\varphi_\mu}\right| < \eta\right\},$$

then

(i) *$z(t) \in \Omega_\mu$ for $t \leqq t_1$ or $t \geqq t_1$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}|\,z(t)\,| > 0 \qquad or \qquad \frac{\mathrm{d}}{\mathrm{d}t}|\,z(t)\,| < 0,$$

respectively;

(ii) *for the continuous determination $\varphi(t)$ of* Arg $z(t)$ *there hold the inequalities*

$$\left(\mathrm{sgn}\,\frac{\mathrm{d}\,|\,z(t)\,|}{\mathrm{d}t}\right)\dot{\varphi}(t) > 0 \qquad whenever \qquad \varphi_\mu + \vartheta < \varphi(t) < \varphi_\mu + 2\arcsin\frac{\eta}{2},$$

$$\left(\mathrm{sgn}\,\frac{\mathrm{d}\,|\,z(t)\,|}{\mathrm{d}t}\right)\dot{\varphi}(t) < 0 \qquad whenever \qquad \varphi_\mu - 2\arcsin\frac{\eta}{2} < \varphi(t) < \varphi_\mu - \vartheta.$$

Proof. Denote $f(z) = i\overline{kh^{(n)}(0)}\,h(z)$, $\varrho(t) = |\,z(t)\,|$. Then $z(t) = \varrho(t)\,e^{i\varphi(t)}$. It follows from the equation (3) that the functions $\varrho(t)$, $\varphi(t)$ are solutions of

$$\dot{\varrho}e^{i\varphi} + i\varrho e^{i\varphi}\dot{\varphi} = f(\varrho e^{i\varphi}).$$

Consider the corresponding system of two real equations

$$\dot{\varrho} = \mathrm{Re}\,[e^{-i\varphi}f(\varrho e^{i\varphi})], \\ \varrho\dot{\varphi} = \mathrm{Im}\,[e^{-i\varphi}f(\varrho e^{i\varphi})]. \tag{4}$$

Taking into account that $f(0) = \ldots = f^{(n-1)}(0) = 0$, $f^{(n)}(0) \neq 0$, we can write the system (4) in the form

$$\dot{\varrho} = \frac{\varrho^n}{n!}\,\mathrm{Re}\,[f^{(n)}(0)\,e^{i(n-1)\varphi}] + o(\varrho^n), \\ \dot{\varphi} = \frac{\varrho^{n-1}}{n!}\,\mathrm{Im}\,[f^{(n)}(0)\,e^{i(n-1)\varphi}] + o(\varrho^{n-1}). \tag{5}$$

Furthermore, we have

$$(5') \qquad \dot{\varrho} = \frac{\varrho^n}{n!}[(-1)^\mu |f^{(n)}(0)| + o(1)] + o(\varrho^n),$$

$$\dot{\varphi} = \frac{\varrho^{n-1}}{n!}[(n-1)(-1)^\mu |f^{(n)}(0)| (\varphi - \varphi_\mu) + o(|\varphi - \varphi_\mu|)] + o(\varrho^{n-1}).$$

Since $\operatorname{Im}[f^{(n)}(0)\, e^{i(n-1)\varphi}] = 0$ if and only if $\varphi = \varphi_\mu = (n-1)^{-1}[\mu\pi - \operatorname{Arg} f^{(n)}(0)]$, both the parts of Lemma 3 can be easily derived from the relations (5), (5').

Now, we are prepared to prove the following

Lemma 4. *Let Γ be any simply connected region such that $\Gamma \subset \Omega$, $0 \in \Gamma$. For $M > 0$ put*

$$(6) \qquad \Gamma_M = \{z \in \Gamma: \inf_{z^* \in \operatorname{Bd} \Gamma} |z - z^*| < M^{-1}\} \cup \{z \in \Gamma: |z| > M\}.$$

Denote

$$\lambda_+^\Gamma = \liminf_{M \to \infty} \inf_{z \in \Gamma_M} W(z).$$

If $0 < \lambda < \lambda_+^\Gamma$, then the set $\{z \in \Gamma: W(z) = \lambda\}$ is the union of a certain nonempty system $\mathscr{L}^+$ of geometric images of curves with the following properties:

1° if $\mathscr{K}^* \in \mathscr{L}^+$, then $\mathscr{K} = \mathscr{K}^* \cup \{0\}$ is the geometric image of a Jordan curve and

$$(7) \qquad \operatorname{Int} \mathscr{K} \subset \{z \in \Gamma: W(z) < \lambda\};$$

2° if $\mathscr{K}^* \in \mathscr{L}^+$, $\mathscr{K} = \mathscr{K}^* \cup \{0\}$ and $0 < \lambda_1 < \lambda$, then the set $\{z \in \operatorname{Int} \mathscr{K}: W(z) = \lambda_1\} \cup \{0\}$ is the geometric image of a Jordan curve;

3° if $\mathscr{K}^* \in \mathscr{L}^+$, $\lambda < \lambda_2 < \lambda_+^\Gamma$, then there is a Jordan curve with the geometric image $\mathscr{K}_1$ such that $\mathscr{K}^* \subset \operatorname{Int} \mathscr{K}_1$ and $W(z) = \lambda_2$ for $z \in \mathscr{K}_1 - \{0\}$.

Proof. Because of Lemma 2 the function $W(z)$ is a first integral for (3) on $\Omega - \{0\}$. We shall show that there is no closed trajectory of (3) lying in Γ. If this is not true, there exists a trajectory of (3) which is a Jordan curve lying in Γ. Its interior must contain the point $z = 0$ with the index equal to 1. However, using Theorem 1 of [9], we have $\operatorname{Ind}_f(0) = n > 1$, a contradiction. Hence there is no closed trajectory of (3) lying in Γ.

The function $w(z)$ is holomorphic in $\Gamma - \{0\}$. Since $a_1 \neq 0$, the function $w(z)$ has an essential singularity at $z = 0$. Choose λ, $0 < \lambda < \lambda_+^\Gamma$. In view of Picard's theorem, there is a $z_1 \in \Gamma - \{0\}$ such that $W(z_1) = \lambda$.

Let z_1 be any point with the mentioned property. There is a unique trajectory of (3) passing through z_1. This trajectory corresponds with a solution $z(t)$ of the initial value problem (3), $z(0) = z_1$. Clearly, $W(z(t)) = \lambda$ for all t for which $z(t)$ is defined. There exists an $M > 0$ such that the considered trajectory is contained in the compact set $\Gamma - \Gamma_M$. Suppose that the set of ω-limit points or the set of α-limit points of the solution $z(t)$ does not contain the point $z = 0$. Then, owing to the Poincaré – Bendixson theorem, the set of ω-limit points or the set of α-limit points of the solution $z(t)$ is the set of points z on a periodic solution $z = z_0(t)$ of (3). The trajectory corresponding to this solution is a closed curve lying in Γ, and we get a contradiction. Thus the set of ω-limit points and the set of α-limit points of the solution $z(t)$ must contain the point $z = 0$.

We claim that

(8) $$\lim_{t\to\infty} z(t) = \lim_{t\to-\infty} z(t) = 0.$$

If it is not the case, then the set of ω-limit points or that of α-limit points of the solution $z(t)$ of (3) consists of the point $z = 0$ and of the points of a certain nonempty system of trajectories $\{z : z = z_0(t),\ t \in (-\infty, \infty)\}$ such that the corresponding solutions $z_0(t)$ satisfy

$$\lim_{t\to\infty} z_0(t) = \lim_{t\to-\infty} z_0(t) = 0.$$

([1, Theorem VII.4.2]). From the continuity it follows that $W(z_0(t)) = \lambda$ for $t \in (-\infty, \infty)$, which, in view of Lemma 2, contradicts the implicit function theorem. This proves (8).

In the following, $\mathcal{K}_{z(t)}$ and $\varphi_{z(t)}(t)$ will denote the trajectory corresponding to $z(t)$ and the continuous determination of Arg $z(t)$, respectively. It is clear that $\mathcal{K}_{z(t)} \cup$ $\cup \{0\}$ is the geometric image of a Jordan curve. By virtue of [1, Theorem VIII.2.1] and Lemma 3 we have

$$\lim_{t\to\infty} \varphi_{z(t)}(t) = \varphi_{\mu_1}, \qquad \lim_{t\to-\infty} \varphi_{z(t)}(t) = \varphi_{\mu_2},$$

where $\varphi_{\mu_1}, \varphi_{\mu_2}$ are characteristic directions for (3) such that $\varphi_{\mu_1} \neq \varphi_{\mu_2} \pmod{2\pi}$.

We shall prove that $\varphi_{\mu_1}, \varphi_{\mu_2}$ are consecutive characteristic directions, i.e. that $|\varphi_{\mu_1} - \varphi_{\mu_2}| = \pi(n-1)^{-1}$. Suppose for the sake of argument that this assertion is false. Then there are solutions $z_1(t)$, $z_2(t)$ with the property $z_1(t) \in \Gamma$, $z_2(t) \in \Gamma$ for $t \in (-\infty, \infty)$, $z_j(t) \to 0$ as $t \to \pm\infty$ $(j = 1, 2)$,

$$\lim_{t\to\infty} \varphi_{z_1(t)}(t) = \varphi_{\mu_3}, \qquad \lim_{t\to-\infty} \varphi_{z_1(t)}(t) = \varphi_{\mu_4},$$
$$\lim_{t\to\infty} \varphi_{z_2(t)}(t) = \varphi_{\mu_5}, \qquad \lim_{t\to-\infty} \varphi_{z_2(t)}(t) = \varphi_{\mu_6},$$
$$\mathcal{K}_{z_2}(t) \subset \operatorname{Int}[\mathcal{K}_{z_1(t)} \cup \{0\}] \quad \text{and} \quad |\varphi_{\mu_5} - \varphi_{\mu_6}| = \pi(n-1)^{-1},$$

where $\varphi_{\mu_3}, \varphi_{\mu_4}$ are not consecutive characteristic directions. Let $\mathcal{F}$ be the set of all solutions $u(t)$ of (3) such that $u(t) \to 0$ as $t \to \pm\infty$,

$$\lim_{t\to\infty} \varphi_{u(t)}(t) = \varphi_{\mu_3} \pmod{2\pi}, \qquad \lim_{t\to-\infty} \varphi_{u(t)}(t) = \varphi_{\mu_4} \pmod{2\pi}$$

and

$$\mathcal{K}_{z_2(t)} \subset \operatorname{Int}[\mathcal{K}_{u(t)} \cup \{0\}].$$

For each $u(t) \in \mathcal{F}$ there is a $z^* \in \mathcal{K}_{u(t)}$ for which $|z^*| = \max\{|u(t)| : t \in (-\infty, \infty)\}$.

Denote by $\mathcal{G}$ the set of all such points z^*. Put $\nu = \inf\{|z^*| : z^* \in \mathcal{G}\}$. Obviously, $\nu > 0$ and there exists a convergent sequence $\{z_j^*\}$, $z_j^* \in \mathcal{G}$ $(i = 1, 2, \ldots)$ such that

$$\lim_{j\to\infty} z_j^* = z_0, \qquad \text{where} \qquad |z_0| = \nu.$$

Because of Lemma 3 and the continuous dependence on initial values, every solution $u(t)$ of (3) for which $u(0)$ is close enough to z_0, satisfies $u(t) \to 0$ as $t \to \pm\infty$,

$$\lim_{t\to\infty} \varphi_{u(t)}(t) = \varphi_{\mu_3}, \qquad \lim_{t\to-\infty} \varphi_{u(t)}(t) = \varphi_{\mu_4},$$

which contradicts the definition of ν.

We claim that

$$(9) \qquad W(z) < \lambda \qquad \text{for } z \in \operatorname{Int}\left[\hat{K}_{z(t)} \cup \{0\}\right].$$

If this is not true, there exists a $z_0 \in \operatorname{Int}\left[\hat{K}_{z(t)} \cup \{0\}\right]$ such that $\lambda \leqq W(z_0) = \lambda^* < \lambda_+^\Gamma$. The solution $z_0(t)$ of an initial value problem (3), $z(0) = z_0$ satisfies $z_0(t) \to 0$ as $t \to \pm\infty$,

$$\lim_{t\to\infty} \varphi_{z_0(t)}(t) = \varphi_{\mu_1}, \qquad \lim_{t\to-\infty} \varphi_{z_0(t)}(t) = \varphi_{\mu_2}.$$

Let δ, η be as in Lemma 3. There are unambiguously determined points $z_1, z_2 \in \Omega_{\mu_1}$ and $z_3, z_4 \in \Omega_{\mu_2}$ such that $z_1, z_3 \in \hat{K}_{z(t)} \cap \{z : |z| = \delta/2\}$, $z_2, z_4 \in \hat{K}_{z_0(t)} \cap \cap \{z : |z| = \delta/2\}$. Let K^* denote the set consisting of the points of the part of $\hat{K}_{z(t)}$ lying between the points z_1, z_3, of the points of the part of $\hat{K}_{z_0(t)}$ lying between the points z_2, z_4 and of the points of two disjoint arcs $\widehat{z_1z_2}$, $\widehat{z_3z_4}$ of the circle $|z| = \delta/2$. Clearly, K^* is the geometric image of a Jordan curve.

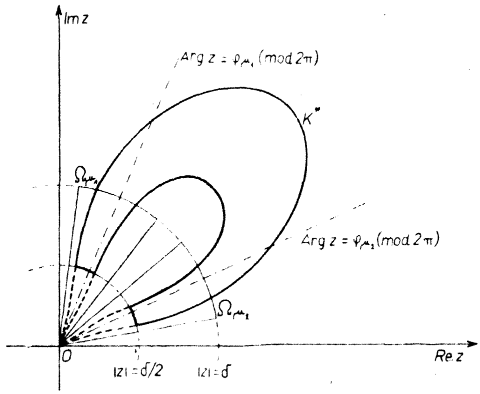

Fig. 1.

In view of Lemma 3 and the maximum modulus theorem, all the points $z^* \in \operatorname{Cl}\operatorname{Int} K^*$ with the property $W(z^*) = \max\{W(z) : z \in \operatorname{Cl}\operatorname{Int} K^*\}$ or $W(z^*) = \min\{W(z) : z \in \operatorname{Cl}\operatorname{Int} K^*\}$ must lie on $\hat{K}_{z(t)}$ or $\hat{K}_{z_0(t)}$. Since $W(z)$ is not constant, we have $\lambda < W(z) < \lambda^*$ for $z \in \operatorname{Int} K^*$. Let $\mathscr{F}$ be the set of all solutions $u(t)$ of (3) such that $u(t) \in \Gamma$ for $t \in (-\infty, \infty)$, $u(t) \to 0$ as $t \to \pm\infty$,

$$\lim_{t\to\infty} \varphi_{u(t)}(t) = \varphi_{\mu_1}, \qquad \lim_{t\to-\infty} \varphi_{u(t)}(t) = \varphi_{\mu_2}$$

and

$$\mathscr{K}_{z(t)} \subset \operatorname{Int}\left[\mathscr{K}_{u(t)} \cup \{0\}\right].$$

By virtue of Lemma 3 and the continuous dependence on initial values we infer that $\mathscr{F} \neq \emptyset$. If $u(t) \in \mathscr{F}$, then $W(u(t)) < \lambda$ for $t \in (-\infty, \infty)$ and there exists an $M > 0$ such that $\mathscr{K}_{u(t)} \subset \Gamma - \Gamma_M$ for any $u(t) \in \mathscr{F}$. Moreover, there is a $z^* \in \mathscr{K}_{u(t)}$ for which $|z^*| = \max\{|u(t)| : t \in (-\infty, \infty)\}$. Denote by $\mathscr{G}$ the set of all such points z^*. Put $v = \sup\{|z^*| : z^* \in \mathscr{G}\}$. Obviously, $v > 0$ and there exists a convergent sequence $\{z_j^*\}$, $z_j^* \in \mathscr{G}$ $(j = 1, 2, \ldots)$ such that

$$\lim_{j\to\infty} z_j^* = z_0^*, \qquad \text{where} \qquad |z_0^*| = v.$$

Further, $0 < W(z_0^*) = \lim_{j\to\infty} W(z_j^*) \leqq \lambda < \lambda_+^\Gamma$. Because of Lemma 3 and the continuous dependence on initial values, every solution $u(t)$ of (3) for which $u(0)$ is close enough to z_0^* satisfies $u(t) \to 0$ as $t \to \pm\infty$,

$$\lim_{t\to\infty} \varphi_{u(t)}(t) = \varphi_{\mu_1}, \qquad \lim_{t\to-\infty} \varphi_{u(t)}(t) = \varphi_{\mu_2},$$

which contradicts the definition of v. Therefore $W(z) < \lambda$ for $z \in \operatorname{Int}\left[\mathscr{K}_{z(t)} \cup \{0\}\right]$.

Now, we want to prove that to any λ_2, $\lambda < \lambda_2 < \lambda_+^\Gamma$, there is a solution $z^*(t)$ of (3) such that $W(z^*(t)) = \lambda_2$ for $t \in (-\infty, \infty)$ and $\mathscr{K}_{z(t)} \subset \operatorname{Int}\left[\mathscr{K}_{z^*(t)} \cup \{0\}\right]$. Suppose not. Denoting by $\mathscr{F}$ the system of all solutions $u(t)$ of (3) such that $W(u(t)) < \lambda_+^\Gamma$ for $t \in (-\infty, \infty)$ and $\mathscr{K}_{z(t)} \subset \operatorname{Int}\left[\mathscr{K}_{u(t)} \cup \{0\}\right]$, we observe that $\mathscr{F} \neq \emptyset$ and there is an $M > 0$ such that $\mathscr{K}_{u(t)} \subset \Gamma - \Gamma_M$ for any $u(t) \in \mathscr{F}$. Proceeding analogously as before and using Lemma 3 and the continuous dependence on initial values, we obtain a contradiction which proves the existence of the solution $z^*(t)$ with the properties $W(z^*(t)) = \lambda_2$ for $t \in (-\infty, \infty)$ and $\mathscr{K}_{z(t)} \subset$ $\subset \operatorname{Int}\left[\mathscr{K}_{z^*(t)} \cup \{0\}\right]$.

Finally, we shall prove that to any λ_1, $0 < \lambda_1 < \lambda$ there is a solution $z^*(t)$ of (3) such that $\mathscr{K}_{z^*(t)} \subset \operatorname{Int}\left[\mathscr{K}_{z(t)} \cup \{0\}\right]$ and $W(z^*(t)) = \lambda_1$ for $t \in (-\infty, \infty)$. It is sufficient to show that there exists a $z^* \in \operatorname{Int}\left[\mathscr{K}_{z(t)} \cup \{0\}\right]$ with the property $W(z^*) \leqq \lambda_1$. Putting $\varphi^* = (\varphi_{\mu_1} + \varphi_{\mu_2})/2$, we obtain

$$\lim_{s\to0+} W(se^{i\varphi^*}) = \lim_{s\to0+} |w(se^{i\varphi^*})| = \lim_{s\to0+} \left| \exp\left[-k^* \frac{a_1}{(n-1)s^{n-1}e^{i(n-1)\varphi^*}}\right] \right| =$$

$$= \lim_{s\to0+} \left| \exp\left[-k^* \frac{1}{(n-1)s^{n-1}} e^{i[\operatorname{Arg}(i\bar{k}) - (\mu_1+\mu_2)\pi/2]}\right] \right| =$$

$$= \lim_{s\to0+} \left| \exp\left[\varepsilon |k^*| \frac{1}{(n-1)s^{n-1}}\right] \right|,$$

where $\varepsilon = -1$ or $\varepsilon = +1$. In view of (9), the second case is impossible, whence

$$\lim_{s\to 0+} W(se^{i\varphi^*}) = 0.$$

Thus the existence of $z^* \in \operatorname{Int}[\hat{K}_{z(t)} \cup \{0\}]$ with the property $|W(z^*)| \leqq \lambda_1$ is proved. The proof is complete.

Quite analogously we can prove the following

Lemma 5. *Let Γ be any simply connected region such that $\Gamma \subset \Omega$, $0 \in \Gamma$. For $M > 0$ put*

$$\Gamma_M = \{z \in \Gamma : \inf_{z^* \in \operatorname{Bd}\Gamma} |z - z^*| < M^{-1}\} \cup \{z \in \Gamma : |z| > M\}.$$

Denote

$$\lambda_-^{\Gamma} = \lim_{M\to\infty} \sup_{z\in\Gamma_M} W(z).$$

If $\lambda_-^{\Gamma} > \lambda < \infty$, than the set $\{z \in \Gamma : W(z) = \lambda\}$ is the union of a certain nonempty system $\mathscr{L}^-$ of geometric images of curves with the following properties:

1° *if $\hat{K}^* \in \mathscr{L}^-$, then $\hat{K} = \hat{K}^* \cup \{0\}$ is the geometric image of a Jordan curve and*

$$\operatorname{Int}\hat{K} \subset \{z \in \Gamma : W(z) > \lambda\};$$

2° *if $\hat{K}^* \in \mathscr{L}^-$, $\hat{K} = \hat{K}^* \cup \{0\}$ and $\lambda < \lambda_1 < \infty$, then the set $\{z \in \operatorname{Int}\hat{K} : W(z) = \lambda_1\} \cup \{0\}$ is the geometric image of a Jordan curve;*

3° *if $\hat{K}^* \in \mathscr{L}^-$, $\lambda_-^{\Gamma} < \lambda_2 < \lambda$, then there is a Jordan curve with the geometric image $\hat{K}_1$ such that $\hat{K}^* \subset \operatorname{Int}\hat{K}_1$ and $W(z) = \lambda_2$ for $z \in \hat{K}_1 - \{0\}$.*

Let Ξ be the system of all simply connected regions $\Gamma \subset \Omega$ such that $0 \in \Gamma$. For any $\Gamma \in \Xi$ put

$$\lambda_+^{\Gamma} = \lim_{M\to\infty} \inf_{z\in\Gamma_M} W(z), \qquad \lambda_-^{\Gamma} = \lim_{M\to\infty} \sup_{z\in\Gamma_M} W(z),$$

where Γ_M is defined by (6). Denote

$$\lambda_+ = \sup_{\Gamma\in\Xi} \lambda_+^{\Gamma}, \qquad \lambda_- = \inf_{\Gamma\in\Xi} \lambda_-^{\Gamma}.$$

Obviously, $0 < \lambda_+ \leqq \infty$, $0 \leqq \lambda_- < \infty$. Moreover, in view of the implicit function theorem, Lemma 2, Lemma 4 and Lemma 5, the inequality $\lambda_+ \leqq \lambda_-$ must hold. For $0 < \lambda < \lambda_+$ and $\lambda_- < \lambda < \infty$, respectively, we define $\mathscr{K}^+(\lambda) = \{z \in \Gamma : W(z) = \lambda\}$, where Γ is any element from Ξ such that $\lambda_+^{\Gamma} > \lambda$ and $\mathscr{K}^-(\lambda) = \{z \in \Gamma : W(z) = \lambda\}$, where Γ is any element from Ξ such that $\lambda_-^{\Gamma} < \lambda$. It follows from Lemma 4 and Lemma 5 that $\mathscr{K}^+(\lambda)$, $\mathscr{K}^-(\lambda)$ are well-defined. Indeed, if e.g. $\mathscr{K}^+(\lambda)$ is not well-defined, then there exist $\Gamma_1, \Gamma_2 \in \Xi$ satisfying $\lambda_+^{\Gamma_1} > \lambda$, $\lambda_+^{\Gamma_2} > \lambda$ and $\mathscr{K}_1^+ = \mathscr{K}_{\Gamma_1}^+(\lambda) \neq \mathscr{K}_2^+ = \mathscr{K}_{\Gamma_2}^+(\lambda)$. Suppose for definiteness that there is a $z^* \in \mathscr{K}_1^+$ so that $z^* \notin \mathscr{K}_2^+$. Owing to Lemma 4 we conclude that there exists a set $\hat{K}$ which is the geometric image of a Jordan curve such that $z^* \in \mathcal{K} \subset \mathcal{K}^+ \cup \{0\}$. Let $\mathcal{O}$ be a neighbourhood of the origin with the property $\mathcal{O} \subset \Gamma_1 \cap \Gamma_2$. Clearly, $W(z) < \lambda$ for $z \in \mathcal{O} \cap \operatorname{Int} \mathcal{K}$. If $z_0^* \in \mathcal{O} \cap \operatorname{Int} \mathcal{K}$ and $W(z_0^*) = \lambda_1$, then, in view of Lemma 4, there is a $\mathcal{K}_1 \subset \Gamma_2$ which is the geometric image of a Jordan curve such that $z_0^* \in \mathcal{K}_1$ and $W(z) = \lambda_1$ for $0 \neq z \in \mathcal{K}_1$. Using Lemma 4, we observe that there is a Jordan curve such that, for its geometric image $\mathcal{K}_2$, conditions $\mathcal{K}_2 \subset \mathcal{K}_2^+ \cup \{0\}$ and $\mathcal{K}_1 - \{0\} \subset \operatorname{Int} \mathcal{K}_2$ are fulfilled. Considering $\mathcal{K}_1 - \{0\} \subset \operatorname{Int} \mathcal{K}$, we have $\mathcal{K}_2 - \{0\} \subset \mathcal{K}$ or $\mathcal{K} - \{0\} \subset \mathcal{K}_2$, which is a contradiction, because of $\operatorname{Int} \mathcal{K} \subset \{z \in \Omega : W(z) < \lambda\}$ and $\operatorname{Int} \mathcal{K}_2 \subset \{z \in \Omega : W(z) < \lambda\}$.

Let $\mathcal{T}^+$ and $\mathcal{T}^-$ be the system of all geometric images of Jordan curves which are contained in $\mathcal{K}^+(\lambda) \cup \{0\}$, $0 < \lambda < \lambda_+$, and $\mathcal{K}^-(\lambda) \cup \{0\}$, $\lambda_- < \lambda < \infty$, respectively. Consider the relation φ defined on $\mathcal{T}^+$ and $\mathcal{T}^-$ in the following way:

$$\mathcal{K}_1 \varphi \mathcal{K}_2 \Leftrightarrow [\mathcal{K}_1 - \{0\} \subset \operatorname{Int} \mathcal{K}_2 \quad \text{or} \quad \mathcal{K}_2 - \{0\} \subset \operatorname{Int} \mathcal{K}_1 \quad \text{or} \quad \mathcal{K}_1 = \mathcal{K}_2].$$

It can be easily verified by means of Lemma 4 and Lemma 5 that φ is an equivalence relation. For decompositions $\mathcal{T}^+/\varphi$ and $\mathcal{T}^-/\varphi$ we obtain the following two statements:

Theorem 1. *If* $\mathcal{S} \in \mathcal{T}^+/\varphi$, *then* $\mathcal{S} = \{\mathcal{K}(\lambda) : 0 < \lambda < \lambda_+\}$, *where*
1° $\mathcal{K}(\lambda)$ *is the geometric image of a Jordan curve for any* λ, $0 < \lambda < \lambda_+$;
2° $\mathcal{K}(\lambda) \subset \mathcal{K}^+(\lambda) \cup \{0\}$;
3° $\mathcal{K}(\lambda_1) - \{0\} \subset \operatorname{Int} \mathcal{K}(\lambda_2)$ *for* $0 < \lambda_1 < \lambda_2 < \lambda_+$.

Theorem 2. *If* $\mathcal{S} \in \mathcal{T}^-/\varphi$, *then* $\mathcal{S} = \{\mathcal{K}(\lambda) : \lambda_- < \lambda < \infty\}$, *where*
1° $\mathcal{K}(\lambda)$ *is the geometric image of a Jordan curve for any* λ, $\lambda_- < \lambda < \infty$;
2° $\mathcal{K}(\lambda) \subset \mathcal{K}^-(\lambda) \cup \{0\}$;
3° $\mathcal{K}(\lambda_2) - \{0\} \subset \operatorname{Int} \mathcal{K}(\lambda_1)$ *for* $\lambda_- < \lambda_1 < \lambda_2 < \infty$.

Remark. It can be easily seen that the trajectories of (2) cut the curves $\mathcal{K}(\lambda)$ with the constant angle ψ such that

$$\cos \psi = \frac{|\operatorname{Re} [ik\overline{h^{(n)}(0)}]|}{|k| \, |h^{(n)}(0)|}, \qquad \sin \psi = \frac{|\operatorname{Im} [ik\overline{h^{(n)}(0)}]|}{|k| \, |h^{(n)}(0)|}.$$

3. Examples

In this section we shall illustrate the results of Section 2 by the following two examples.

Example 1. Let $\Omega = \{z \in C : \alpha < \operatorname{Re} [bz] < \beta\}$, where $b \in C$, $b \neq 0$ and $-\infty \leqq \alpha < 0 < \beta \leqq \infty$. Put $h(z) = bz^2$. Then $h'(z) = 2bz$, $h''(z) \doteq 2b$, $h'''(z) = 0$. Further we obtain $a_1 = 1$, $a_2 = a_3 = 0$, $\Theta = 1$, $k = \bar{b}$, $r(z) = 0$, $w(z) =$

$= \exp\left[-2\bar{b}z^{-1}\right]$, $W(z) = \exp\{\mathrm{Re}\left[-2\bar{b}z^{-1}\right]\}$. Moreover, $0 < \lambda_+ = \exp\left[-2\,|b|^2\beta^{-1}\right] \leqq 1 \leqq \exp\left[-2\,|b|^2\alpha^{-1}\right] = \lambda_- < \infty$. The sets $\hat{K}(\lambda) \cup \{0\}$, where $0 < \lambda < \lambda_+$ or $\lambda_- < \lambda < \infty$, are circles with centres $\left[-\mathrm{Re}\, b \ln^{-1}\lambda,\ \mathrm{Im}\, b \ln^{-1}\lambda\right]$ and radii $|\ln \lambda|^{-1}\,|b|$.

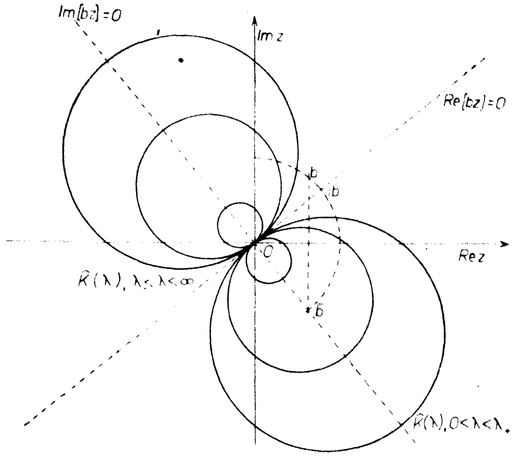

Fig. 2.

Example 2. Let $\Omega = \boldsymbol{C}$, $h(z) = b(z - a)\,z^2$, $a \in \boldsymbol{C}$, $b \in \boldsymbol{C}$, $a \neq 0 \neq b$. Then $h'(z) = b(3z - 2a)\,z$, $h''(z) = 2b(3z - a)$, $h'''(z) = 6b$, $h^{(4)}(z) = 0$. Furthermore we have $a_1 = 1$, $a_2 = a^{-1}$, $a_3 = a^{-2}$, $\Theta = |a|^{-2}$, $k = a/2$, $r(z) = \left[a(a - z)\right]^{-1}$, $w(z) = az(a - z)^{-1}\exp\left[-az^{-1}\right]$, $W(z) = |a|\,|z|\,|z - a|^{-1}\exp\{\mathrm{Re}\left[-az^{-1}\right]\}$, $\lambda_+ = \lambda_- = |a|$. The sets $\hat{K}(\lambda)$, where $0 < \lambda < \lambda_+$ or $\lambda_- < \lambda < \infty$, are sketched in Fig. 3.

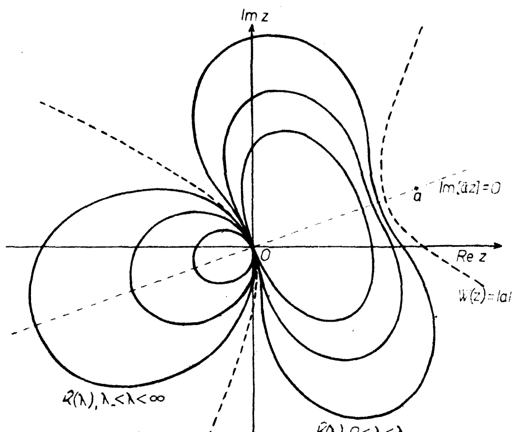

Fig. 3.

REFERENCES

[1] Hartman, P.: "*Ordinary Differential Equations*", Wiley, New York/London/Sydney, 1964.

[2] Kalas, J.: *Asymptotic behaviour of the solutions of the equation* $dz/dt = f(t, z)$ *with a complex-valued function f*, Proceedings of the Colloquium on Qualitative Theory of Differential Equations, August 1979, Szeged-Hungary, Seria Colloquia Mathematica Societatis János Bolyai & North-Holland Publishing Company, pp. 431—462.

[3] Kalas, J.: *On the asymptotic behaviour of the equation* $dz/dt = f(t, z)$ *with a complex-valued function f*, Arch. Math. (Brno) 17 (1981), 11—22.

[4] Kalas, J.: *On certain asymptotic properties of the solutions of the equation* $\dot{z} = f(t, z)$ *with a complex-valued function f*, Czech. Math. Journal, to appear.

[5] Kalas, J.: *Asymptotic properties of the solutions of the equation* $\dot{z} = f(t, z)$ *with a complex-valued function f*, Arch. Math. (Brno) 17 (1981), 113—124.

[6] Kalas, J.: *Asymptotic behaviour of equations* $\dot{z} = q(t, z) - p(t) z^2$ *and* $\ddot{x} = x\varphi(t, \dot{x}x^{-1})$, Arch. Math. (Brno) 17 (1981), 191—206.

[7] Ráb, M.: *The Riccati differential equation with complex-valued coefficients*, Czech. Math. Journal 20 (1970), 491—503.

[8] Ráb, M.: *Geometrical approach to the study of the Riccati differential equation with complex-valued coefficients*, J. Diff. Equations 25 (1977), 108—114.

[9] Sverdlove, R.: *Vector fields defined by complex functions*, J. Differential Equations 34 (1979), 427—439.

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