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Martin E. Muldoon

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ON THE ZEROS OF A FUNCTION RELATED TO BESSEL FUNCTIONS*

M. E. MULDOON

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1. INTRODUCTION

The function $\mu J_\nu(t) + tJ'_\nu(t)$, for general μ and ν, arises in connection with the so-called Dini series [28, Chapter 18] and in the solution of Laplace's equation for a sphere with mixed boundary conditions [5, p. 304]. For special values of μ and ν, it arises in applications to heat conduction [4] and elasticity [3, 23]. In these investigations, a knowledge of the zeros of this function is of importance and some of these zeros have been tabulated [1, 3, 4, 23]. K. D. Graham [7] has recently encountered the function $\mu J_\nu(t) + tJ'_\nu(t)$ in control theory and discussed the spacing of its positive zeros. R. Spigler [24] has considered these zeros also and the zero-free regions of a more general function have been investigated by A. D. Rawlins [22].

In the present paper we examine the spacing of the positive zeros of the function $C(\mu, \nu, t) = \mu C_\nu(t) + tC'_\nu(t)$ where μ and ν are real and where $C_\nu(t)$ denotes a real linear combination of the Bessel functions $J_\nu(t)$ and $Y_\nu(t)$. We will be particularly interested in the circumstances under which we can say that the sequence of zeros is strictly convex (i.e. the spacing is increasing) or strictly concave.

Results of this kind are well-known in some special cases. For example if $\mu = \nu$ then using a recurrence relation [28, p. 245] we get $t^{\nu-1}C(\nu, \nu, t) = t^\nu C_{\nu-1}(t)$ and an old result of Sturm [25, pp. 174–175] shows that the spacing of the zeros decreases to, is equal to, or increases to the value π according as $|\nu - 1|$ is greater than, equal to or less than 1/2. In the case $\mu = 0$ it follows from more general results [19, Theorem 7.2 and Remark (iii), p. 365] on zeros of $C'_\nu(t)$ that, *for all real* ν, the spacing of the zeros which exceed $|\nu|$ decreases to its asymptotic value π. In case $\mu = 1/2$ and $|\nu| > 1/2$ the sequence of differences decreases also; this follows from [19, Theorem 7.1].

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Another easy case occurs when $\nu = 1/2$. The zeros in question are then the positive zeros of $(\mu - 1/2) \tan t + t$ and it is easy to see that the spacing decreases or increases according as $\mu < 1/2$ or $\mu > 1/2$. There is a similar argument in the case $\nu = -1/2$. (In this paragraph we consider $c_\nu = J_\nu$ only.)

For general values of μ and ν we use $t_{\mu,\nu,k}$ to denote the k-th positive zero of $C(\mu, \nu, t)$. We show that the sequence $\{\Delta t_{\mu,\nu,k}\} = \{t_{\mu,\nu,k+1} - t_{\mu,\nu,k}\}$ is ultimately decreasing or ultimately increasing according as the point (μ, ν) is outside or inside the parabola $2\mu = \nu^2 + 3/4$ in the (μ, ν) plane. The question of when one can say that the sequence $\{\Delta t_{\mu,\nu,k}\}$ is monotonic from the start ($k = 1$) or nearly from the start is more difficult. Some partial results in this direction are given.

In the special case $C_\nu(t) = J_\nu(t)$, the spacing of the zeros of $C(\mu, \nu, t)$ was considered by K. D. Graham [7]. He was led to this function in a study of the spacing of the square roots of the eigenvalues of the wave equation in a hyperspherical domain in N dimensions under mixed homogeneous boundary conditions; see also [8] and [9]. In terms of the present notation, Graham's Theorem 1.1 asserts that (in case $C_\nu(t) = J_\nu(t)$),

(i) $\{\Delta t_{\mu,\nu,k}\}$ decreases to π if $\nu \geqq 1/2$, $\mu + \nu > 0$, $\mu \neq 1/2$;

(ii) $\Delta t_{1/2,\nu,k} = \pi, \qquad k = 1, 2, \dots .$

It will be clear from our § 2 that neither of these assertions is valid for the full range of the parameters for which the assertions are made. In fact, (ii) is not true without the additional assumption $|\nu| = 1/2$ while (i) is contradicted even in some of the special cases enumerated above ($\nu = 1/2$, $\mu > 1/2$ or $1/2 < \mu = \nu < 3/2$). However in the application [7, Theorem 1.4] which Graham makes it is only the asymptotic spacing of the zeros which is used.

Graham's results are derived from a long series of lemmas whose validity is difficult to check. In the present paper the main results are found by using G. Szegö's elegant formulation of a simple consequence of the Sturm comparison theorem. It runs as follows [26; 27, p. 20].

Lemma 1.1 *If f increases on the interval (a, b) and if a solution y of $y'' + f(x)\,y = 0$ has consecutive zeros at x_1, x_2, x_3 on (a, b), then*

$$x_3 - x_2 < x_2 - x_1 .$$

If f decreases on (a, b) the inequality sign in the result is reversed.

A more general formulation of this lemma involving the "halfwaves" of the graph of y has been given by E. Makai [20]; earlier, some results on the "quarter-waves" of y were given by P. Hartman and A. Wintner [10]. In spite of the elegance of the method based on Lemma 1.1, the calculations involved are somewhat tedious. However, the results obtained are reasonably sharp; it seems to be scarcely worthwhile, from the point of view of the present application to use one of the sharpened forms of Lemma 1.1 discussed in [27, p. 20] and [20]. Lemma 1.1 and the sharpened forms just mentioned depend for their validity on the classical Sturm comparison theorem; improved versions of this theorem are available [13, 14, 15, 16] but there does not appear to be any obvious way to get an improved version of Lemma 1.1 from them.

2. THE PRINCIPAL RESULTS

The function $w(x) = xC_\nu'(x) + \mu C_\nu(x)$ satisfies [6, p. 13]

$$x^2(x^2 - \nu^2 + \mu^2)\, w'' - x[x^2 + \nu^2 - \mu^2]\, w' + \\ + [(x^2 - \nu^2)^2 + 2\mu x^2 + \mu^2(x^2 - \nu^2)]\, w = 0.$$

By means of a well-known transformation [27, pp. 16–17, for example] we find that $u(x) = x^{1/2}(x^2 - M)^{-1/2}\, w(x)$ satisfies

$$u'' + \lambda(x)\, u = 0, \qquad x > M_1^{1/2}$$

where $M_1 = \max)M, 0)$,

$$M = \nu^2 - \mu^2$$

and

(2.1) $$\lambda(x) = 1 + (1/4 - \nu^2)\, x^{-2} + (2\mu - 1)(x^2 - M)^{-1} - 3M(x^2 - M)^{-2}.$$

$u(x)$ and $w(x)$ have the same zeros on $(M_1^{1/2}, \infty)$.

From (2.1), we get

(2.2) $$(2x)^{-1}(M + \eta)^2 \eta^3 \lambda'(x) = (\nu^2 - 2\mu + 3/4)\, \eta^3 + 4M(2 - \mu)\, \eta^2 + \\ + (13 - 2\mu)\, \eta M^2 + 6M^3, \qquad x > M_1^{1/2}$$

on writing $\eta = x^2 - M$. It is clear from this that $\lambda'(x)$ is positive or negative for large x depending on whether $\nu^2 - 2\mu + 3/4$ is positive or negative. Thus, by Lemma 1.1, the sequence $\{\Delta t_{\mu,\nu,k}\}$ *ultimately* decreases or increases according as the point (μ, ν) lies outside or inside the parabola $2\mu = \nu^2 + 3/4$. To discover a more precise result we examine the function $\lambda(x)$ more closely. Such an examination leads to the following theorems.

Theorem 2.1. *Suppose that $\nu^2 - \mu^2$ and $\nu^2 - 2\mu + 3/4$ are both > 0 but not both equal to 0. Then the sequence $\{\Delta t_{\mu,\nu,k}\}_{k=\varkappa}^{\infty}$ is decreasing provided that $t_{\mu,\nu,\varkappa}^2 \geqq (\nu^2 - \mu^2)(1 + \alpha)$ where $\alpha = \max(0, \alpha_1)$ and α_1 is the largest value for which*

(2.3) $$f(\mu, \nu, x) = (\nu^2 - 2\mu + 3/4)\, x^3 + 4(2 - \mu)\, x^2 + (13 - 2\mu)\, x + 6$$

changes sign.

Corollary 2.1. *Suppose that the hypotheses of Theorem 1.2 holds and that in addition, $\nu^2 \geqq 2\mu - 3/4 + 4(2\mu + 23)(\mu - 2)^2/243$, if $\mu \geqq 2$. Then the sequence*

$\{\Delta t_{\mu,\nu,k}\}_{k=x}^{\infty}$ *is decreasing provided that*

$$t_{\mu,\nu,x}^2 > \nu^2 - \mu^2.$$

Theorem 2.2. *Suppose that* $\nu^2 \geqq \mu^2$, $\nu^2 < 2\mu - 3/4$. *Then the sequence* $\{\Delta t_{\mu,\nu,k}\}_{k=x}^{\infty}$ *is increasing provided that*

$$t_{\mu,\nu,x}^2 > (\nu^2 - \mu^2)(1 + \alpha)$$

and α *is defined as in Theorem* 2.1.

Theorem 2.3. *Suppose that* $\nu^2 < \mu^2$, $\nu^2 \leqq 2\mu - 3/4$. *In addition, if* $\mu \geqq 2$ *suppose that* $\nu^2 < 2\mu - 3/4$. *Then* $\{\Delta t_{\mu,\nu,k}\}_{k=x}^{\infty}$ *is increasing provided that*

$$t_{\mu,\nu,x}^2 > (\beta - 1)(\mu^2 - \nu^2),$$

where $\beta = \max(1, \beta_1)$ *and* β_1 *is the largest value for which*

$$g(\mu, \nu, x) = (\nu^2 - 2\mu + 3/4)x^3 - 4(2 - \mu)x^2 + (13 - 2\mu)x - 6. \tag{2.4}$$

changes sign.

Corollary 2.3. *If* $|\nu| \leqq 1/2$ *and* $\nu^2 \leqq 2\mu - 3/4$ *then* $\{\Delta t_{\mu,\nu,k}\}_{k=1}^{\infty}$ *is increasing.*

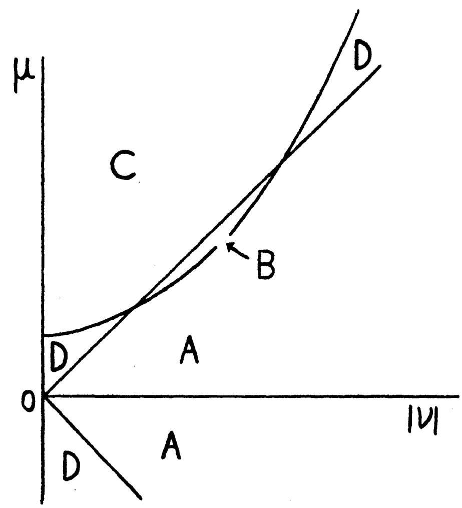

Fig. 1

Theorem 2.4. *Suppose that* $\nu^2 < \mu^2$, $\nu^2 \geqq 2\mu - 3/4$. *In addition, if* $\mu < 2$, *suppose that* $\nu^2 > 2\mu - 3/4$. *Then the sequence* $\{\Delta t_{\mu,\nu,k}\}_{k=x}^{\infty}$ *is decreasing provided*

$$t_{\mu,\nu,x}^2 > (\beta - 1)(\mu^2 - \nu^2)$$

and β *is as in Theorem λ.3.*

Corollary 2.4. *Suppose that $\nu^2 < \mu^2$ that $\nu^2 \geqq 2\mu - 3/4$ and that $\mu \geqq 2$. Then the sequence $\{\Delta t_{\mu,\nu,k}\}_{k=1}^{\infty}$ is decreasing.*

The four theorems in this section cover all real values of μ and ν except those satisfying $\mu = |\nu| = 1/2$ and $\mu = |\nu| = 3/2$. However it is clear that in these cases we are dealing with the positive zeros of $C_{1/2}(t)$ and these are evenly spaced.

We give Figure 1 as a guide to the regions in the (μ, ν) plane which are covered by the four theorems. The curve in figure 1 is the parabola $\nu^2 = 2\mu - 3/4$; it meets the line $\mu = \nu$ at the points (1/2, 1/2) and 3/2, 3/2). The regions marked A, B, C and D in the figure are covered by Theorems 2.1, 2.2, 2.3, and 2.4 respectively. To keep the diagram simple we do not indicate the regions to which the boundaries "belong". This information can be obtained from the detailed statements of the theorems.

In order to see what region in the (μ, ν) plane is covered by Corollary 2.1 we remark that the first quadrant curve represented by the equation

$$\nu^2 = 2\mu - 3/4 + 4(2\mu + 23)(\mu - 2)^2/243 \tag{2.5}$$

is such that μ increases with $|\nu|$ for $\mu \geqq 2$. This curve crosses the line $\mu = |\nu|$ at about $\mu = |\nu| = 22$ and is below that line for all larger $|\nu|$. Clearly, along this curve we have $\mu \to \infty$ as $|\nu| \to \infty$. The region covered by Corollary 2.1 is that covered by Theorem 2.1 with the exception of a "wedge-shaped" region below the line $\mu = \nu$ and above the curve (2.5).

3. PROOFS OF THE PRINCIPAL RESULTS

We first make the remark that it is sufficient to prove our results in the case $\nu \geqq 0$ since any linear combination of J_ν and Y_ν can be written as a linear combination of $J_{-\nu}$ and $Y_{-\nu}$.

The case $\mu = \nu$ of Theorem 2.1 has already been covered in the Introduction, and the case $\mu = -\nu$ can be dealt with similarly. Thus we may suppose that $M = \nu^2 - \mu^2 > 0$. Hence we find from (2.2) and (2.3) that $\lambda(x)$ increases with x for $x > [(1 + \alpha) M]^{1/2}$, provided $f(\mu, \nu, \eta/M) \geqq 0$ for $\eta/M > \alpha$. But this is certainly true since $f(\mu, \nu, x)$ is ultimately positive and its largest change of sign is less than or equal to α. Thus the result of Theorem 2.1 follows from Lemma 2.1.

In view of the Theorem just proved, in order to prove Corollary 2.1 it suffices to show that under the hypotheses of the corollary we have $\alpha_1 \leqq 0$, i.e. that $f(\mu, \nu, x) \geqq 0$ for $x > 0$. We divide the proof of this into three cases.

In the first case we suppose that $|\mu - 1| \geqq 1/2$ $\mu \leqq 2$. Then $f(\mu, \nu, x) \geqq f(\mu, \mu, x) = [(\mu - 1)^2 - 1/4]\, x^3 + 4(2 - \mu)\, x^2 + (13 - 2\mu)\, x + 6$ and this is clearly positive for $x > 0$.

On the other hand if we suppose that $|\mu - 1| < 1/2$ we have

$$f(\mu, \nu, x) \geqq f(\mu, \sqrt{2\mu - 3/4}, x) = 4(2 - \mu)x^2 + (13 - 2\mu)x + 6$$

and this too is positive for $x > 0$.

In the third case we suppose $\mu > 2$. We will show that $f(\mu, \nu, x) \geqq 0$ on $(0, \infty)$ in the case

$$\nu^2 = 2\mu - 3/4 + 4(2\mu + 23)(\mu - 2)^2/243 = \varphi(\mu).$$

It is clear then that it is positive on $(0, \infty)$ for larger values of ν^2. When $\nu^2 = \varphi(\mu)$ we find that

$$\begin{aligned} f((\mu, \nu, x) &= 3[(2/9)(\mu - 2)x - 1]^2[(2\mu + 23)x/9 + 2] \\ &\geqq 0, \qquad 0 < x < \infty. \end{aligned}$$

Thus we find that $f(\mu, \nu, x) \geqq 0$ for $x > 0$ and the proof of Corollary 2.1 is complete.

The proof of Theorem 2.2 follows the same lines as that of Theorem 2.1. In this case $f(\mu, \nu, x)$ is ultimately negative and the result follows from the fact that it is negative for $x > \alpha$.

In dealing with Theorem 2.3 we remark that in this case $M < 0$ and we wish to show that $\lambda(x)$ decreases for $x^2 > -M(\beta - 1)$. This will follow by showing that $f(\mu, \nu, \eta/M) \geqq 0$ for $\eta/M < -\beta$ or, what is the same thing, that $g(\mu, \nu, x) \leqq 0$ for $x > \beta$. But $g(\mu, \nu, x)$ is ultimately negative and its largest sign change does not exceed β. Hence the result follows and Theorem 2.3 is proved.

In order to prove Corollary 2.3 it is only necessary to show that under its hypotheses, $\beta = 1$, that is that $\beta_1 \leqq 1$ or that $g(\mu, \nu, x) \leqq 0$ for $x > 1$.

We consider two cases, $\mu \geqq 1/2$ and $\mu < 1/2$. In the first case we have

$$\begin{aligned} g(\mu, \nu, x) &\leqq g(\mu, 1/2, x) \\ &= (x - 1)^2[(1 - 2\mu)x - 6] \\ &< 0, \qquad x > 1. \end{aligned}$$

In the second case

$$g(\mu, \nu, x) \leqq g(\mu, \sqrt{2\mu - 3/4}, x) = -4(2 - \mu)x^2 + (13 - 2\mu)x - 6$$

and this last expression is seen to have no real zeros for the values of μ considered. Thus $g(\mu, \nu, x)$ is again < 0 for $x > 1$. This completes the proof of Corollary 2.3.

The proof of Theorem 2.4 is similar to that of Theorem 2.3. It is enough to show that $g(\mu, \nu, x) \geqq 0$ for $x > \beta$. But this is obvious by the definition of β and the fact that $g(\mu, \nu, x)$ is ultimately positive. To prove Corollary 2.4 we will show that with the additional assumption $\mu \geqq 2$, we have

$$g(\mu, \nu, x) \geqq 0, \qquad x > 1.$$

Now it is clear that in the region concerned $g(\mu, \nu, x)$ increases with ν, for fixed μ and x so it is enough to show that

$$g(\mu, \sqrt{2\mu - 3/4}, x) \geqq 0, \qquad x > 1.$$

Now

$$g(\mu, \sqrt{2\mu - 3/4}, x) = 4(\mu - 2)x^2 + (13 - 2\mu)x - 6$$

and it is a simple matter to show that the largest real zero of this quadratic function does not exceed 1 for each $\mu \geqq 2$. This completes the proof of Corollary 2.4.

4. THE CASE $C_\nu = J_\nu$

From the regions described in Corollaries 2.3 and 2.4 we have monotonicity "from the start" even for general C_ν. We turn attention now to regions where we can get monotonicity from the start for J_ν even if it is not possible to get it for general C_ν. We first prove some lemmas.

Lemma 4.1. For $\nu > 0$ each positive zero of $\mu J_\nu'(x) + xJ_\nu'(x)$ increases with ν.

Proof. The zeros in question are the eigenvalues of the boundary value problem

$$(4.1) \qquad (-xy')' + \nu^2 x^{-1} y = \lambda^2 xy$$

$$(4.2) \qquad y(0) = 0$$

$$(4.3) \qquad \mu y(1) + y'(1) = 0$$

Corresponding to an eigenvalue λ_ν, we have an eigenfunction $y_\nu = J_\nu(\lambda_\nu x)$. Following a method outlined in [17], [11, § 4] and [21] we multiply the equations

$$(-xy_\nu')' + \nu^2 x^{-1} y_\nu = \lambda_\nu^2 x y_\nu$$

$$(-xy_{\nu+\varepsilon}')' + (\nu + \varepsilon)^2 x^{-1} y_{\nu+\varepsilon} = \lambda_{\nu+\varepsilon}^2 x y_{\nu+\varepsilon}$$

by $y_{\nu+\varepsilon}, y_\nu$ respectively subtract, integrate between 0 and 1, divide by ε and let $\varepsilon \to 0^+$ to get

$$\frac{d\lambda_\nu^2}{d\nu} = 2\nu \int_0^1 x^{-1} y_\nu^2(x)\,dx \Big/ \int_0^1 x y_\nu^2(x)\,dx.$$

The result of the lemma follows easily.

Lemma 4.2. In case $\nu \geqq |\mu|$, the smallest positive zero of

$$xJ_\nu'(x) + \mu J_\nu(x)$$

exceeds $(\nu^2 - \mu^2)^{1/2}$.

Proof. If we write $w(x) = x^\mu J_\nu(x)$, we find that we are dealing with the smallest positive zero x_0 of $w'(x)$ and w satisfies [12, p. 156]

$$(4.4) \qquad w''(x) + (1 - 2\mu)x^{-1}w' + [1 + (\mu^2 - \nu^2)x^{-2}]\,w = 0$$

Now we have $w''(x_0) < 0$, $w(x_0) > 0$ and $w'(x_0) = 0$. The result follows from (4.4).

Remarks. Graham has this result but it is proved in a more complicated way [7, p. 335]. Here we use the method of L. Lorch [18] who proved this result in the case $\mu = 1/2$. Recently, Ahmed and Calogero [2, p. 314] have proved the stronger result that the zero in question exceeds

$$[(1 + \nu^{-1})(\nu^2 - \mu^2)]^{1/2}.$$

The main results of this section are stated in the form of two theorems.

Theorem 4.1. *Suppose that $\nu - |\mu|$ and $\nu^2 - 2\mu + 3/4$ are both $\geqq 0$ but not both equal to zero and that*

$$\nu^2 \geqq 2\mu - 3/4 + 4(2\mu + 23)(\mu - 2)^2/243, \qquad \textit{if } \mu \geqq 2.$$

Then the sequence of spaces between the positive zeros of $\mu J_\nu(x) + xJ_\nu'(x)$ is decreasing.

Proof. The hypotheses imply those in Corollary 2.1 so it is simply a matter of showing that the zeros in question exceed $(\nu^2 - \mu^2)^{1/2}$. But this follows from Lemma 4.2.

Theorem 4.2. *Suppose that $\mu < -\nu$, $\nu \geqq 1/2$. Then the sequence of spaces between the positive zeros of $\mu J_\nu(x) + xJ_\nu'(x)$ is decreasing.*

Proof. In view of Theorem 2.4, we have to show that

$$x_{\nu,1}^2 > (\beta_1 - 1)(\mu^2 - \nu^2), \tag{4.5}$$

where $x_{\nu,1}$ is the first positive zero of $\mu J_\nu(x) + xJ_\nu'(x)$ and β_1 is the largest value for which $g(\mu, \nu, x)$ changes sign. It suffices to consider $\beta_1 > 1$. We see that, for the range of values of μ and ν in question, $g(\mu, \nu, x)$ is ultimately positive and that for each fixed μ and x, $g(\mu, \nu, x)$ increases with ν. Thus β_1 decreases with ν. Hence, for fixed μ, the right-hand-side of (4.5) decreases as ν increases and, from Lemma 4.1, its left-hand-side increases with ν. This means that it is enough to prove (4.5) in the special case $\nu = 1/2$. We have $g(\mu, 1/2, x) = (x - 1)^2[(1 - 2\mu)x - 6]$ so that $\beta_1 = 6/(1 - 2\mu)$. Hence proving (4.5) reduces to showing that

$$x_{1/2,1}^2 > 1 - (2\mu + 3)^2/4, \qquad \mu < -1/2.$$

But this is easy to show since $x_{1/2,1}$ is the smallest positive root of the equation

$$(1/2 - \mu) \tan x = x$$

and this exceeds π for $\mu \leqq -1/2$. This completes the proof of Theorem 4.2.

Theorems 4.1 and 4.2, together with Corollary 2.4 enable us to describe a fairly wide region in the (μ, ν) plane in which the spacing of the positive zeros of $\mu J_\nu(x) + x J'_\nu(x)$ decreases starting with the first positive zero. This region is sketched in Figure 2 which is not drawn to scale. It is the region to the right of the piecewise

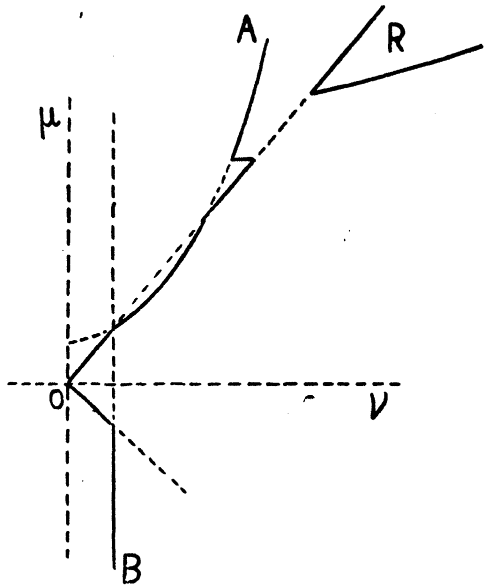

Fig. 2

continuous line AB apart from the "wedge-shaped" region R below the line $\mu = \nu$ and above the curve (2.5). AB consists of the line $\nu = 1/2$ (for $\mu < -1/2$), the lines $\nu = |\mu|$, (for $|\mu| \leqq 1/2$ and $3/2 \leqq \mu \leqq 2$), the curve $\nu^2 = 2\mu - 3/4$ (for $1/2 < \mu < 3/2$ and $\mu \geqq 2$) and the line $\mu = 2$ (for $\sqrt{13}/2 < \nu < 2$). It seems likely that this region can be improved.

5. SOME REMARKS ON THEOREMS 2.2 AND 2.4

We have seen in the previous sections, that for many values of μ and ν we can get monotonicity from the start of the spacing between the zeros in question. On the other hand we will show here that there are certain values of μ and ν for which no such regular behaviour can be expected.

Theorem 2.2. refers to the region $\nu^2 \gneqq \mu^2$, $\nu^2 < 2\mu - 3/4$ and shows that for (μ, ν) in this region the spacing ultimately increases. We will show here that close to the boundary curve $\nu^2 = 2\mu - 3/4$ the spacing starts off decreasing and in fact by going close enough to that boundary we may have decrease for an arbitrarily large number of zeros. [This is not surprising since on the boundary we have decreasing spacing ultimately (Theorem 2.1)].

To prove this it is only necessary to show that close to the boundary $\nu^2 = 2\mu - 3/4$ we have

$$f(\mu, \nu, x) > 0, \qquad 0 < x < x_0$$

where x_0 can be made arbitrarily large by choosing μ, ν close enough to the boundary curve $\nu^2 = 2\mu - 3/4$.

We suppose that $\nu^2 = 2\mu - 3/4 - \varepsilon$, $\varepsilon > 0$. We then have

$$f(\mu, \nu, x) = -\varepsilon x^3 + 4(2 - \mu) x^2 + (13 - 2\mu) x + 6 > \\ > x^2[-\varepsilon x + 4(2 - \mu)] > 0$$

if $0 < x < 4(2 - \mu)/\varepsilon$. It is clear now that this interval may be made arbitrarily large by choosing ε small enough. To show that an arbitrarily large interval contains arbitrarily many zeros of $xJ_\nu(x) + \mu J_\nu'(x)$ (and, hence, by the Sturm separation theorem, of $xC_\nu(x) + \mu C_\nu'(x)$) one has only to recall the form of the graph of $xJ_\nu'(x)/J_\nu(x)$ [24] and the known distribution of zeros of $J_\nu(x)$.

Throughout much of the region $\mu > |\nu|$, $\nu^2 > 2\mu - 3/4$ it seems likely that the spacing of the zeros of $\mu J_\nu(x) + xJ_\nu'(x)$ decreases from the start. One expects this property to break down at some points close to the boundary curve $\nu^2 = 2\mu - 3/4$. To get some idea of the actual situation we consider the case $\nu = 0$. It follows from the table in [4, p. 493] that the spacing starts out decreasing (beginning with the first positive zero) – as well as being ultimately decreasing – for several values of μ, including $\mu = 0.3$, in the interval $0 \leqq \mu < 3/8$.

Let us now consider what happens in the case $\nu = 0$ when we let μ approach 3/8 from below. If we let $\mu = 3/8 - \varepsilon$, $\varepsilon > 0$ we find $g(3/8 - \varepsilon, 0, x) = 2x(x - 1)^2 + h(x)$ where

$$h(x) = -(13/2)x^2 + (49/4)x - 6 < 1/4, \qquad x \geqq 1,$$

so it is clear that $g(3/8 - \varepsilon, 0, x)$ is negative on $1 < x < x_0$ where $x_0 \to \infty$ as $\varepsilon \to 0^+$. Thus for $\nu = 0$ and $\mu = 3/8 - \varepsilon$ with ε sufficiently small we find that for the first arbitrarily many zeros the spacing is increasing.

6. OTHER RESULTS

(a) The above results on the spacing of zeros are only some of those which may be obtained by the methods of the present paper. We may also obtain results relating to the maxima of

$$(6.1) \qquad \left| \left(\frac{x}{x^2 - M} \right)^{1/2} [xC_\nu'(x) + \mu C_\nu(x)] \right|, \qquad M = \nu^2 - \mu^2.$$

Each of the theorems and corollaries of section 2 has a counterpart for the maxima of (6.1). Thus, for example, we find as in Corollary 2.3 that if $|\nu| \leqq 1/2$ and $\nu^2 \leqq \leqq 2\mu - 3/4$ then the maxima of the function in (6.1) which exceed its first zero form an i~~ncreasing sequence~~. Results of this kind ("Sonin's Theorem") follow from our hypotheses in the manner outlined by Makai [20].

(b) More generally one may seek to find conditions under which certain general expressions have *higher monotonicity* properties.

We find that if $|\nu| \geqq 1/2$ $\mu \leqq 1/2$ and $M \geqq 0$ then $\lambda(x) \to 1$ as $x \to \infty$ and $\lambda'(x)$ is completely monotonic on $(M^{1/2}, \infty)$. Hence using [19, Theorem 3.1], we find that if $W(x)$ is completely monotonic the sequence

$$\left\{\int_{t_{\mu,\nu,k}}^{t_{\mu,\nu,k+1}} W(x)\left|\left(\frac{x}{x^2-M}\right)^{1/2}[xC_\nu'(x)+\mu C_\nu(x)]\right|^{\lambda} \mathrm{d}x\right\}, \qquad \lambda > -1,$$

$k = \varkappa, \varkappa + 1, \ldots$ is completely monotonic as long as $t^2_{\mu,\nu,\varkappa} > M$. In particular, if we take $\lambda = 0$, $W(x) = 1$ we find that for $|\nu| > 1/2$, $\mu < 1/2$ and $M > 0$) the sequence $\{\Delta t_{\mu,\nu,k}\}_{k=\varkappa}^{\infty}$ is not only decreasing (as follows from Corollary 2.1) but completely monotonic.

(c) We have confined our attention to positive zeros but in some cases a zero occurring at the origin may be included in the results by taking account of the possibility of including an end-point zero in Lemma 1.1; see [27, p. 20].

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REFERENCES

[1] M. Abramowitz and I. A. Stegun, eds.: *Handbook of Mathematical Functions*, Applied Mathematics Series, vol. 55, National Bureau of Standards, Washington, 1964.

[2] S. Ahmed and F. Calogero: *On the zeros of Bessel Functions. — III*, Lett. Nuovo Cimento, 21 (1978), 311—314.

[3] J. R. Airey: *The vibrations of cyl~~inders and~~ cylindrical shells*, Arch. d. Math. u. Phys. (3) 20 (1913), 289—294.

[4] H. S. Carslaw and J. C. Jaeger: *Conduction of heat in solids*, 2nd ed., Clarendon Press, Oxford, 1959.

[5] R. Courant and D. Hilbert: *The methods of mathematical physicsl* vol. 1, 1st English edition, Wiley-Interscience, New York, 1953.

[6] A. Erdélyi et al.: *Higher transcendental functions*, vol. 2, McGraw—Hill, New York, 1954.

[7] K. D. Graham: *Separation of eigenvalues of the wave equation for the unit ball in R^N*, Studies Appl. Math. 52 (1973), 329—343.

[8] K. D. Graham: *On boundary value control of symmetric hyperbolic systems*, Ph. D thesis, University of Minnesota, Minneapolis, 1973.

[9] K. D. Graham and D. L. Russell: *Boundary value control of the wave equation in a spherical region*, SIAM. J. Control 13 (1975), 174—196.

[10] P. Hartman and A. Wintner: *On nonconservative linear oscillators of low frequency*, Amer. J. Math. 70 (1945), 529—539.

[11] M. E. H. Ismail and M. E. Muldoon: *Monotonicity of the zeros of a cross-product of Bessel functions*, SIAM. J. Math. Anal. 9 (1978), 759—767.

[12] E. Jähnke, F. Emde and F. Lösch: *Tables of higher functions*, 6th ed., McGraw—Hill, New York, 1960.

[13] K. Kreith: *Comparison theorems for non-self-adjoint differential equations based on integral inequalities*, Proc. Amer. Math. Soc. 34 (1972), 105—109.

[14] W. Leighton: *A substitute for the Picone formula*, Bull. Amer. Math. Soc. 55 (1949), 325—328.

[15] W. Leighton: *Comparison theorems for linear differential equations of the second order*, Proc. Amer. Math. Soc. 13 (1962), 603—610.

[16] A. Yu. Levin: *A comparison principle for second-order differential equations*, Soviet Math. Doklady 1 (1960), 1313—1316. [Russian original, Dokl. Akad. Nauk SSSR 136 (1960), 783—786.]

[17] J. T. Lewis and M. E. Muldoon: *Monotonicity and convexity properties of zeros of Bessel functions*, SIAM. J. Math. Anal. 8 (1977), 171—178.

[18] L. Lorch: *Comparison of two formulations of Sonin's theorem and of their respective applications to Bessel functions*, Studia Sci. Math. Hungar. 1 (1966), 141—145.

[19] L. Lorch, M. E. Muldoon and P. Szego: *Higher monotonicity properties of certain* Sturm—*Liouville functions*, IV, Canad. J. Math. 24 (1972), 349—365.

[20] E. Makai: *On a monotonic property of certain Sturm—Liouville functions*, Acta. Math. Acad. Sci. Hungar. 3 (1952), 165—172.

[21] M. E. Muldoon: *On the zeros of a cross-product of Bessel functions of different orders*, Z. Angew. Math. Mech., 59 (1979), 272—273.

[22] A. D. Rawlins: *A new class of Bessel function inequality useful for investigating the roots of a class of transcendental equation involving Bessel functions*, Proc. Roy. Soc. Edinburgh 74A (1974—1975), 231—238.

[23] R. Ruedy: *On the propagation of longitudinal waves in cylindrical rods*, Canad. J. Research, 5 (1931), 149—155.

[24] R. Spigler: *Sulle radici dell' equazione*: $AJ_\nu(x) + BxJ'_\nu(x) = 0$, Atti Sem. Mat. Fis. Univ. Modena 24 (1975), 399—419.

[25] Ch. Sturm: *Mémoire sur les équations différentielles du second ordre*, J. Math. Pures Appl. 1 (1836), 106—186.

[26] G. Szegö: *Inequalities for the zeros of Legendre polynomials and related functions*, Trans. Amer. Math. Soc. 39 (1936), 1—17.

[27] G. Szegö: *Orhogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 4th ed., 1975.

[28] G. N. Watson: *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge University Press, 1944.

M. E. Muldoon
Department of Mathematics, York University
Downsview, Ontario,
CANADA M3J 1P3