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ON SEMIMODULAR LATTICES OF GENERATING SYSTEMS

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0. INTRODUCTION

A subset of a complete lattice L closed under formation of arbitrary g.l. bounds is called a closure system on L and the complete lattice of closure systems on L, ordered by inclusion, is denoted by $\mathfrak{C}(L)$. The following results are obtained. A principal filter in $\mathfrak{C}(L)$ is semimodular iff it is meet infinitely distributive. Under certain conditions, $\mathfrak{C}(L)$ does not contain the "diamond". An example showing that these conditions cannot be omitted is presented and some corollaries concerning lattices of generating systems, called briefly gs-lattices in [4] and [5], are formulated.

For the motivation of the study of gs-lattices the reader may look at [5]. This study can be included into the general treatment of lattices of topologies on a set introduced in [9], but the properties of gs-lattices differ essentially from the properties of lattices of topologies in the sense of [2]. This fact can be observed by comparison of the results from [4] and this paper with those from [7] and [8]. An extensive list of results concerning lattices of topologies can be found in [6].

1. THE SEMIMODULARITY OF $\mathfrak{C}(L, N)$

The symbol $\emptyset$ will signify the empty set. For a set A we denote by *card*(A) the cardinality of A and by id_A the identity relation on A.

If P is a poset then the ordering on P will be denoted by $\leqq$, the covering relation by $\prec$, the incomparability relation by $\parallel$ and $a \preceq b$ will abbreviate $a \prec b$ or $a = b$. As it is usual, $(a]$, $[a)$ will denote the principal ideal, principal filter in P generated by a, respectively, and $[a, b]$ the interval $[a) \cap (b]$ for all $a, b \in P$, $a \leqq b$. A set $Q \subseteq P$ will be called *hereditary* in P if $a \in Q$, $b \leqq a$ imply $b \in Q$. The set of hereditary subsets in P will be denoted by $\mathrm{H}(P)$ and the *normal completion* of P by $\mathrm{N}(P)$ or, more exactly, by $\mathrm{N}(P, \leqq)$. It is the least subset of $\mathrm{H}(P)$ containing P as well as all principal ideals in P which is closed under intersection. If $A \subseteq P$ then $(A]$ will denote the least hereditary subset of P containing A, i.e. $(A] = \emptyset$ if $A = \emptyset$ and $(A] = \bigcup (a]$ otherwise. Finally, $\bigwedge A$, $a \wedge b$ and $\bigvee A$, $a \vee b$ will be a notation for the g.l. bound of A, $\{a, b\}$ and the l.u. bound of A, $\{a, b\}$ in P, respectively.

1.1. Definition. A subset C of a complete lattice L is said to be a *closure system* on L if $\bigwedge A \in C$ for each $A \subseteq C$. ($\bigwedge \emptyset$ is the greatest element in L.)

We denote by $\mathfrak{C}(L)$ the set of closure systems on L and by $\mathfrak{C}(L, N)$ the set $\{C \in \mathfrak{C}(L) \mid N \subseteq C\}$ for each $N \in \mathfrak{C}(L)$.

1.2. Remark. (i) In the following, both $\mathfrak{C}(L)$ and $\mathfrak{C}(L, N)$ will be considered to be complete lattices in which L is the greatest element and the g.l. bound of every nonempty subset is its intersection.

(ii) Important special cases of $\mathfrak{C}(L, N)$ are lattices $\mathfrak{C}(\mathbf{H}(P), \mathbf{N}(P))$, where P is a poset, which are called *lattices of generating systems* and denoted by Gs(P) in [3], [4], [5].

1.3. Definition. If $C \in \mathfrak{C}(L)$ then we put $\varphi_C(a) = \bigwedge \{b \in C \mid a \leqq b\}$ for each $a \in L$.

1.4. Lemma. *If $C \in \mathfrak{C}(L)$ then φ_C is an isotone, extensive and idempotent map of L into L (a closure operator on L) and $C = \{a \in L \mid a = \varphi_C(a)\}$.*

1.5. Lemma. *The following assertions hold for all $C, D \in \mathfrak{C}(L)$.*

(i) $C \vee D = \{c \wedge d \mid c \in C$ *and* $d \in D\}$.

(ii) $\varphi_{C \vee D}(a) = \varphi_C(a) \wedge \varphi_D(a)$ *for each* $a \in L$.

(iii) $C \subseteq D \Rightarrow \varphi_D(a) \leqq \varphi_C(a)$ *for each* $a \in L$.

1.6. Corollary. $a \in C \vee D$ *iff* $a = \varphi_C(a) \wedge \varphi_D(a)$ *for all* $a \in L$ *and* $C, D \in \mathfrak{C}(L)$.

1.7. Definition. We denote by $\langle A \rangle$ the least $C \in \mathfrak{C}(L)$ satisfying $A \subseteq C$ for any complete lattice L and $A \subseteq L$.

If $C \in \mathfrak{C}(L)$ and $\{a_1, a_2, \ldots, a_n\} \subseteq L$ then it is possible to write $\langle C, a_1, a_2, \ldots, a_n \rangle$ instead of $\langle C \cup \{a_1, a_2, \ldots, a_n\} \rangle$.

1.8. Lemma. *Let L be a complete lattice. Then the following assertions hold.*

(i) $\langle A \rangle = \{\bigwedge B \mid B \subseteq A\}$ *for each* $A \subseteq L$.

(ii) $\langle C, a \rangle \subseteq C \cup (a]$ *for all* $C \in \mathfrak{C}(L)$, $a \in L$.

(iii) $\langle C, a \rangle - \{a\} \in \mathfrak{C}(L)$ *for all* $C \in \mathfrak{C}(L)$, $a \in L - C$.

1.9. Lemma. *If $B, C \in \mathfrak{C}(L)$ and $a \in L - C$ then $a \in B \vee C$ implies $\varphi_B(a) \notin C$.*

Proof. $a \in B \vee C \Rightarrow a = \varphi_B(a) \wedge \varphi_C(a)$ regarding 1.6. By this and by $\varphi_B(a) \in C$ we obtain $a \in C$ which is a contradiction.

1.10. Definition. A complete lattice L is said to be

(i) *semimodular* if $a \prec b$ implies $a \vee x \preceq b \vee x$ for each $x \in L$.

(ii) *meet infinitely distributive* if $a \vee \bigwedge B = \bigvee_{b \in B} (a \vee b)$ for all $a \in L$ and $B \subseteq L$.

(iii) *upper continuous* if $a \wedge \bigvee B = \bigwedge_{b \in B} (a \wedge b)$ for all $a \in L$ and all chains B in L.

1.11. Theorem. *Let L be a complete lattice and $N \in \mathfrak{C}(L)$. Then the following assertions are equivalent.*

(i) $\mathfrak{C}(L, N)$ *is semimodular.*

(ii) $\mathfrak{C}(L, N)$ *is meet infinitely distributive.*

(iii) $C \vee D = C \cup D$ *for all* $C, D \in \mathfrak{C}(L, N)$.

(iv) $[a, \varphi_N(a)]$ *is a chain for each* $a \in L$.

Proof. (i) $\Rightarrow$ (iii): If (iii) is not true then there exist $E, F \in \mathfrak{C}(L, N)$ and $a \in \in (E \vee F) - (E \cup F)$. For $b = \varphi_E(a)$, $c = \varphi_F(a)$ it holds $a < b$, $a < c$ and $a = = b \wedge c$ by 1.4, 1.6. If we put

$$B = \langle N, b \rangle, \qquad C = \langle N, c \rangle, \qquad A = B - \{b\}$$

hen $A \in \mathfrak{C}(L, N)$ by 1.8 (iii) and by the validity of $b \notin N$. Indeed, $b \notin F$ by 1.9 and $N \subseteq F$.

It follows by $b \notin N$, $b \nleq c$ and $C \subsetneqq N \cup (c]$, see 1.8 (ii), that $b \in L - C$. Moreover, $b \notin A \Rightarrow b < \varphi_A(b) \in A \subseteq B \subseteq N \cup (b] \Rightarrow \varphi_A(b) \in N \subseteq C$. The last two conclusions and 1.9 give $b \notin A \vee C$. Further, $N \subsetneqq E$, $b \in E$ imply $A \subseteq B = \langle N, b \rangle \subseteq \subseteq E$. Then $b = \varphi_E(a) \leqq \varphi_A(a)$ by 1.5 (iii) and this fact together with $\varphi_A(a) \in \in A = B - \{b\} \subseteqq (N \cup (b]) - \{b\}$ imply $\varphi_A(a) \in N \subsetneqq C$. By this, $a \notin F \supseteq C$ and by 1.9 we obtain $a \notin A \vee C$. As $b \in B \vee C$ obviously and $a \in B \vee C$ according to $a = b \wedge c$, $b \in B$, $c \in C$, it holds $\{a, b\} \subseteq (B \vee C) - (A \vee C)$.

If we denote $D = \langle A \vee C, a \rangle$ then $D \subseteq B \vee C$ and $b \notin D$ regarding $D \subseteq \subseteq (A \vee C) \cup (a]$, $a < b$. Hence $A \vee C \subset D \subset B \vee C$ and we have not $A \vee C \prec \prec B \vee C$. Since $A \prec B$ obviously, (i) does not hold for $\mathfrak{C}(L, N)$.

(iii) $\Rightarrow$ (iv): Let us admit that $[a, \varphi_N(a)]$ is not a chain for some $a \in L$. Then there exist $b, c \in [a, \varphi_N(a)]$ such that $b \parallel c$. If we denote $B = \langle N, b \rangle$ and $C = \langle N, c \rangle$ then, according to $\varphi_B(a) \in B$ and 1.8 (i), we can find $Q \subsetneqq N \cup \{b\}$ satisfying $\varphi_B(a) = \bigwedge Q$. We have $\varphi_B(a) \geqq \bigwedge(Q - \{b\}) \wedge b \geqq \varphi_N(a) \wedge b = b$ because of $Q - \{b\} \subseteq N$ and $a \leqq x$ for all $x \in Q - \{b\}$. By $b \leqq \varphi_B(a)$ and by $b \parallel c$, $a \leqq \leqq b \wedge c$ we obtain $b \wedge c < b \leqq \varphi_B(a) \leqq \varphi_B(b \wedge c)$. In the same way we prove $b \wedge c < \varphi_C(b \wedge c)$.

These two relations and 1.4 say $b \wedge c \notin B \cup C$. As $b \wedge c \in B \vee C$, we have $B \cup C \neq B \vee C$.

(iv) $\Rightarrow$ (iii): Let us now suppose that $[a, \varphi_N(a)]$ is a chain for each $a \in L$ and take $C, D \in \mathfrak{C}(L, N)$, $a \in C \vee D$ arbitrarily. Then $a = \varphi_C(a) \wedge \varphi_D(a)$ according to 1.6. It follows by $N \subseteq C$, $N \subsetneqq D$ and 1.4, 1.5 (iii) that $\varphi_C(a), \varphi_D(a) \in [a, \varphi_N(a)]$. Hence $\varphi_C(a)$ is comparable with $\varphi_D(a)$ and either $a = \varphi_C(a)$ or $a = \varphi_D(a)$. As this is equivalent to $a \in C \cup D$, we have $C \vee D \subseteq C \cup D$; the converse inclusion is true trivially.

1.12. Corollary. *Let L be a complete lattice. Then $\mathfrak{C}(L)$ is semimodular iff L i a chain.*

2. ON A LATTICE $\mathfrak{C}(L)$ CONTAINING M_3

2.1. Definition. Let V be a set and o, i elements such that $card(V) > 1$, $o \neq i$ and $V \cap \{o, i\} = \emptyset$. We denote by M_V the lattice $V \cup \{o, i\}$ provided by the following ordering. $o \leqq x \leqq i$ and $x \parallel y$ for all $x, y \in V$, $x \neq y$.

We write M_3 instead of $M_{\{a,b,c\}}$.

2.2. Definition. We say that a complete lattice L *contains* M_V whenever there is an embedding (an injective lattice-homomorphism) of M_V into L.

2.3. Definition. A closure system C on a complete lattice L is called inductive in L if $\bigvee \{a_i \mid i = 0, 1, \ldots\} \in C$ for each chain $a_0 < a_1 < \ldots$ in C.

2.4. Theorem. *Let L be an upper continuous complete lattice, N a closure system on L and let every element of $\mathfrak{C}(L, N)$ be inductive in L. Then $\mathfrak{C}(L, N)$ does not contain M_3.*

Proof. Let us admit that ι: $M_3 \to \mathfrak{C}(L, N)$ is an embedding and put $\iota x = X$ for $x = o, i, a, b, c$. Then $A \cap B = B \cap C = C \cap A = 0$, $A \vee B = B \vee C = C \vee A = I$ and $\Delta_X = X - 0 \neq \emptyset$ for $X = A, B, C$.

Choose $a \in \Delta_A$ arbitrarily. Then $a \in A \subseteq B \vee C$ implies $a = \varphi_B(a) \wedge \varphi_C(a)$ and, as $a \notin B$, $a < \varphi_B(a)$. Moreover, $a \in L - B$, $a \in B \vee C$ and 1.9 imply $\varphi_B(a) \notin C$. Hence $\varphi_B(a) \in \Delta_B$. If we take $\varphi_B(a)$ instead of a and change the roles of A, B in the previous consideration then we get

$$\varphi_B(a) = \varphi_A\varphi_B(a) \wedge \varphi_C\varphi_B(a), \qquad \varphi_B(a) < \varphi_A\varphi_B(a) \qquad \text{and} \qquad \varphi_A\varphi_B(a) \in \Delta_A.$$

Further, $a = \varphi_B(a) \wedge \varphi_C(a) = \varphi_A\varphi_B(a) \wedge \varphi_C\varphi_B(a) \wedge \varphi_C(a) = \varphi_A\varphi_B(a) \wedge \varphi_C(a)$ according to $\varphi_C(a) \leqq \varphi_C\varphi_B(a)$. Hence $a < \varphi_B(a) < \varphi_A\varphi_B(a)$ and $\varphi_A\varphi_B(a) \wedge \varphi_C(a) = a$. By induction we obtain

$$a < \varphi_B(a) < \varphi_A\varphi_B(a) < \ldots < \varphi_B(\varphi_A\varphi_B)^k(a) < (\varphi_A\varphi_B)^{k+1}(a) < \ldots$$

and

$$(\varphi_A\varphi_B)^n(a) \wedge \varphi_C(a) = a \qquad \text{for } n = 1, 2, \ldots$$

If we put $Q = \{(\varphi_A\varphi_B)^n(a) \mid n = 1, 2, \ldots\}$, $R = \{\varphi_B(\varphi_A\varphi_B)^n(a) \mid n = 1, 2, \ldots\}$ and $b = \bigvee Q$ then $b = \bigvee R$ obviously. By this, $Q \subseteq A$, $R \subseteq B$ and by the inductivity of A, B we obtain $b \in A \cap B = 0$. As, moreover, $a < b$, we have $\varphi_0(a) \leqq b$. At the same time, $a < \varphi_C(a)$ and $\varphi_C(a) \leqq \varphi_0(a)$ hold with respect to $a \notin C$ and $0 \subseteq C$. Then $a < \varphi_C(a) = b \wedge \varphi_C(a)$. But $b \wedge \varphi_C(a) = \bigvee Q \wedge \varphi_C(a) = \bigvee \{(\varphi_A\varphi_B)^n(a) \wedge \varphi_C(a) \mid n = 1, 2, \ldots\} = a$ and we have a contradiction.

We shall now prove that there exists a complete lattice L such that $\mathfrak{C}(L)$ contains M_V for an arbitrary given set V with the property $card(V) > 1$.

2.5. Definition. Let $V \neq \emptyset$ be a set. We denote by V^* the free monoid over V and by e its unit. If $u \in V^*$ then there are $m \geqq 0$ and $a_1, a_2, \ldots, a_m \in V$ with the property $a_1a_2 \dots a_m = u$ (we set $a_1a_2 \dots a_m = e$ for $m = 0$). We call the symbol $a_1a_2 \dots a_m$ a *decomposition* of u (in V) and m a *length* of u; we write $m = |u|$. If $u, v \in V^*$ then the symbol $v_0a_1v_1 \dots a_mv_m$ is said to be a *u-decomposition* of v whenever $a_1a_2 \dots a_m$ is a decomposition of u, $v_0, v_1, \dots, v_m \in V^*$ and $v_0a_1v_1 \dots a_mv_m = v$. For arbitrary $u, v \in V^*$ we put

$$u \leqq v \qquad \text{if there is a } u\text{-decomposition of } v.$$

One can easily see that $\leqq$ is an ordering on V^*.

In lemma 2.6 we repeatedly use the following obvious fact. If $V \neq \emptyset$, $u_1, u_2, v_1, v_2 \in V^*$ and $u_1u_2 = v_1v_2$ then $|v_1| \leqq |u_1|$, $|v_1| < |u_1|$ if and only if there exists $z \in V^*$, $z \in V^* - \{e\}$, respectively, such that $u_1 = v_1z$.

2.6. Lemma. *If $V \neq \emptyset$, $v_i \in V^*$ for $i = 0, 1, \dots, m$, and $a_i \in V$ are such that $a_i \nleqq v_{i-1}$ for $i = 1, 2, \dots, m + 1$ then*

$$a_1a_2 \dots a_{m+1} \nleqq v_0a_1v_1 \dots a_mv_m.$$

Proof. Let us denote $v = v_0a_1v_1 \dots a_mv_m$ and admit that $a_1a_2 \dots a_{m+1} \leqq v$. Then there is an $a_1a_2 \dots a_{m+1}$-decomposition $w_0a_1w_1 \dots a_{m+1}w_{m+1}$ of v. Let us put $\bar{x}_i = x_0a_1x_1 \dots a_ix_i$ for $x = v, w$ and $i = 0, 1, \dots, m$ and

$$S = \{i \mid |\bar{v}_i| \leqq |\bar{w}_i|\}.$$

(a) $0 \in S$: If $0 \notin S$ then $|w_0| = |\bar{w}_0| < |\bar{v}_0| = |v_0|$. Thus $|w_0a_1| \leqq |v_0|$ and we can find $z \in V^*$ such that $w_0a_1z = v_0$. But then $a_1 \leqq v_0$, a contradiction.

(b) $m \notin S$: $|\bar{w}_m| < |v| = |\bar{v}_m|$.

The statements (a) and (b) say that S is a nonempty subset of $\{0, 1, \dots, m - 1\}$. If we denote by k the greatest integer in S then $|\bar{v}_k| \leqq |\bar{w}_k|$, $|\bar{w}_{k+1}| < |\bar{v}_{k+1}|$. Hence there exist $z_1 \in V^*$, $z_2 \in V^* - \{e\}$ satisfying $\bar{w}_k = \bar{v}_kz_1$, $\bar{v}_{k+1} = \bar{w}_{k+1}z_2$. By this and by $\bar{w}_{k+1} = \bar{w}_ka_{k+1}w_{k+1}$ we obtain

(c) $\bar{v}_{k+1} = \bar{w}_{k+1}z_2 = \bar{w}_ka_{k+1}w_{k+1}z_2 = \bar{v}_kz_1a_{k+1}w_{k+1}z_2$.

Since $|a_{k+2}| \leqq |z_2|$, it holds $|\bar{v}_kz_1a_{k+1}w_{k+1}a_{k+2}| \leqq |\bar{v}_kz_1a_{k+1}w_{k+1}z_2|$. This implies $\bar{v}_kz_1a_{k+1}w_{k+1}a_{k+2}z_3 = \bar{v}_kz_1a_{k+1}w_{k+1}z_2$ for some $z_3 \in V^*$. Then $a_{k+2}z_3 = z_2$ and by this, (c), $\bar{v}_{k+1} = \bar{v}_ka_{k+1}v_{k+1}$ we obtain $z_1a_{k+1}w_{k+1}a_{k+2}z_3 = a_{k+1}v_{k+1}$. As, simultaneously, $|a_{k+1}| \leqq |z_1a_{k+1}|$, there is $z_4 \in V^*$ with the property $a_{k+1}z_4 = z_1a_{k+1}$. But then $a_{k+1}z_4w_{k+1}a_{k+2}z_3 = a_{k+1}v_{k+1}$ implies $z_4w_{k+1}a_{k+2}z_3 = v_{k+1}$ which means $a_{k+2} \leqq v_{k+1}$. This is a contradiction.

2.7. Definition. Suppose that $V \neq \emptyset$ and $G \subseteq V^*$. We say that

(i) G is *locally complete* if $G \cap [u)$ has a least element, which we denote by u_G, for each $u \in V^*$.

(ii) G is *closed under submerging* whenever

$$u_0a_1u_1 \dots a_mu_m \in G, \qquad v_0a_1v_1 \dots a_mv_m \in G \Rightarrow u_0v_0a_1u_1v_1 \dots a_mu_mv_m \in G$$

for arbitrary $m \geqq 0$, $a_1, a_2, \dots, a_m \in V$ and $u_0, v_0, u_1, v_1, \dots, u_m, v_m \in V^*$.

2.8. Lemma. *Suppose that $V \neq \emptyset$, $G \subseteq V^*$ is closed under submerging, $0 < k$, $s_1 \leqq s_2 \leqq \dots \leqq s_k = s$ are integers and $a_1, a_2, \dots, a_{s+1} \in V$. Further, let $u_0^i, u_1^i, \dots, u_{s_i}^i \in V^*$ be such that $u_0^i a_1 u_1^i \dots a_{s_i} u_{s_i}^i \in G$, $u_{s_i+1}^i = \dots = u_s^i = e$ for $i = 1, 2, \dots, k$ and $v_j = u_j^1 u_j^2 \dots u_j^k$ for $j = 0, 1, \dots, s$. Then $v_0 a_1 v_1 \dots a_s v_s \in G$.*

Proof. (a) If $k = 1$ then $v_0 a_1 v_1 \dots a_s v_s = u_0^1 a_1 u_1^1 \dots a_s u_s^1 \in G$.

(b) Assume that $k > 1$ and $v_0' a_1 v_1' \dots a_t v_t' \in G$ for $v_j' = u_j^1 u_j^2 \dots u_j^{k-1}$ and $j = 1, 2, \dots, s_{k-1} = t$. If we put $\bar{u}_t^k = u_t^k a_{t+1} u_{t+1}^k \dots a_s u_s^k$ then also $u_0^k a_1 u_1^k \dots a_t \bar{u}_t^k \in G$ and, as G is closed under submerging, we have $v_0' u_0^k a_1 v_1' u_1^k \dots a_t v_t' \bar{u}_t^k \in G$. But $v_j' u_j^k = v_j$ for $j = 0, 1, \dots, t - 1$ and $v_t' \bar{u}_t^k = v_t' u_t^k a_{t+1} u_{t+1}^k \dots a_s u_s^k = v_t a_{t+1} v_{t+1} \dots a_s v_s$ because $v_t' u_t^k = v_t$ and, regarding $s_j < t + 1$, $u_{t+1}^j = \dots = u_s^j = e$ for $j = 1, 2, \dots, k - 1$. Hence $v_0 a_1 v_1 \dots a_s v_s \in G$.

2.9. Theorem. *Suppose that $V \neq \emptyset$ and $G \subseteq V^*$ is locally complete and closed under submerging. Then*

$$\langle\{(u] \mid u \in G\}\rangle = \{V^*\} \cup \{(F] \mid \emptyset \neq F \subseteq G \text{ and } F \text{ is finite}\}.$$

Proof. Let us denote $C_G = \langle\{(u] \mid u \in G\}\rangle$ and $L_G = \{V^*\} \cup \{(F] \mid \emptyset \neq F \subseteq G$ and F is finite$\}$.

(a) $C_G \subseteq L_G$: If we take an arbitrary $P \in C_G$ then, by 1.8 (i), there is $Q \subseteq G$ such that $P = \bigwedge\{(q] \mid q \in Q\}$. In case $Q = \emptyset$ we have $P = V^* \in L_G$. Otherwise $P = \bigcap\{(q] \mid q \in Q\} = \{u \in V^* \mid u \leqq q$ for all $q \in Q\}$. One can easily see that $(q]$ is finite and $e_G \in (q] \cap G$ for every $q \in Q$. Since, at the same time, $P \subseteq (q]$ for at least one $q \in Q$, we obtain that $F_P = P \cap G$ is a finite nonempty subset of G. The validity of $(F_P] \subseteq P$ is a consequence of $F_P \subseteq P$, $P \in \mathrm{H}(V^*)$. For the proof of the converse inclusion consider $u \in P$ arbitrarily. Since $Q \subseteq G \cap [u)$, we have $u_G \leqq q$ for all $q \in Q$. This and $u_G \in G$ imply $u_G \in F_P$. Then $u \in (u_G] \subseteq (F_P]$.

(b) $L_G \subseteq C_G$: Clearly, $V^* \in C_G$. If $P \in L_G - \{V^*\}$ then there is a finite nonempty set $\{u^1, u^2, \dots, u^k\} \subseteq G$ satisfying $P = \bigcup_{i=1}^{k} (u^i]$. We prove that $P = \bigcap\{(w] \mid w \in W\}$ where

$$W = \{w \mid u^i \leqq w \text{ for } i = 1, 2, \dots, k \text{ and } w \in G\}.$$

The inclusion $P \subseteq \bigcap\{(w] \mid w \in W\}$ being trivial, consider an arbitrary $z = a_1 a_2 \dots a_m \in V^*$ and suppose that $z \notin P$. Then, for $i = 1, 2, \dots, k$, we have $z \nleqq u^i$ which is equivalent to $a_1 a_2 \dots a_{s_i} \leqq u^i$, $a_1 a_2 \dots a_{s_i+1} \nleqq u^i$ for some s_i, $0 \leqq s_i < m$. Without loss of generality we assume that $s_1 \leqq s_2 \leqq \dots \leqq s_k$ and put $s = s_k$. Obviously, there exists such an $a_1 a_2 \dots a_{s_i}$-decomposition $u_0^i a_1 u_1^i \dots a_{s_i} u_{s_i}^i$ of u^i that $a_j \nleqq u_{j-1}^i$ for $j = 1, 2, \dots, s_i$; $a_{s_i+1} \nleqq u_{s_i}^i$ is now a consequence of $a_1 a_2 \dots a_{s_i+1} \nleqq u^i$ for $i = 1, 2, \dots, k$.

Let $u_j^i = e$ for $j = s_i + 1, \dots, s$, $i = 1, 2, \dots, k$ and $v_j = u_j^1 u_j^2 \dots u_j^k$ for $j = 0, 1, \dots, s$. Further, let $v = v_0 a_1 v_1 \dots a_s v_s$. Then $v \in G$ by 2.8 and $u^i \leqq v$ for $i = 1, 2, \dots, k$. Indeed, since $u_j^i \leqq v_j$ for $j = 0, 1, \dots, s$ obviously, we have $u^i =$

$= u_0^i a_1 u_1^i \dots a_{s_i} u_{s_i}^i \leqq v_0 a_1 v_1 \dots a_{s_i} v_{s_i} \leqq v_0 a_1 v_1 \dots a_s v_s = v$. Hence $v \in W$ and, as $a_j \nleqq u_{j-1}^i$ for $i = 1, 2, \dots, k$, we have $a_j \nleqq v_{j-1}$ for all $j \in \{1, 2, \dots, s + 1\}$. But then $a_1 a_2 \dots a_{s+1} \nleqq v$ by 2.6 and we have $z \nleqq v$.

2.10. Definition. If $V \neq \emptyset$ and $a \in V$ then we put

$$V^*a = \{ua \mid u \in V^*\}, \qquad L_a = \langle\{(u] \mid u \in V^*a\}\rangle.$$

2.11. Lemma. *If $V \neq \emptyset$ then V^* and V^*a for every $a \in V$ are locally complete and closed under submerging.*

Proof. V^*a is locally complete for each $a \in V$: Let $u \in V^*$ be arbitrary. In case $u \in V^*a$ we have $u_{V^*a} = u$. If $u \in V^* - V^*a$ then we show $u_{V^*a} = ua$. As $ua \in [u) \cap V^*a$ obviously, consider $v \in [u) \cap V^*a$ arbitrarily. Then there is a u-decomposition $v_0 a_1 v_1 \dots a_m v_m$ of v. It holds $a_m \neq a$ according to $u \notin V^*a$. By this and by $v \in V^*a$ there exists $\bar{v}_m \in V^*$ satisfying $v_m = \bar{v}_m a$. But then $v_0 a_1 v_1 \dots a_m \bar{v}_m ae$ is a ua-decomposition of v so that $ua \leqq v$.

All the remaining statements of this lemma are true trivially.

2.12. Corollary. *If $V \neq \emptyset$ then $\mathrm{N}(V^*, \leqq) = \{V^*\} \cup \{(A] \mid \emptyset \subset A \subseteq V^* \text{ is finite}\}$ and $L_a = \{V^*\} \cup \{(A] \mid \emptyset \subset A \subseteq V^*a \text{ is finite}\}$ for each $a \in V$.*

2.13. Lemma. *If $a, b \in V$, $a \neq b$ and $v \in V^*$ then $v = va \wedge vb$.*

Proof. v is a lower bound of $\{va, vb\}$ obviously. Suppose that $u \leqq va$ and $u \leqq vb$ for some $u \in V^*$ and denote by $v_0 a_1 v_1 \dots a_m v_m$, $v_0' a_1 v_1' \dots a_m v_m'$ the u-decomposition of va, vb, respectively. Since $a \neq b$, either $a_m \neq a$ or $a_m \neq b$ is true. In the first case there is $\bar{v}_m \in V^*$ satisfying $v_m = \bar{v}_m a$ and, clearly, $v_0 a_1 v_1 \dots a_m \bar{v}_m$ is a u-decomposition of v so that $u \leqq v$. In the second case we obtain $u \leqq v$, too.

2.14. Theorem. *For every set V satisfying $\mathrm{card}(V) > 1$ there exists a complete lattice L such that $\mathfrak{C}(L)$ contains M_V.*

Proof. Let us put $L = \mathrm{N}(V^*, \leqq)$, $\iota o = \{V^*\}$, $\iota i = L$ and $\iota x = L_x$ for each $x \in V$.

(a) $L_a \wedge L_b = \{V^*\}$ for arbitrary $a, b \in V$, $a \neq b$: $\{V^*\} \subseteq L_a \wedge L_b$ by 2.12. For the proof of the converse inclusion, consider $P \in L_a - \{V^*\}$ arbitrarily. Then, regarding 2.12, there is a finite nonempty set $F \subseteq V^*a$ with the property $P = (F]$. By this and by the finiteness of principal ideals in V^* we obtain that P is finite and nonempty. Hence P is uniquely determined by the antichain $A \neq \emptyset$ of its maximal elements. It follows immediately by $P = (F]$ that $A \subseteq F \subseteq V^*a$. If we admit $P \in L_b$ then we get $A \subseteq V^*b$ in the same way. But this implies $\emptyset \subset A \subseteq V^*a \cap V^*b$ which is a contradiction.

(b) $L_a \vee L_b = L$ for arbitrary $a, b \in V$, $a \neq b$: Since $L_a \vee L_b \in \mathfrak{C}(L)$ and $L = \mathrm{N}(V^*, \leqq)$, it is sufficient to prove $(u] \in L_a \vee L_b$ for every $u \in V^*$: As $u = ua \wedge ub$ regarding 2.13, we obtain $(u] = (ua] \cap (ub]$. This, $(ua] \in L_a$, $(ub] \in L_b$ and 1.5(i) imply $(u] \in L_a \vee L_b$.

3. COROLLARIES ON LATTICES OF GENERATING SYSTEMS

As it is usual, we write $(A^*)_*$ instead of $\varphi_{\mathbf{N}(P)}(A)$ for arbitrary poset P and $A \in \mathbf{H}(P)$.

3.1. Theorem. *If P is a poset then the following statements are equivalent.*

(i) $\mathrm{Gs}(P)$ *is semimodular.*

(ii) $\mathrm{Gs}(P)$ *is meet infinitely distributive.*

(iii) $\mathfrak{G} \vee \mathfrak{H} = \mathfrak{G} \cup \mathfrak{H}$ *for all* $\mathfrak{G}, \mathfrak{H} \in \mathrm{Gs}(P)$.

(iv) $(A^*)_* - A$ *is a chain in P for each* $A \in \mathbf{H}(P)$.

Proof. Regarding 1.11 we only have to prove that $(A^*)_* - A$ is a chain in $P \Leftrightarrow [A, (A^*)_*]$ is a chain in $\mathbf{H}(P)$ for all posets P and $A \in \mathbf{H}(P)$.

If $[A, (A^*)_*]$ is not a chain then there exist $B, C \in [A, (A^*)_*]$ such that $B \parallel C$. Clearly, there are $b \in B - C$ and $c \in C - B$; but then $b \parallel c$ and $b, c \in (A^*)_* - A$. Conversely, if there exist $b, c \in (A^*)_* - A$ such that $b \parallel c$ then we have $B \parallel C$ and $B, C \in [A, (A^*)_*]$ for $B = A \cup (b]$, $C = A \cup (c]$.

3.2. Theorem. *If* $\mathrm{Gs}(P)$ *is finite then it does not contain* M_3.

Proof. This is a consequence of 2.4.

3.3. Theorem. *For every set V satisfying card$(V) > 1$ there exists a poset P such that* $\mathrm{Gs}(P)$ *contains* M_V.

Proof. If we consider V^* ordered by $\omega = id_{V^*}$ then, evidently, $\mathbf{N}(V^*, \omega) = \{\emptyset, V^*\} \cup \{\{u\} \mid u \in V^*\}$. Using 2.14 (a), (b), one can easily see that $\iota\colon M_V \to \mathrm{Gs}(V^*)$, defined by $\iota o = \mathbf{N}(V^*, \omega)$, $\iota i = \mathbf{N}(V^*, \leqq) \cup \mathbf{N}(V^*, \omega)$ and $\iota x = L_x \cup \mathbf{N}(V^*, \omega)$ for every $x \in V$, is an embedding.

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