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FIXED EDGE THEOREMS FOR COMPLETE LATTICES

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1. INTRODUCTION

In this paper there is introduced a concept of a *fixed edge* for the mapping of a poset into itself. The aim of this paper is to investigate conditions under which a mapping of a poset into itself has a fixed edge. The *Fixed Edge Theorem* is proved: *Every antitone mapping of a complete lattice L into itself has a fixed edge.* The set $E(f)$ of all fixed edges of a mapping f is considered. The partial ordering on the set $E(f)$ is defined and it is shown that this poset forms a complete lattice. We prove generalizations and extensions of these results for a family of mappings. Further the existence of a fixed edge of various mappings on complete lattices is proved. Especially the Fixed Edge Theorem will be extended to include multifunctions on complete lattices. We also extend our results about fixed edges to include mappings which are not antitone in general.

In the following P will denote a partially ordered set (poset) with a partial order $\leqq$. If X is a nonempty subset in a poset P, the least upper bound of a subset X (if exists) is denoted by $\sup X$. Analogously, the term greatest lower bound will be abbreviated to $\inf X$. The least and greatest elements (if exist) are denoted by 0 and 1, respectively. Two elements x and y in a poset P are called comparable if either $x \leqq y$ or $y \leqq x$. For $x \leqq y$ in a poset P, $[x, y]$ denotes an interval $\{z \in P \mid x \leqq z \leqq y\}$. A mapping f of a poset P into the poset Q is called *antitone* (*isotone*) if and only if for all $x, y \in P$, $x \leqq y$ it implies $f(y) \leqq f(x)$ ($f(y) \geqq f(x)$). An element that covers the least element of a lattice L will be referred to as an atom of L. It will be convenient to use the terminology concerning lattices, see [1].

2. A FIXED EDGE THEOREM

Definition. Let f be a mapping of a poset P into itself and let $x \leqq y$ be elements of P. An ordered pair (x, y) is called a *fixed edge of f* if $f(x) = y$ and $f(y) = x$.

Theorem 1 (*Fixed Edge Theorem*). *Let L be a complete lattice and f an antitone mapping of L into itself. Then there exists a fixed edge of f. In particular,* (u, v) *is the fixed edge of f, where*

$$u = \inf\{y \in L \mid y \geqq f^2(y)\}, \qquad v = \sup\{x \in L \mid x \leqq f^2(x)\}$$

and u is the least element in L such that $(u, f(u))$ *is the fixed edge of f.*

Proof. Let $X = \{x \in L \mid x \leqq f^2(x)\}$ and $Y = \{y \in L \mid y \geqq f^2(y)\}$. The set $X(Y)$ is nonempty since it contains an element $0(1)$. Let $u = \inf Y$ and $v = \sup X$. According to Tarski's theorem [5, Theorem 1] we can see that $u = f^2(u)$ and $v = f^2(v)$, since f^2 is an isotone mapping. Hence $v \in Y$ and therefore $u \leqq v$. The last equations imply that $f(u) = f^2(f(u))$ and $f(v) = f^2(f(v))$, whence $f(u) \in X$ and $f(v) \in Y$. Hence $f(u) \leqq v$ and $u \leqq f(v)$. On the other hand, we have $f(v) \leqq f^2(u) = u$, and $f^2(v) = v \leqq f(u)$ since f is antitone. It implies $u = f(v)$ and $v = f(u)$, i.e. (u, v) is a fixed edge of f. If (x, y) is any fixed edge of f then $x \in Y$. Hence $u \leqq x$, which completes the proof.

Theorem 2. *Let L be a complete lattice and f an antitone mapping of L into itself. Then there exists a maximal element p in L such that* $(p, f(p))$ *is a fixed edge of f.*

Proof. Let $X = \{x \in L \mid x \leqq f^2(x) \leqq f(x)\}$. Clearly X is nonempty since $0 \in X$. Let C be a maximal chain in the set X and let $p = \sup C$. If $x, y \in C$, then x and y are comparable since C is a chain. If $x \leqq y$, then by antitonicity of f we have $x \leqq y \leqq f(y) \leqq f(x)$ since $C \subseteq X$. If $y \leqq x$, then we analogously obtain $y \leqq x \leqq f(x) \leqq f(y)$. Hence $x \leqq f(y)$ for each $x, y \in C$ and it implies $p \leqq f(x)$ for all $x \in C$. As f is antitone, we have $x \leqq f^2(x) \leqq f(p)$ for all $x \in C$, and hence $p \leqq f(p)$. Further from $p \leqq f(p) \leqq f(x)$ for all $x \in C$, we have $x \leqq f^2(x) \leqq f^2(p) \leqq f(p)$ for all $x \in C$. Hence $p \leqq f^2(p) \leqq f(p)$ and since f^2 is isotone $f^2(p) \leqq f^2(f^2(p)) \leqq f^3(p)$. Thus p and $f^2(p)$ belong to X. Since $p \leqq f^2(p)$, then $C \cup \{f^2(p)\}$ is a chain, whence by the maximality of C, $f^2(p) \in C$. This yields $f^2(p) \leqq p$ and hence $p = f^2(p)$. We have proved that $(p, f(p))$ is the fixed edge and p is the maximal element in X. If (r, s) is another fixed edge of f and $p < r$, then r belongs to X and the chain $C \cup \{r\}$ properly contains C, contradicting the maximality of C. The proof of the theorem is thereby complete.

Remark. Theorem 2 is a generalisation of a result described in [3]. The greatest element x in L, such that $(x, f(x))$ is the fixed edge of f, need not exist. It can be demonstrated by the following.

Example. Assume that $L = \{o, a, b, c, i\}$ is the non-modular pentagon, where $b < a$, $\sup\{c, b\} = i$, $\inf\{a, c\} = o$ and define an antitone mapping f of L into itself as follows: $f(o) = i$, $f(i) = o$, $f(b) = a$, $f(a) = b$ and $f(c) = c$.

3. COMMON FIXED EDGES

Definition. Let F be a family of mappings of a poset P into itself and let $x \leqq y$ be elements of P. An ordered pair (x, y) is said to be a *common fixed edge of a family* F if $f(x) = y$ and $f(y) = x$ for all $f \in F$.

Speaking about *commuting family* of mappings we mean that composition is commutative. The lattice L is said to be *atomic* if it has the least element 0 and the sublattice $[0, a]$ contains an atom for each element $a > 0$. Throughout this section let F be a family of single valued mappings.

Theorem 3 (*Generalized Fixed Edge Theorem*). *Let L be a complete lattice and F a commuting family of antitone mappings of L into itself. Then F has a common fixed edge. In particular, (u, v) is a common fixed edge of F, where*

$$u = \inf \{y \in L \mid y \geqq fg(y) \text{ for all } f, g \in F\},$$

$$v = \sup \{x \in L \mid x \leqq fg(x) \text{ for all } f, g \in F\},$$

and u is the least element in L such that (u, v), $v = f(u)$ for all $f \in F$ is a common fixed edge of F.

Proof. Let $X = \{x \in L \mid x \leqq fg(x)$ for all $f, g \in F\}$ and $Y = \{y \in L \mid y \geqq fg(y)$ for all $f, g \in F\}$. Clearly $0(1)$ belongs to $X(Y)$ and hence X and Y are nonempty. Let $u = \inf Y$, $v = \sup X$. According to Tarski's theorem [5, Theorem 2] we can see that $u = fg(u)$, $v = fg(v)$ for all $f, g \in F$, which implies $u \in X$ and therefore $u \leqq v$. Hence $h(u) = hfg(u)$, $h(v) = hfg(v)$ for all $f, g, h \in F$. Using commutativity of F, we get $h(u) = fgh(u)$, $h(v) = fgh(v)$ proving that $h(u) \in X$, $h(v) \in Y$ for all $h \in F$. Then since any g of F is antitone, we have $u \leqq g(v) \leqq g(u) \leqq v$ for all $g \in F$. On the other hand, using antitonicity of $f \in F$ we get $f(v) \leqq fg(u) = u \leqq fg(v) = v \leqq f(u)$ for all $f, g \in F$. Because of antisymmetry, $v = f(u)$ and $u = f(v)$ for all $f \in F$. Then it follows that (u, v) is a common fixed edge of F. If (x, y) is any common fixed edge of F, then x belongs to Y, hence $u \leqq x$, completing the proof.

Theorem 4. *Let L be a complete lattice and F a commuting family of antitone mappings of L into itself. Then there exists a maximal element p in L such that (p, q), $q = f(p)$ for all $f \in F$, is a common fixed edge of F.*

Proof. Let $X = \{x \in L \mid x \leqq fg(x) \leqq f(x)$ for all $f, g \in F\}$. Clearly X is nonempty since $0 \in X$. Let C be a maximal chain in X and let $p = \sup C$. Next it will be shown that $p \in X$. Let $x, y \in C$; since C is a chain, we infer that either $x \leqq y$ or $y \leqq x$. If the first alternative holds, then the fact that $x, y \in X$ and antitonicity of $f \in F$ implies that $x \leqq y \leqq f(y) \leqq f(x)$. The second alternative, analogously, implies that $y \leqq$ $\leqq x \leqq f(x) \leqq f(y)$. Hence $x \leqq f(y)$ for all $x, y \in C$ and $f \in F$ and it implies $p \leqq f(x)$ for all $x \in C$ and $f \in F$. Since $g \in F$ is antitone we have $x \leqq gf(x) \leqq g(p)$ for all $x \in C$, $f, g \in F$ and hence $p \leqq g(p)$ for all $g \in F$. Furthermore, the inequality $p \leqq$

$\leqq g(p) \leqq g(x)$ for all $x \in C$, $g \in F$ implies that $x \leqq fg(x) \leqq fg(p) \leqq f(p)$ for all $x \in C, f, g \in F$. Hence p belongs to X.

To prove that $p = fg(p)$ for all $f, g \in F$, assume that there are $f_0, g_0 \in F$ such that $p \neq f_0g_0(p)$. Let $f_0g_0(p) = p_0$. Also from the fact that $p \leqq fg(p) \leqq f(p)$ for all $f, g \in F$ and hk is isotone for all $h, k \in F$, we get that $hk(p) \leqq hkfg(p) \leqq hkf(p)$ for all $f, g, h, k \in F$. Replacing now h by f_0 and k by g_0, interchanging hk and fg (hk and f), we get $p_0 \leqq fg(p_0) \leqq f(p_0)$ for all $f, g \in F$, because F is a commuting family. Hence $p_0 \in X$ and $p < p_0$. Then $C \cup \{p_0\}$ is a chain in X properly containing C, contradicting the maximality of C. This contradiction shows that $p = fg(p)$ for all $f, g \in F$. Hence we get $h(p) = hfg(p) = f(hg(p)) = f(p)$ for all $f, h \in F$. This means that all elements $f(p)$, $f \in F$, are equal to the same element, say q, and we conclude that (p, q) is a common fixed edge of F. Suppose (r, s) is another common fixed edge of F and suppose $p < r$. Then r belongs to the set X and the chain $C \cup \{r\}$ properly contains C. Again, this contradicts the maximality of C, which completes the proof.

Remark. Theorem 1 (Theorem 2) is a collary of Theorem 3 (Theorem 4), since every set consisting of a single mapping is obviously commutative. Obviously, the dual Theorem 3 and Theorem 4 – replacing the least by the greatest and the maximal by the minimal – is also true.

4. STRUCTURE OF THE SET OF ALL FIXED EDGES

Let P be a poset and F a family of mappings of P into itself. By $E(F)$ we denote the set of all common fixed edges of F. Partially order $E(F)$ by defining $(a, b) \leqq \leqq (c, d)$ if $c \leqq a$ and $b \leqq d$. Adjoin a new element Θ to $E(F)$ in such a way that Θ is the least element of the poset $E(F)$.

Notation. If F is a one-element family containing a mapping f then we will use $E(f)$ instead of $E(F)$.

Theorem 5. *Let L be a complete lattice and F a commuting family of antitone mappings of L into itself. Then $E(F)$ is a complete and atomic lattice.*

Proof. A. *Completeness.* It is evident that the fixed edge (u, v) from the Theorem 3 is the greatest element of $E(F)$. Let (x_i, y_i) be a common fixed edge of F for each $i \in I$ and set $Z = \bigcap_{i \in I}[x_i, y_i]$. If Z is an empty set then Θ is the infimum of $\{(x_i, y_i)\}_{i \in I}$ in $E(F)$. In the opposite case denote $x = \sup \{x_i\}_{i \in I}$ and $y = \inf \{y_i\}_{i \in I}$. Since each element of Z is an upper bound for $\{x_i\}_{i \in I}$ and a lower bound for $\{y_i\}_{i \in I}$, $x_i \leqq y \leqq y_i$ for all $i \in I$. It implies $f(y_i) = x_i \leqq f(y) \leqq f(x) \leqq y_i = f(x_i)$ for all $i \in I$ and $f \in F$. This implies that f maps $[x, y]$ into itself for all $f \in F$. If we denote the restriction of f to $[x, y]$ by f' for each $f \in F$, then $F' = \{f' \mid f \in F\}$ is a commuting family of antitone mappings of the complete lattice $[x, y]$ into itself. Applying the Theorem 3 to the complete lattice $[x, y]$ and to the family F', we get a common fixed edge (u', v').

According to Theorem 3 and its dual an element $u'(v')$ is the least (greatest) element in $[x, y]$ such that (u', v') is the common fixed edge of F. It is evident that (u', v') is the infimum of $\{(x_i, y_i)\}_{i \in I}$ in $E(F)$.

B. *Atomicity.* Choose any nonzero element (x, y) in $E(F)$. It is easy to see that each $f \in F$ maps $[x, y]$ into itself. Then the restriction of f to the interval $[x, y]$ is the antitone mapping of the complete lattice $[x, y]$ into itself for all $f \in F$. Applying the Theorem 4 and its dual to the complete lattice $[x, y]$ and to the partial mappings of F, we get a maximal element p and a minimal element q in $[x, y]$ such that (p, q) is a common fixed edge. Now $\Theta \leqq (p, q) \leqq (x, y)$, and the maximality of p and the minimality of q ensure that there is no element (r, s) in $E(F)$ with $\Theta < (r, s) < (p, q)$. Thus every nonzero element in $E(F)$ contains at least one element covering Θ. The proof of the theorem is complete.

Corollary 6. *Let L be a complete lattice and f an antitone mapping of L into itself. Then $E(f)$ is a complete atomic lattice.*

Proof follows immediately from Theorem 5.

5. FIXED EDGES OF MULTIFUNCTIONS

In this section we prove an extension of the Fixed Edge Theorem to include multifunctions. A *multivalued function* F on a set X into a set Y is a point to set correspondence, i.e. $F(x)$ is a nonempty subset of Y for each $x \in X$. The term *multifunction* is used as a contraction of multivalued function. Of course any single valued mapping is a multifunction. Now we generalize the concept of antitone mapping and fixed edge. Observe that for single valued mappings the following definitions are reduced to the prior ones.

We say that a *multifunction* F on a poset P into a poset Q is *antitone* if $x_1, x_2 \in P$, $x_1 \leqq x_2$ implies $y_2 \leqq y_1$ for all $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$. Let F be a multifunction on a poset P and let $x \leqq y$ be elements in P. An ordered pair (x, y) is called a *fixed edge of F* if $y \in F(x)$ and $x \in F(y)$. A family $\mathscr{F}$ of multifunctions is commuting in case $FG = GF$ for all $F, G \in \mathscr{F}$ where $FG(x) = F(G(x)) = \bigcup \{F(y) \mid y \in G(x)\}$.

Theorem 7. *Let L be a complete lattice and F an antitone multifunction of L into itself such that* $\sup F(x) \in F(x)$ *for all $x \in L$. Then there exists a fixed edge of F.*

Proof. We define a single valued mapping f from L to L by setting $f(x) = \sup F(x)$ for all $x \in L$. Clearly f is well-defined, since $\sup F(x)$ exists for each $x \in L$. To show that f is an antitone mapping, let x_1, x_2 be in L with $x_1 \leqq x_2$. Then $y_2 \leqq \sup F(x_1)$ for all $y_2 \in F(x_2)$ and hence $f(x_2) \leqq f(x_1)$, since $\sup F(x_2) \in F(x_2)$. Thus f is antitone mapping of L into itself. According to Theorem 1 there exist $u, v \in L$, $u \leqq v$ such that $u = f(v)$, $v = f(u)$. But this implies that $u \in F(v)$ and $v \in F(u)$, which completes the proof.

Theorem 8. *Let L be a complete lattice and $\mathscr{F}$ a commuting family of antitone multifunctions on L. If each $F \in \mathscr{F}$ satisfies* $\sup F(x) \in F(x)$ *for each $x \in L$, then there is a common fixed edge for the members of $\mathscr{F}$.*

Proof. For each $F \in \mathscr{F}$ we define a single valued mapping f as follows. For each $x \in L$, let $f(x) = \sup F(x)$. f be clearly well-defined, antitone single valued mapping of L into itself for all $F \in \mathscr{F}$. Let $\mathscr{F}_0 = \{f \mid f$ be single valued mapping associated with $F, F \in \mathscr{F}\}$. We need only show that $\mathscr{F}_0$ is a commuting family. Let $f, g \in \mathscr{F}_0$ where $f(x) = \sup F(x)$ and $g(x) = \sup G(x)$ for all $x \in L$. Then $f(g(x)) \in F(G(x)) = G(F(x))$. Thus there exists $y \in F(x)$ such that $f(g(x)) \in G(y)$. Further, $y \leqq f(x) = \sup F(x)$, and by antitonicity of G we have for each $z \in G(f(x))$, $z \leqq f(g(x))$. Therefore $g(f(x)) \leqq f(g(x))$. We also get $f(g(x)) \leqq g(f(x))$ by a similar argument. Thus $fg = gf$. Then we apply the Theorem 3 to obtain the elements $u, v \in L$, $u \leqq v$ such that $u = f(v)$, $v = f(u)$ for all $f \in \mathscr{F}_0$. Hence we get $u \in F(v)$ and $v \in F(u)$ for all $F \in \mathscr{F}$. This completes the proof of the theorem.

6. FIXED EDGES OF WEAKLY ANTITONE MAPPINGS

Definition. A mapping f from a poset P into a poset Q is called *weakly antitone* if $x, y \in P$, $x \leqq y$ implies $f(y) \leqq f(x)$ whenever $x \leqq f(x)$ or $f(y) \leqq y$.

It is clear that each antitone mapping is weakly antitone. But there are weakly antitone mappings which are not antitone. Let us consider, for instance, a four-element chain $\{1, 2, 3, 4\}$ with usually ordering and let us suppouse $f(1) = f(3) = 4$. $f(2) = f(4) = 1$.

Theorem 9. *Let L be a complete lattice and f a weakly antitone mapping of L into itself. Then:*

(i) *there exists a maximal element p in L such that $(p, f(p))$ is a fixed edge of f;*

(ii) *there exists the least element u in L such that $(u, f(u))$ is a fixed edge of f;*

(iii) *$E(f)$ is a complete and atomic lattice.*

Proof. (i) Let $X = \{x \in L \mid x \leqq f^2(x) \leqq f(x)\}$. Clearly 0 is in X and therefore X is nonempty. Let C be a maximal chain in X and let $p = \sup C$. For each x in C we have $x \leqq p$, $x \leqq f(x)$ and hence $f(p) \leqq f(x)$ for all $x \in C$. This last relation together with $f^2(x) \leqq f(x)$ imply that $f^2(x) \leqq f^2(p)$ for all $x \in C$, showing that $f^2(p)$ is an upper bound of C. Hence $p \leqq f^2(p)$. Let $x, y \in C$. If $x \leqq y$ then $x \leqq f(y)$, since f is weakly antitone mapping and $x \leqq f(x)$. If $y \leqq x$ then also $x \leqq f(y)$, by a similar argument. Thus $x \leqq f(y)$ for all $x, y \in C$. Hence $p \leqq f(x)$ for all $x \in C$, moreover $f^2(x) \leqq f(x)$, and this implies that $f^2(x) \leqq f(p)$ for all $x \in C$. Consequently, $f(p)$ is an upper bound of C, i.e. $p \leqq f(p)$ and it implies $f^2(p) \leqq f(p)$. This means that p belongs to X. Next it will be shown that also $f^2(p)$ belongs to X. The inequality $p \leqq f^2(p)$ together with $p \leqq f(p)$ yield that $f^3(p) \leqq f(p)$. This implies that $f^2(p) \leqq$

$\leqq f^4(p)$, since $f^2(p) \leqq f(p)$. Further the inequality $f^2(p) \leqq f(p)$, together with the assumption on f yield that $f^2(p) \leqq f^3(p)$ and this implies $f^4(p) \leqq f^3(p)$. Hence $f^2(p)$ belongs to X. If $p < f^2(p)$, then the chain C is not maximal, contrary to assumption. Therefore $p = f^2(p)$ and $(p, f(p))$ is a fixed edge of f. Suppose (r, s) is another fixed edge of f and suppose $p < r$. Then r belongs to the set X and the chain $C \cup \{r\}$ properly contains C, which gives a contradiction.

(ii) Let $S = \{x \in L \mid x \leqq f^2(x) \leqq f(x),\ x \leqq y$ for all $y \in L$ such that $(y, f(y))$ is a fixed edge of $f\}$. $0 \in S$. Let C be a maximal chain in S and let $\sup C = u$. Then the proof can be modelled on the proof used in (i) – replacing the set X by the set S.

The proof of (iii) can be modelled on the proof used in the Theorem 5.

Theorem 10. *Let L be a complete lattice and F a commuting family of weakly antitone mappings of L into itself. Then:*

(i) *there exists a maximal element p in L such that (p, q), $q = f(p)$ for all $f \in F$, is a common fixed edge of F;*

(ii) *there exists the least element u in L such that (u, v), $v = f(u)$ for all $f \in F$, is a common fixed edge of F;*

(iii) *$E(F)$ is a complete and atomic lattice.*

Proof. In fact, the proof of the theorem can be modelled on an idea used by the proofs of Theorem 9 and Theorem 4. We leave the details to the reader.

Remark. McShane [2] has introduced the following concept of *Dedekind completeness* for posets which is a generalization of the concept of completeness for lattices.

A poset P is called *Dedekind complete* if every up-directed subset of P has a sup in P and every down-directed subset has an inf in P.

Recall that a subset S of a poset P is said to be *up-directed* (*down-directed*) if for all $x \in S$ and $y \in S$ there exists $z \in S$ such that $x \leqq z$, $y \leqq z$ ($z \leqq x$, $z \leqq y$). For properties of Dedekind complete posets, the reader is referred to [6].

It is not difficult to see that all results of this section are true for Dedekind complete posets with 0 and 1.

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