Archivum Mathematicum

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Archivum Mathematicum, Vol. 17 (1981), No. 4, 191--206

Persistent URL: `http://dml.cz/dmlcz/107111`

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVII: 191—206, 1981

ASYMPTOTIC BEHAVIOUR OF EQUATIONS $\dot{z} = q(t, z) - p(t) z^2$ AND $\ddot{x} = x\varphi(t, \dot{x}x^{-1})$

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(Received December 10, 1980)

1. INTRODUCTION

During the last few years, a good deal of research activity has been concentrated on the investigation of the asymptotic behaviour of the solutions of an equation

$$(1) \qquad \dot{z} = f(t, z),$$

where f is a complex-valued function of a real variable t and a complex variable z. The global asymptotic properties of the Riccati equation

$$(2) \qquad \dot{z} = q(t) - p(t) z^2$$

are described in detail by M. Ráb in papers [5], [6]. Papers [1], [2], [3], [4] contain a considerable amount of results related to the equation

$$(3) \qquad \dot{z} = G(t, z) [h(z) + g(t, z)],$$

where G is a real-valued function and h, g are complex-valued functions, h being holomorphic in certain simply-connected region Ω. By virtue of general results referring to the equation (3), in [1], [2], [3] there are derived several results dealing with the global asymptotic character of the equation

$$(4) \qquad \dot{z} = q(t, z) - p(t) z^2.$$

The technique of the proofs of the majority of these results is based on the Liapunov function method and the Ważewski topological principle. Considering the associated Riccati differential equation and using the above methods, M. Ráb studies the asymptotic nature of the solutions of the linear second order differential equation

$$(5) \qquad \ddot{x} + p(t) \dot{x} + q(t) x = 0$$

with complex-valued coefficients p, q in paper [7].

The purpose of the present paper is to generalize the results concerning the asymptotic behaviour of the solutions of (4) and to extend some results of [7] to an equation

$$\ddot{x} = x\varphi(t, \dot{x}x^{-1}), \tag{6}$$

where $\varphi(t, z)$ is a continuous complex-valued function defined for all real numbers t and all complex numbers z. In what follows we use the notation from [1] (see also [2], [3], [4]). In particular, $\boldsymbol{C}$ denotes the set of all complex numbers, $\boldsymbol{N}$ the set of positive integers, I the interval $[t_0, \infty)$, $C(\Gamma)$ the class of all continuous real-valued functions defined on the set Γ, and $\bar{C}(\Gamma)$ the class of all continuous complex-valued functions defined on the set Γ. By $\bar{C}^1(I)$ we denote the class of all continuously differentiable complex-valued functions defined on I.

For brevity, we shall omit sometimes the independent variable, writing e.g. α instead of $\alpha(t)$ etc. Throughout the paper we shall assume that $q \in \bar{C}(I \times \boldsymbol{C})$, $p \in \bar{C}(I)$.

2. PRELIMINARY RESULTS

Suppose that $\alpha(t), \beta(t) \in \bar{C}^1(I)$, $\varrho(t) \in \bar{C}(I)$ and that $\beta(t) \neq 0$ for $t \in I$. The following lemma can be easily verified and therefore its proof is omitted.

Lemma. *Put*

$$\begin{aligned} p &= \beta^{-1} + \varrho, \\ q(t, z) &= \beta\varphi(t, (z + \alpha)\beta^{-1}) + \varrho z^2 + (\dot{\beta} - 2\alpha)\beta^{-1}z + (\dot{\beta} - \alpha)\alpha\beta^{-1} - \dot{\alpha}. \end{aligned} \tag{7}$$

i) *A function $z(t)$ is a solution of* (4) *defined on an interval $J \subset I$, if and only if,*

$$z(t) = \beta(t)\dot{x}(t)x^{-1}(t) - \alpha(t),$$

where $x(t)$ is a solution of (6) *on J.*

ii) *A function $x(t)$ is a solution of* (6) *defined on $J \subset I$, if and only if,*

$$x(t) = \Theta \exp\left[\int_\omega^t [z(s) + \alpha(s)]\beta^{-1}(s)\,ds\right],$$

where Θ is a constant different from zero, $\omega \in J$, and $z(t)$ is a solution of (4) *on J.*

In view of Lemma we shall obtain the results concerning the asymptotic behaviour of the solutions of (6) as the immediate consequences of the results referring to the solutions of the equation (4). If $a, b \in \boldsymbol{C}$, $b \neq 0$, $\psi \in C(I)$ and $\psi(t) > 0$ for $t \geqq t_0$, then (4) may be written in the form

$$\begin{aligned} \dot{z} = \psi(t)\{-2\bar{b}[(z - a)^2 - b^2] + q(t, z)\psi^{-1}(t) - p(t)\psi^{-1}(t)z^2 + \\ + 2\bar{b}[(z - a)^2 - b^2]\}. \end{aligned} \tag{8}$$

Substituting $z_1 = z - a - b$ or $z_2 = z - a + b$, we get

$$(9_1) \qquad \dot{z}_1 = G_1(t, z_1)\,[h_1(z_1) + g_1(t, z_1)]$$

or

$$(9_2) \qquad \dot{z}_2 = G_2(t, z_2)\,[h_2(z_2) + g_2(t, z_2)],$$

respectively, where

$$G_1(t, z_1) = G_2(t, z_2) = \psi(t),$$
$$h_1(z_1) = -2\bar{b}z_1(z_1 + 2b), \qquad h_2(z_2) = -2\bar{b}z_2(z_2 - 2b),$$
$$g_1(t, z_1) = q(t, z_1 + a + b)\,\psi^{-1}(t) - p(t)\,\psi^{-1}(t)\,(z_1 + a + b)^2 + 2\bar{b}z_1(z_1 + 2b),$$
$$g_2(t, z_2) = q(t, z_2 + a - b)\,\psi^{-1}(t) - p(t)\,\psi^{-1}(t)\,(z_2 + a - b)^2 + 2\bar{b}z_2(z_2 - 2b).$$

Put

$$\Omega_1 = \{z_1 \in C\colon \operatorname{Re}[\bar{b}z_1] > -|b|^2\},$$
$$\Omega_2 = \{z_2 \in C\colon \operatorname{Re}[\bar{b}z_2] < |b|^2\}.$$

I. First we shall consider the equation (9_1) on the set $I \times \Omega_1$. $W(z)$, λ_0, $K(\lambda_0)$ and $\hat{K}(\lambda)$ from [1] (see also [2], [3], [4]) are of the following form:

$$W(z_1) = 2\,|b|\,|z_1|\,|z_1 + 2b|^{-1}, \qquad \lambda_0 = 2\,|b|,$$
$$K(\lambda_0) = \Omega_1, \qquad \hat{K}(\lambda) = \{z_1 \in \Omega_1 : 2\,|b|\,|z_1| = \lambda\,|z_1 + 2b|\}.$$

For $t \geqq t_0$, $z_1 \in \Omega_1$, we get

$$\operatorname{Re}\left\{h_1'(0)\left[1 + \frac{g_1(t, z_1)}{h_1(z_1)}\right]\right\} = \operatorname{Re}\left[g_1(t, z_1)\frac{h_1'(0)}{h_1(z_1)}\right] - 4\,|b|^2 =$$
$$= \psi^{-1}(t)\operatorname{Re}\left\{[q(t, z_1 + a + b) + (a^2 - b^2)\,p(t) - 4p(t)\,(z_1 + a + b)\frac{ab}{z_1(z_1 + 2b)}\right\} -$$
$$- 2\psi^{-1}(t)\operatorname{Re}[bp(t)].$$

Suppose there are $H_1, H_2 \in C(I)$ such that

$$|\,q(t, z_1 + a + b) + (a^2 - b^2)\,p(t) - 2ap(t)\,(z_1 + a + b)\,| \leqq |\,z_1 + b\,|\,H_1(t) + H_2(t)$$

for $t \geqq t_0$, $z_1 \in \Omega_1$. It is clear that H_1, H_2 must be nonnegative.

1° Assume that

$$(10) \qquad \operatorname{Re}[bp(t)] > 0 \quad \text{for } t \geqq t_0,$$

$$(11) \qquad \int_{t_0}^{\infty} \operatorname{Re}[bp(t)]\,\mathrm{d}t = \infty$$

and

$$(12) \qquad \sup_{t \geqq t_0} \frac{|b|\,H_1(t) + 2H_2(t)}{\operatorname{Re}[bp(t)]} < 2\,|b|.$$

f $\delta \leqq 2\,|b|$ is defined by

$$\text{(13)} \qquad \sup_{t \geqq t_0} \frac{|b| H_1(t) + 2H_2(t)}{\operatorname{Re}[bp(t)]} = \frac{8\delta |b|^2}{4|b|^2 + \delta^2},$$

then $0 \leqq \delta < 2|b| = \lambda_0$. Put $\psi(t) \equiv 1$. We have

$$\operatorname{Re}\left\{h_1'(0)\left[1 + \frac{g_1(t, z_1)}{h_1(z_1)}\right]\right\} \leqq [|z_1 + b| H_1(t) + H_2(t)] \frac{2|b|}{|z_1| \, |z_1 + 2b|} -$$

$$- 2 \operatorname{Re}[bp(t)] =$$

$$= [|z_1 + b| H_1(t) + H_2(t)] \frac{W^2(z_1) + 4|b|^2}{2W(z_1)[|z_1 + b|^2 + |b|^2]} - 2 \operatorname{Re}[bp(t)].$$

Denote $\delta_n = [2|b| + (2n - 1)\delta](2n)^{-1}$ for $n \in N$ and choose $\xi_n > 1$ so that

$$\text{(14)} \qquad \sup_{t \geqq t_0} \frac{\xi_n[|b| H_1(t) + 2H_2(t)]}{\operatorname{Re}[bp(t)]} \leqq \frac{8\delta_n |b|^2}{4|b|^2 + \delta_n^2}.$$

It can be easily verified that there are constants $\mu_n, \nu_n \in (0, 1)$ such that

$$\frac{2|b| \, |z_1 + b|}{|z_1 + b|^2 + |b|^2} \leqq \mu_n \xi_n, \qquad \frac{|b|^2}{|z_1 + b|^2 + |b|^2} \leqq \nu_n \xi_n$$

for $z_1 \in K(\delta_n, \lambda_0) = \{z_1 \in \Omega_1 : \delta_n < 2|b| \, |z_1| \, |z_1 + 2b|^{-1} < \lambda_0\}$, $n \in N$. Therefore

$$\operatorname{Re}\left\{h_1'(0)\left[1 + \frac{g_1(t, z_1)}{h_1(z_1)}\right]\right\} \leqq$$

$$\leqq \xi_n[\mu_n |b| H_1(t) + 2\nu_n H_2(t)] \frac{W^2(z_1) + 4|b|^2}{4|b|^2 W(z_1)} - 2 \operatorname{Re}[bp(t)] \leqq$$

$$\leqq \xi_n[|b| H_1(t) + 2H_2(t)] \frac{W^2(z_1) + 4|b|^2}{4|b|^2 W(z_1)} \max(\mu_n, \nu_n) - 2 \operatorname{Re}[bp(t)]$$

for $t \geqq t_0$, $z_1 \in K(\delta_n, \lambda_0)$, $n \in N$. Making use of (14) we get

$$\operatorname{Re}\left\{h_1'(0)\left[1 + \frac{g_1(t, z_1)}{h_1(z_1)}\right]\right\} \leqq$$

$$\leqq 2\left[\frac{8\delta_n |b|^2 (W^2(z_1) + 4|b|^2)}{(4|b|^2 + \delta_n^2) 8|b|^2 W(z_1)} \max(\mu_n, \nu_n) - 1\right] \operatorname{Re}[bp(t)] \leqq$$

$$\leqq 2[\max(\mu_n, \nu_n) - 1] \operatorname{Re}[bp(t)].$$

Now, we can apply Theorem 2.3 and Theorem 2.4 of [1], where $\vartheta = \lambda_0$, $s_n = t_0$, $G(t, z) \equiv 1$, $E_n(t) = 2[\max(\mu_n, \nu_n) - 1] \operatorname{Re}[bp(t)]$ (see also Theorem 3.5 and Theorem 3.6 of [4]), thus we obtain the following statement:

If a solution $z_1(t)$ of (9_1) satisfies $\operatorname{Re}[\bar{b} z_1(t_1)] > -|b|^2$, *where $t_1 \geqq t_0$, then to ny $\varepsilon > \delta$ there is a $T > 0$ such that $2|b| \, |z_1(t)| < \varepsilon |z_1(t) + 2b|$ for $t \geqq t_1 + T$.*
a

If, in addition,

$$\int_s^{s+t} \operatorname{Re}\,[bp(\tau)]\,\mathrm{d}\tau \to \infty \qquad \textit{as } t \to \infty$$

uniformly for $s \in I$, then T is independent of t_1 and of $z_1(t)$.

2° Suppose that (10), (11) hold and that

$$\text{(15)} \qquad \int_{t_0}^{\infty} H_1(t)\,\mathrm{d}t < \infty, \qquad \int_{t_0}^{\infty} H_2(t)\,\mathrm{d}t < \infty.$$

Let $s_n \geqq t_0$ be such that

$$\int_{s_n}^{\infty} [\,|\,b\,|\,H_1(t) + H_2(t)]\,\mathrm{d}t < |\,b\,|\,(2ne)^{-1}, \qquad n \in N.$$

Put $\psi(t) \equiv 1$ and $\delta_n = 2\,|\,b\,|\,(ne)^{-1}$ for $n \in N$. Then

$$\operatorname{Re}\left\{h_1'(0)\left[1 + \frac{g_1(t, z_1)}{h_1(z_1)}\right]\right\} \leqq$$

$$\leqq 2\,|\,b\,|\,\frac{|\,z_1 + b\,|\,H_1(t) + H_2(t)}{|\,z_1\,|\,|\,z_1 + 2b\,|} - 2\operatorname{Re}\,[bp(t)] \leqq$$

$$\leqq |\,b\,|\left[\frac{H_1(t)}{|\,z_1 + 2b\,|} + \frac{H_1(t)}{|\,z_1\,|} + \frac{2H_2(t)}{|\,z_1\,|\,|\,z_1 + 2b\,|}\right] - 2\operatorname{Re}\,[bp(t)] \leqq$$

$$\leqq |\,b\,|\left[\frac{H_1(t)}{|\,b\,|} + \frac{2\,|\,b\,| + \delta_n}{2\,|\,b\,|\,\delta_n}H_1(t) + \frac{2\,|\,b\,| + \delta_n}{|\,b\,|^2\,\delta_n}H_2(t)\right] - 2\operatorname{Re}\,[bp(t)] \leqq$$

$$\leqq \frac{4}{\delta_n}[\,|\,b\,|\,H_1(t) + H_2(t)] - 2\operatorname{Re}\,[bp(t)]$$

for $t \geqq s_n$, $z_1 \in K(\delta_n, \lambda_0)$, $n \in N$. Using Theorem 2.3 and Theorem 2.4 of [1], where $\vartheta = \lambda_0$, $G(t, z) \equiv 1$, $E_n(t) = 4[\,|\,b\,|\,H_1(t) + H_2(t)]/\delta_n - 2\operatorname{Re}\,[bp(t)]$, we get the assertion:

If a solution $z_1(t)$ of (9_1) satisfies

$$|\,z_1(t_1)\,| < \exp\left\{-\frac{2e}{|\,b\,|}\int_{s_1}^{\infty}[\,|\,b\,|\,H_1(t) + H_2(t)]\,\mathrm{d}t\right\}|\,z_1(t_1) + 2b\,|,$$

where $t_1 \geqq s_1$, then

$$\lim_{t\to\infty} z_1(t) = 0.$$

If, in addition,

$$\int_s^{s+t} \operatorname{Re}\,[bp(\tau)]\,\mathrm{d}\tau \to \infty \qquad \textit{as } t \to \infty$$

uniformly for $s \in I$, then to any $\varepsilon > 0$ there is a $T > 0$ independent of t_1 and of $z_1(t)$ such that $|\,z_1(t)\,| < \varepsilon$ for $t \geqq t_1 + T$.

3° Suppose there is a $\varkappa > 0$ such that

$$\operatorname{Re}[bp(t)] \geqq \varkappa \qquad \text{for } t \geqq t_0$$

and assume that

$$\lim_{t\to\infty} \int_t^{t+1} H_1(s)\,ds = \lim_{t\to\infty} \int_t^{t+1} H_2(s)\,ds = 0.$$

Put $\psi(t) \equiv 1$ and $\delta_n = 2\,|\,b\,|/(n+1)$ for $n \in N$. Denoting

$$g_{11}(t, z_1) = q(t, z_1 + a + b) + (a^2 - b^2)\,p(t) - 2ap(t)\,(z_1 + a + b),$$
$$g_{12}(t, z_1) = -2bp(t)\,z_1 - p(t)\,z_1^2 + 2\bar{b}z_1(z_1 + 2b),$$

we have

$$g_1(t, z_1) = g_{11}(t, z_1) + g_{12}(t, z_1).$$

For $t \geqq t_0$, $z_1 \in K(\delta_n, \lambda_0)$, $n \in N$, we obtain

$$\operatorname{Re}\left[g_{11}(t, z_1)\frac{h_1'(0)}{h_1(z_1)}\right] \leqq \frac{4}{\delta_n}[\,|\,b\,|\,H_1(t) + H_2(t)],$$

$$\operatorname{Re}\left\{h_1'(0)\left[1 + \frac{g_{12}(t, z_1)}{h_1(z_1)}\right]\right\} = -2\operatorname{Re}[bp(t)] \leqq -2\varkappa.$$

Using Remark 2.1 of [1], where $G(t, z) \equiv 1$, $\vartheta = \lambda_0$, $\Theta_n = -2\varkappa$, $F_n(t) = 4[\,|\,b\,|\,H_1(t) + H_2(t)]/\delta_n$, $\sigma_n = t_0$, we observe that to any $\varepsilon > 0$ there are sequences $\{s_n\}$, $\{E_n(t)\}$ such that $s_n \geqq t_0$, $E_n \in C(I)$ and the assumptions of Theorem 2.4 of [1] are fulfilled with $\varkappa_1 < \varepsilon$, and

$$\liminf_{n\to\infty}\,[\delta_n e^{\varkappa_n}] = 0.$$

In view of Theorem 2.4 of [1] we have the assertion:

To any ϑ^*, $0 < \vartheta^* < \lambda_0$, there is an $S \geqq t_0$ such that for any $\varepsilon > 0$ and any solution $z_1(t)$ of (9_1) satisfying $2\,|\,b\,|\;|\,z_1(t_1)\,| < \vartheta^*\,|\,z_1(t_1) + 2b\,|$, where $t_1 \geqq S$, there is a $T > 0$ independent of t_1 and of $z_1(t)$ such that $|z_1(t)\,| < \varepsilon$ for $t \geqq t_1 + T$.

4° Assume that the conditions (10), (11) and (15) are fulfilled. Put

$$\psi(t) = \frac{\operatorname{Re}[bp(t)]}{2\,|\,b\,|^2}.$$

It holds that

$$W(z_1)\,\psi(t)\operatorname{Re}\left[g_1(t, z_1)\frac{h_1'(0)}{h_1(z_1)}\right] \leqq \frac{4\,|\,b\,|^2}{|\,z_1 + 2b\,|^2}[\,|\,z_1 + b\,|\,H_1(t) + H_2(t)] \leqq$$
$$\leqq 2\left[\frac{|\,b\,|^2\,|\,z_1\,|}{|\,z_1 + 2b\,|^2} + \frac{|\,b\,|^2}{|\,z_1 + 2b\,|}\right]H_1(t) + \frac{4\,|\,b\,|^2}{|\,z_1 + 2b\,|^2}H_2(t) \leqq$$
$$\leqq 4[\,|\,b\,|\,H_1(t) + H_2(t)]$$

for $t \geqq t_0$, $z_1 \in K(0, \lambda_0)$. Applying Theorem 3.3 of [3], where $\vartheta = \lambda_0$, $D(t) = G(t, z) = \psi(t), E(t) = 4[| b | H_1(t) + H_2(t)]$, we obtain:

If a solution $z_1(t)$ *of* (9_1) *satisfies* $\operatorname{Re}[\bar{b}z_1(t)] > -| b |^2$ *for* $t \geqq t_1$, *where* $t_1 \geqq t_0$, *then*

$$\int_{t_1}^{\infty} \operatorname{Re}[bp(t)] \, | z_1(t) | \, dt < \infty$$

and

$$\lim_{t \to \infty} z_1(t) = 0.$$

II. Consider the equation (9_2) on the set $I \times \Omega_2$. $W(z)$, λ_0, $K(\lambda_0)$ and $\hat{K}(\lambda)$ from [1] are of the following form:

$$W(z_2) = 2 | b | \, | z_2 | \, | z_2 - 2b |^{-1}, \qquad \lambda_0 = 2 | b |,$$

$$K(\lambda_0) = \Omega_2, \qquad \hat{K}(\lambda) = \{z_2 \in \Omega_2 : 2 | b | \, | z_2 | = \lambda | z_2 - 2b |\}.$$

Assume there are $H_1, H_2 \in C(I)$ such that

$$| q(t, z_2 + a - b) + (a^2 - b^2) p(t) - 2ap(t)(z_2 + a - b) | \leqq | z_2 - b | H_1(t) + H_2(t)$$

for $t \geqq t_0$, $z_2 \in \Omega_2$. Obviously, H_1 and H_2 must be nonnegative.

5° Let (10), (11), (12) be fulfilled. Define $\delta \leqq 2 | b |$ by (13). Put $\psi(t) \equiv 1$ and choose $\varkappa \in (\delta, \lambda_0)$. There is a $\xi > 1$ with the property

$$\sup_{t \geqq t_0} \frac{\xi[| b | H_1(t) + 2H_2(t)]}{\operatorname{Re}[bp(t)]} \leqq \frac{8\varkappa | b |^2}{4 | b |^2 + \varkappa^2}.$$

Analogously as in 1°, it can be verified that there exist constants $\mu, \nu \in (0, 1)$ such that

$$-\operatorname{Re}\left\{h_2'(0)\left[1 + \frac{g_2(t, z_2)}{h_2(z_2)}\right]\right\} \leqq 2[\max(\mu, \nu) - 1] \operatorname{Re}[bp(t)]$$

for $t \geqq t_0$, $z_2 \in K(\varkappa, \lambda_0) = \{z_2 \in \Omega_2 : \varkappa < 2 | b | \, | z_2 | \, | z_2 - 2b |^{-1} < \lambda_0\}$. By use of Theorem 2.2 and Theorem 2.5 of [1] (see also Theorem 2.2 and Theorem 2.4 of [3]), we get the following assertion:

If a solution $z_2(t)$ *of* (9_2) *satisfies* $2 | b | \, | z_2(t_1) | > \delta | z_2(t_1) - 2b |$, *where* $t_1 \geqq t_0$, *then to any* ε, $0 < \varepsilon < \lambda_0$ *there is a* $T > 0$ *such that* $2 | b | \, | z_2(t) | > \varepsilon | z_2(t) - 2b |$ *for all* $t \geqq t_1 + T$ *for which* $z_2(t)$ *is defined. Moreover,* $2 | b | \, | z_2(t) | > \delta | z_2(t) - 2b |$ *for all* $t \geqq t_1$ *for which* $z_2(t)$ *is defined.*

6° Suppose that (10), (11), (12) hold. Putting

$$\psi(t) = \frac{\operatorname{Re}[bp(t)]}{2 | b |^2}$$

and proceeding similarly as in 5°, we obtain

$$-\operatorname{Re}\left[g_2(t, z_2)\frac{h_2'(0)}{h_2(z_2)}\right] \leqq 4[\max(\mu, \nu) - 1] \mid b \mid^2 + 4 \mid b \mid^2 \leqq$$
$$\leqq 4 \max(\mu, \nu) \mid b \mid^2 < 4 \mid b \mid^2 = \operatorname{Re} h_2'(0)$$

for $t \geqq t_0$, $z_2 \in K(\varkappa, \lambda_0)$. Applying Theorem 2.3 of [2], where $G(t, z) = \psi(t)$, we have:

For any γ, $\delta < \gamma < \lambda_0$, and for any $S > t_0$ there exists a solution $z_2(t)$ of (9_2) *such that* $2 \mid b \mid \mid z_2(t) \mid < \gamma \mid z_2(t) - 2b \mid$ *for all* $t \geqq S$.

7° Assume that (10), (11) and (15) are fulfilled. Put $\psi(t) \equiv 1$ and choose $\delta \in \in (0, 2 \mid b \mid e^{-1})$. Let $S \geqq t_0$ be such that

$$\int_S^\infty [\mid b \mid H_1(t) + H_2(t)] \,\mathrm{d}t < \delta/4.$$

For $t \geqq S$ and $z_2 \in K(\delta, \lambda_0)$ it holds that

$$-\operatorname{Re}\left\{h_2'(0)\left[1 + \frac{g_2(t, z_2)}{h_2(z_2)}\right]\right\} \leqq \frac{4}{\delta}[\mid b \mid H_1(t) + H_2(t)] - \operatorname{Re}[bp(t)].$$

Using Theorem 2.2 of [1] with $\vartheta = \lambda_0$, $E(t) = 4[\mid b \mid H_1(t) + H_2(t)]/\delta - 2\operatorname{Re}[bp(t)]$, $G(t, z) \equiv 1$ we get:

If a solution $z_2(t)$ of (9_2) *satisfies* $2 \mid b \mid \mid z_2(t_1) \mid > \delta e \mid z_2(t_1) - 2b \mid$, *where* $t_1 \geqq S$, *then* $2 \mid b \mid \mid z_2(t) \mid > \delta \mid z_2(t) - 2b \mid$ *for all* $t \geqq t_1$ *for which* $z_2(t)$ *is defined.*

8° Let (10), (11) and (15) hold. Putting

$$\psi(t) = \frac{\operatorname{Re}[bp(t)]}{2 \mid b \mid^2},$$

we obtain

$$-W(z_2)\,\psi(t)\operatorname{Re}\left[g_2(t, z_2)\frac{h_2'(0)}{h_2(z_2)}\right] \leqq 4[\mid b \mid H_1(t) + H_2(t)]$$

for $t \geqq t_0$, $z_2 \in K(0, \lambda_0)$. From Theorem 3.3 of [3], where $\vartheta = \lambda_0$, $D(t) = G(t, z) = = \psi(t)$, $E(t) = 4[\mid b \mid H_1(t) + H_2(t)]$, it follows:

If a solution $z_2(t)$ of (9_2) *satisfies* $\operatorname{Re}[\bar{b}z_2(t)] < \mid b \mid^2$ *for* $t \geqq t_1$, *where* $t_1 \geqq t_0$, *then*

$$\int_{t_1}^\infty \operatorname{Re}[bp(t)] \mid z_2(t) \mid \mathrm{d}t < \infty$$

and

$$\lim_{t \to \infty} z_2(t) = 0.$$

3. MAIN RESULTS

Considering that $\tilde{K}(\lambda)$ are circles with centres $2b\lambda^2(4 \mid b \mid^2 - \lambda^2)^{-1}$ or $-2b\lambda^2(4 \mid b \mid^2 - \lambda^2)^{-1}$ and radii $4 \mid b \mid^2 \lambda(4 \mid b \mid^2 - \lambda^2)^{-1}$, and applying 1° and 5°, we obtain the following generalization of Theorem 3.1 of [1]:

Theorem 1. *Suppose there are* $a, b \in \mathbf{C}$ and $H_1, H_2 \in C(I)$ *such that*

$$(16) \qquad | q(t, z) + (a^2 - b^2) p(t) - 2ap(t) z | \leqq | z - a | H_1(t) + H_2(t)$$

for $t \geqq t_0$, $z \in \mathbf{C}$,

$$(17) \qquad \operatorname{Re}[bp(t)] > 0 \qquad \textit{for } t \geqq t_0,$$

$$(18) \qquad \int_{t_0}^{\infty} \operatorname{Re}[bp(t)] \, dt = \infty$$

and

$$(19) \qquad \sup_{t \geqq t_0} \frac{|b| H_1(t) + 2H_2(t)}{\operatorname{Re}[bp(t)]} < 2|b|.$$

Let $\delta \in [0, 1)$ *be defined by*

$$(20) \qquad \sup_{t \geqq t_0} \frac{|b| H_1(t) + 2H_2(t)}{\operatorname{Re}[bp(t)]} = \frac{4|b|\delta}{1 + \delta^2}.$$

Assume that a complete solution $z(t)$ *of* (4) *defined on* $[t_1, \omega)$, *where* $t_1 \geqq t_0$, *satisfies*

$$(21) \qquad | z(t_1) - a + (1 + \delta^2)(1 - \delta^2)^{-1} b | > 2|b| \delta(1 - \delta^2)^{-1}.$$

If $\omega = \infty$, *then*

$$(22) \qquad \limsup_{t \to \infty} | z(t) - a - (1 + \delta^2)(1 - \delta^2)^{-1} b | \leqq 2|b| \delta(1 - \delta^2)^{-1}.$$

If $\omega < \infty$, *then* $\operatorname{Re}[\bar{b}(z(t) - a)] < 0$ *for* $t \in [t_1, \omega)$ *and*

$$\lim_{t \to \omega} | z(t) | = \infty.$$

If, in addition,

$$\int_s^{s+t} \operatorname{Re}[bp(\tau)] \, d\tau \to \infty \qquad \textit{as } t \to \infty$$

uniformly for $s \in I$ *and* $\operatorname{Re}[\bar{b}(z(t_1) - a)] \geqq 0$, *then to any* $\varepsilon > 2|b|\delta(1 - \delta^2)^{-1}$ *there is a* $T > 0$ *independent of* t_1 *and of* $z(t)$ *such that*

$$| z(t) - a - (1 + \delta^2)(1 - \delta^2)^{-1} b | < \varepsilon$$

for $t \geqq t_1 + T$.

Proof. Let $\varepsilon > 2|b|\delta(1 - \delta^2)^{-1}$ be arbitrary. Put $\Delta = [(1 - \delta^2)\varepsilon + 2\delta^2 |b|] \times [(1 - \delta^2)\varepsilon + 2|b|]^{-1}$. Clearly $\delta < \Delta < 1$. Using 1°, we obtain: If $\operatorname{Re}[\bar{b}(z(t_1) - a)] > 0$, then there is a $T > 0$ such that $| z(t) - a - b | < \Delta | z(t) - a + b |$ for $t \geqq t_1 + T$. Hence

$$| z(t) - a - (1 + \delta^2)(1 - \delta^2)^{-1} b | \leqq$$
$$\leqq | z(t) - a - (1 + \Delta^2)(1 - \Delta^2)^{-1} b | + |b| \times$$
$$\times [(1 + \Delta^2)(1 - \Delta^2)^{-1} - (1 + \delta^2)(1 - \delta^2)^{-1}] <$$

$$< |b| [(1 + \Delta)^2 (1 - \Delta^2)^{-1} - (1 + \delta^2)(1 - \delta^2)^{-1}] \leqq \varepsilon$$

for $t \geqq t_1 + T$. We shall prove that this assertion remains true if $\mathrm{Re}\,[\bar{b}(z(t_1) - a)] = 0$.

It suffices to show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Re}\,[\bar{b}(z(t_1) - a)] > 0. \tag{23}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Re}\,[\bar{b}(z(t) - a)] = \mathrm{Re}\,\{\bar{b}[q(t, z) - p(t) z^2]\} =$$
$$= \mathrm{Re}\,\{\bar{b}[q(t, z) + (a^2 - b^2) p(t) - 2ap(t) z]\} +$$
$$+ \mathrm{Re}\,\{\bar{b}[2ap(t) z - (a^2 - b^2) p(t) - p(t) z^2]\} \geqq$$
$$\geqq -|b| [\gamma(t) H_1(t) + H_2(t)] + |b|^2 \vartheta(t) - \mathrm{Re}\,[\bar{b}(z - a)^2 p(t)],$$

where $\gamma(t) = |z(t) - a|$ and $\vartheta(t) = \mathrm{Re}\,[bp(t)]$. In view of (19) there exists a $\xi \in (0, 1)$ such that

$$H_1(t_1) < 2\xi\, \mathrm{Re}\,[bp(t_1)], \qquad H_2(t_1) < (1 - \xi) |b|\, \mathrm{Re}\,[bp(t_1)].$$

This together with

$$\mathrm{Re}\,[\bar{b}(z(t_1) - a)^2 p(t_1)] = -\gamma^2(t_1)\, \vartheta(t_1)$$

yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Re}\,[\bar{b}(z(t_1) - a)] > B(t_1),$$

where

$$B(t) = -|b| [2\xi\gamma(t) + (1 - \xi) |b|]\, \vartheta(t) + |b|^2 \vartheta(t) + \gamma^2(t)\, \vartheta(t) =$$
$$= [-2\xi |b| \gamma(t) + \xi |b|^2 + \gamma^2(t)]\, \vartheta(t) \geqq$$
$$\geqq \xi[\gamma(t) - |b|]^2\, \vartheta(t) \geqq 0,$$

from which (23) follows.

Now, it is clear that $\mathrm{Re}\,[\bar{b}(z(t) - a)] < 0$ and

$$\lim_{t \to \omega} |z(t)| = \infty,$$

provided that $\omega < \infty$. Assume $\omega = \infty$. It is to show that (22) holds. It is sufficient to prove that there exists a $t_2 \geqq t_1$ with the property $\mathrm{Re}\,[\bar{b}(z(t_2) - a)] \geqq 0$. Suppose conversely that $\mathrm{Re}\,[\bar{b}(z(t) - a)] < 0$ for $t \geqq t_1$. By 5° we know that to any ε, $0 < \varepsilon < 1$, there is a $T > 0$ such that $|z(t) - a + b| > \varepsilon |z(t) - a - b|$ for $t \geqq t_1 + T$. Consequently, there exists a $T_1 > t_1$ with the following property:

$$\frac{|z(t) - a|}{|z(t) - a - b|\, |z(t) - a + b|} < \frac{1 + \delta^2}{|b| (1 + \delta)^2},$$

$$\frac{1}{|z(t) - a - b|\;|z(t) - a + b|} < \frac{2(1 + \delta^2)}{|b|^2 (1 + \delta)^2},$$

for $t \geqq T_1$. Further, denoting

$$\Theta(t) = \frac{|z(t) - a - b|}{|z(t) - a + b|},$$

we get

$$\frac{d}{dt}\Theta(t) =$$

$$= \Theta^{-1}(t)\frac{|z - a + b|^2 \operatorname{Re}[(\bar{z} - \bar{a} - \bar{b})\dot{z}] - |z - a - b|^2 \operatorname{Re}[(\bar{z} - \bar{a} + \bar{b})\dot{z}]}{|z - a + b|^4} =$$

$$= \Theta(t) \operatorname{Re}\left\{\frac{2b}{(z - a - b)(z - a + b)}[q(t, z) + (a^2 - b^2)p(t) - 2ap(t)z] + \right.$$

$$\left. + \frac{2b}{(z - a - b)(z - a + b)}[2ap(t)z - (a^2 - b^2)p(t) - p(t)z^2]\right\} \leqq$$

$$\leqq 2\Theta(t)\left[-\vartheta(t) + \frac{|b|\;|q(t, z) + (a^2 - b^2)p(t) - 2ap(t)z|}{|z - a - b|\;|z - a + b|}\right] \leqq$$

$$\leqq 2\Theta(t)\left[-\vartheta(t) + |b|\frac{|z - a|H_1(t) + H_2(t)}{|z - a - b|\;|z - a + b|}\right] \leqq$$

$$\leqq 2\Theta(t)\left\{-\vartheta(t) + |b|\left[\frac{(1 + \delta^2)H_1(t)}{|b|(1 + \delta)^2} + \frac{2(1 + \delta^2)H_2(t)}{|b|^2(1 + \delta)^2}\right]\right\} \leqq$$

$$\leqq 2\Theta(t)\left\{-\vartheta(t) + \frac{1 + \delta^2}{|b|(1 + \delta)^2}[|b|H_1(t) + 2H_2(t)]\right\} \leqq$$

$$\leqq 2\Theta(t)\vartheta(t)[-1 + 4\delta(1 + \delta)^{-2}] \leqq -(1 - \delta)^2(1 + \delta)^{-2}\vartheta(t).$$

Integrating and letting $t \to \infty$, we infer that

$$\lim_{t \to \infty} \Theta(t) = -\infty,$$

which is impossible. Therefore there exists a $t_2 \geqq t_1$ such that $\operatorname{Re}[\bar{b}(z(t_2) - a)] \geqq 0$. The rest of the proof results from (23) and 1°.

Applying 6° and using Theorem 1, we can generalize Theorem 3.1 of [2]:

Theorem 2. *Let the asumptions of Theorem 1 be fulfilled. Then to any $S > t_0$ there is a solution $z(t)$ of* (4) *such that*

$$|z(t) - a + (1 + \delta^2)(1 - \delta^2)^{-1} b| \leqq 2|b|\delta(1 - \delta^2)^{-1}$$

for $t \geqq S$.

By virtue of Theorem 1 and Theorem 2 we obtain the following generalization of Theorem 3.2 of [2]:

Theorem 3. *Suppose there are* $a, b \in \mathbf{C}$ *and* $H_1, H_2 \in C(I)$ *such that the conditions* (16), (17), (18) *and*

$$\limsup_{t\to\infty} \frac{|b| H_1(t) + 2H_2(t)}{\operatorname{Re}[bp(t)]} < 2|b| \tag{24}$$

are fulfilled. Define $\delta \in [0, 1)$ *by*

$$\limsup_{t\to\infty} \frac{|b| H_1(t) + 2H_2(t)}{\operatorname{Re}[bp(t)]} = \frac{4|b|\delta}{1+\delta^2}.$$

Then there is at least one solution $z_0(t)$ *of* (4) *with the property*

$$\limsup_{t\to\infty} |z_0(t) - a + (1+\delta^2)(1-\delta^2)^{-1} b| \leqq 2|b|\delta(1-\delta^2)^{-1}.$$

Let $S \geqq t_0$ *be such that*

$$\sup_{t\geqq S} \frac{|b| H_1(t) + 2H_2(t)}{\operatorname{Re}[bp(t)]} < 2|b|.$$

Then every solution $z(t)$ *of* (4) *satisfying* $\operatorname{Re}[\bar{b}(z(t_1) - a)] \geqq 0$, *where* $t_1 \geqq S$, *is defined for all* $t \geqq t_1$ *and*

$$\limsup_{t\to\infty} |z(t) - a - (1+\delta^2)(1-\delta^2)^{-1} b| \leqq 2|b|\delta(1-\delta^2)^{-1}.$$

If, in addition,

$$\int_s^{s+t} \operatorname{Re}[bp(\tau)]\, d\tau \to \infty \qquad \text{as } t \to \infty$$

uniformly for $s \in I$, *then to any* $\varepsilon > 2|b|\delta(1-\delta^2)^{-1}$ *there is a* $T > 0$ *independent of* t_1 *and of* $z(t)$ *such that*

$$|z(t) - a - (1+\delta^2)(1-\delta^2)^{-1} b| < \varepsilon$$

for $t \geqq t_1 + T$.

Corollary 1. *Let* $\alpha(t)$, $\beta(t)$, $\varrho(t)$ *be as in Section* 2 *and let* $p(t)$, $q(t, z)$ *be defined by* (7). *Suppose there are* $a, b \in C$ *and* $H_1, H_2 \in C(I)$ *such that the conditions* (16), (17), (18) *and* (24) *are fulfilled. Let* $\delta \in [0, 1)$ *be defined as in Theorem* 3. *Then there is a solution* $x_0(t)$ *of* (6) *with the property*

$$\limsup_{t\to\infty} |\beta(t)\dot{x}_0(t)x_0^{-1}(t) - \alpha(t) - a + (1+\delta^2)(1-\delta^2)^{-1} b| \leqq 2|b|\delta(1-\delta^2)^{-1}.$$

If $S \geqq t_0$ *is as in Theorem* 3, *then every solution* $x(t)$ *of* (6) *satisfying* $\operatorname{Re}[\bar{b}\beta(t_1)\dot{x}(t_1)x^{-1}(t_1)] \geqq \operatorname{Re}[\bar{b}(\alpha(t_1) + a)]$, *where* $t_1 \geqq S$, *is defined for all* $t \geqq t_1$, *and*

$$\limsup_{t\to\infty} |\beta(t)\dot{x}(t)x^{-1}(t) - \alpha(t) - a - (1+\delta^2)(1-\delta^2)^{-1} b| \leqq 2|b|\delta(1-\delta^2)^{-1}.$$

If, in addition,

$$\int_{s}^{s+t} \operatorname{Re}\left[b(\beta^{-1}(\tau) + \varrho(\tau))\right] d\tau \to \infty \qquad \text{as } t \to \infty,$$

uniformly for $s \in I$, *then to any* $\varepsilon > 2\,|b|\,\delta(1 - \delta^2)^{-1}$ *there is a* $T > 0$ *independent of* t_1 *and of* $x(t)$ *such that*

$$|\,\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t) - a - (1 + \delta^2)(1 - \delta^2)^{-1}\,b\,| < \varepsilon$$

for $t \geqq t_1 + T$.

Making use of 2°, 4°, 7°, 8°, we can generalize Theorem 4.1 of [3]:

Theorem 4. *Suppose there exist* $a, b \in C$ *and* $H_1, H_2 \in C(I)$ *such that the conditions* (16), (17), (18) *and*

$$\int_{t_0}^{\infty} H_1(t)\,dt < \infty, \qquad \int_{t_0}^{\infty} H_2(t)\,dt < \infty \tag{25}$$

are fulfilled. Then each solution $z(t)$ *of* (4), *defined for* $t \to \infty$, *satisfies either*

$$\lim_{t \to \infty} z(t) = a + b, \qquad \int^{\infty} \operatorname{Re}\left[bp(t)\right]\,|\,z(t) - a - b\,|\,dt < \infty \tag{26}$$

or

$$\lim_{t \to \infty} z(t) = a - b, \qquad \int^{\infty} \operatorname{Re}\left[bp(t)\right]\,|\,z(t) - a + b\,|\,dt < \infty. \tag{27}$$

Let $S \geqq t_0$ *be such that*

$$\int_{S}^{\infty} \left[|\,b\,|\,H_1(t) + H_2(t)\right] dt < |\,b\,|\,(2e)^{-1}.$$

Then any solution $z(t)$ *of* (4) *satisfying*

$$|\,z(t_1) - a - (1 + \varkappa^2)(1 - \varkappa^2)^{-1}\,b\,| < 2\,|\,b\,|\,\varkappa(1 - \varkappa^2)^{-1},$$

where $t_1 \geqq S$ *and*

$$\varkappa = \exp\left\{-\frac{2e}{|\,b\,|} \int_{S}^{\infty} \left[|\,b\,|\,H_1(t) + H_2(t)\right] dt\right\},$$

is defined for all $t \geqq t_1$ *and there holds*

$$\lim_{t \to \infty} z(t) = a + b.$$

If, in addition,

$$\int_{s}^{s+t} \operatorname{Re}\left[bp(\tau)\right] d\tau \to \infty \qquad \text{as } t \to \infty,$$

uniformly for $s \in I$, then to any $\varepsilon > 0$ there is a $T > 0$ independent of t_1 and of $z(t)$ such that $|z(t) - a - b| < \varepsilon$ for $t \geqq t_1 + T$.

Proof. We claim that there is a $\sigma \geqq t_0$ such that

$$\mathrm{Re}\,[\bar{b}(z(t) - a)] > 0 \qquad \text{for } t \geqq \sigma \tag{28}$$

or

$$\mathrm{Re}\,[\bar{b}(z(t) - a)] < 0 \qquad \text{for } t \geqq \sigma. \tag{29}$$

Assuming that this claim is false, there exists a sequence $\{\tilde{t}_n\}$, $\tilde{t}_n \to \infty$ as $n \to \infty$, with the property

$$\mathrm{Re}\,[\bar{b}(z(\tilde{t}_n) - a)] = 0 \qquad \text{for } n \in N. \tag{30}$$

By using 2°, 7°, it can be easily verified that there is an $L > 0$ such that

$$|z(t) - a - b| \geqq L, \qquad |z(t) - a + b| \geqq L$$

or all sufficiently large $t \in I$. Denoting

$$\Theta(t) = \frac{|z(t) - a - b|}{|z(t) - a + b|}, \qquad \vartheta(t) = \mathrm{Re}\,[bp(t)],$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t) \leqq 2\Theta(t)\left[-\vartheta(t) + |b|\frac{|z(t) - a|\,H_1(t) + H_2(t)}{|z(t) - a - b|\;|z(t) - a + b|}\right] \leqq$$

$$\leqq \Theta(t)\left\{-2\vartheta(t) + |b|\left[\frac{H_1(t)}{|z(t) - a - b|} + \frac{H_1(t)}{|z(t) - a + b|} + \right.\right.$$

$$\left.\left. + \frac{2H_2(t)}{|z(t) - a - b|\;|z(t) - a + b|}\right]\right\} \leqq 2\Theta(t) \times$$

$$\times \{-\vartheta(t) + |b|\,[L^{-1}H_1(t) + L^{-2}H_2(t)]\},$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\exp[-2\int_{t_1}^{t}[|b|\,L^{-2}(LH_1(s) + H_2(s)) - \vartheta(s)]\,\mathrm{d}s]\,\Theta(t)\} \leqq 0.$$

Integration and limiting process $t \to \infty$ yield

$$\lim_{t \to \infty} \Theta(t) = 0,$$

which contradicts (30). Hence there is a $\sigma \geqq t_0$ such that (28) or (29) is satisfied for $t \geqq \sigma$. By 4° and 8° there hold the conditions (26) and (27). The rest of the proof follows from 2°.

Corollary 2. *Let $\alpha(t)$, $\beta(t)$, $\varrho(t)$ be as in Section 2 and let $p(t)$, $q(t, z)$ be defined by (7). Suppose there are $a, b \in \mathbf{C}$ and $H_1, H_2 \in C(I)$ such that* (16), (17), (18) *and* (25) *are fulfilled. Then each solution $x(t)$ of* (6) *defined for $t \to \infty$ obeys one of the following two conditions:*

(31) $$\lim_{t\to\infty} [\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t)] = a + b,$$

$$\int^{\infty} \operatorname{Re}[b(\beta^{-1}(t) + \varrho(t))]\,|\,\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t) - a - b\,|\,\mathrm{d}t < \infty,$$

(32) $$\lim_{t\to\infty} [\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t)] = a - b,$$

$$\int^{\infty} \operatorname{Re}[b(\beta^{-1}(t) + \varrho(t))]\,|\,\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t) - a + b\,|\,\mathrm{d}t < \infty.$$

If $S \geqq t_0$ is as in Theorem 4, then any solution $x(t)$ of (6) *satisfying*

$$|\,\beta(t_1)\,\dot{x}(t_1)\,x^{-1}(t_1) - \alpha(t_1) - a - (1 + \varkappa^2)(1 - \varkappa^2)^{-1}\,b\,| < 2\,|\,b\,|\,\varkappa(1 - \varkappa^2)^{-1},$$

where $t_1 \geqq S$ and

$$\varkappa = \exp\left\{-\frac{2e}{|\,b\,|}\int_S^{\infty}[|\,b\,|\,H_1(t) + H_2(t)]\,\mathrm{d}t\right\},$$

is defined for all $t \geqq t_1$ and there holds

$$\lim_{t\to\infty} [\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t)] = a + b.$$

If, in addition,

$$\int_s^{s+t} \operatorname{Re}[b(\beta^{-1}(\tau) + \varrho(\tau))]\,\mathrm{d}\tau \to \infty \qquad \textit{as } t \to \infty,$$

uniformly for $s \in I$, then to any $\varepsilon > 0$ there is a $T > 0$ independent of t_1 and of $x(t)$ such that

$$|\,\beta(t)\,\dot{x}(t)\,x^{-1}(t) - \alpha(t) - a - b\,| < \varepsilon$$

for $t \geqq t_1 + T$.

Application of 3° yields the following generalization of Theorem 3.3 of [1]:

Theorem 5. *Assume there are $a,\ b \in C$, $\varkappa > 0$ and $H_1, H_2 \in C(I)$ such that the conditions* (16),

(33) $$\operatorname{Re}[bp(t)] \geqq \varkappa \qquad \textit{for } t \geqq t_0$$

and

(34) $$\lim_{t\to\infty}\int_t^{t+1} H_1(s)\,\mathrm{d}s = \lim_{t\to\infty}\int_t^{t+1} H_2(s)\,\mathrm{d}s = 0,$$

are fulfilled. Then to any ϑ, $0 < \vartheta < 1$, there is an $S \geqq t_0$ such that for any $\varepsilon > 0$ and for any solution $z(t)$ of (4) *satisfying*

$$|\,z(t_1) - a - (1 + \vartheta^2)(1 - \vartheta^2)^{-1}\,b\,| < 2\,|\,b\,|\,\vartheta(1 - \vartheta^2)^{-1},$$

where $t_1 \geqq S$, *there is a* $T > 0$ *independent of* t_1 *and of* $z(t)$ *such that* $| z(t) - a - b | < \varepsilon$ *for* $t \geqq t_1 + T$.

Corollary 3. *Let* $\alpha(t)$, $\beta(t)$, $\varrho(t)$ *be as in Section* 2 *and let* $p(t)$, $q(t, z)$ *be defined by* (7). *Suppose there are* $a, b \in C$, $\varkappa > 0$ *and* $H_1, H_2 \in C(I)$ *such that* (16), (33) *and* (34) *are fulfilled. Then to any* ϑ, $0 < \vartheta < 1$, *there is an* $S \geqq t_0$ *such that for any* $\varepsilon > 0$ *and for any solution* $x(t)$ *of* (6) *satisfying*

$$| \beta(t_1) \dot{x}(t_1) x^{-1}(t_1) - \alpha(t_1) - a - (1 + \vartheta^2)(1 - \vartheta^2)^{-1} b | < 2 | b | \vartheta(1 - \vartheta^2)^{-1},$$

where $t_1 \geqq S$, *there is a* $T > 0$ *independent of* t_1 *and of* $x(t)$ *such that* $| \beta(t) \dot{x}(t) x^{-1}(t) - \alpha(t) - a - b | < \varepsilon$ for $t \geqq t_1 + T$.

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