Josef Janyška

A remark on linear functions on the sphere

Archivum Mathematicum, Vol. 17 (1981), No. 3, 141--149

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVII: 141—150, 1981

A REMARK ON LINEAR FUNCTIONS ON THE SPHERE

JOSEF JANYŠKA, Brno

(Received July 15, 1980)

1. Let S^n be a unit sphere in E^{n+1}. Let D be a domain in S^n. A function $f : D \to R$ of class C^∞ is called linear if

$$f(M) = \langle m, a\rangle + k, \tag{1}$$

where a is a constant vector, m is the position vector of the point $M \in S^n$ with respect to the centre of S^n, $k \in \mathrm{R}$. The linear function f is called homogeneous or non-homogeneous if $k = 0$ or $k \neq 0$, respectively.

For the case S^2, A. Švec [2] found certain conditions for a function f to be a linear and homogeneous. These conditions are expressed in terms of partial differential equations on some domain D and on the boundary ∂D of D. These results are extended in [1] on the wider class of non-homogeneous linear functions.

The aim of this paper is to prove some assertions for the case of non-homogeneous linear functions on S^3.

2. Let us introduce some notations (see [4]). Consider the unit sphere $S^3 \subset E^4$. To each point M of S^3, let us associate a tangent orthonormal frame $\{m, v_1, v_2, v_3, v_4\}$ such that m is the position vector of $M \in S^3$, $v_1, v_2, v_3 \in T_MS^3$ and $v_4 \in N_MS^3$. Then we have

$$\mathrm{d}m = \omega^i v_i, \qquad \mathrm{d}v_j = \omega^i_j v_i + \omega^j v_4,$$
$$\mathrm{d}v_4 = -\omega^i v_i = -\mathrm{d}m, \qquad \omega^i_j = -\omega^j_i, \quad (i, j = 1, 2, 3),$$

with the usual integrability conditions.

Let $f : S^3 \to R$ be a differentiable function. The covariant derivatives of f with respect to a field of tangent orthonormal frames $\{m, v_k\}$ $(k = 1, 2, 3, 4)$ are defined by the following formulas

$$\mathrm{d}f = f_1\omega^1 + f_2\omega^2 + f_3\omega^3, \tag{2}$$

$$\begin{aligned} \mathrm{d}f_1 - f_2\omega^2_1 - f_3\omega^3_1 &= f_{11}\omega^1 + f_{12}\omega^2 + f_{13}\omega^3, \\ \mathrm{d}f_2 + f_1\omega^2_1 - f_3\omega^3_2 &= f_{12}\omega^1 + f_{22}\omega^2 + f_{23}\omega^3, \\ \mathrm{d}f_3 + f_1\omega^3_1 + f_2\omega^3_2 &= f_{13}\omega^1 + f_{23}\omega^2 + f_{33}\omega^3, \end{aligned} \tag{3}$$

(4) $$\begin{aligned}
&df_{11} - 2f_{12}\omega_1^2 - 2f_{13}\omega_1^3 = A\omega^1 + B\omega^2 + C\omega^3,\\
&df_{12} + (f_{11} - f_{22})\,\omega_1^2 - f_{23}\omega_1^3 - f_{13}\omega_2^3 = (B + f_2)\,\omega^1 + (D + f_1)\,\omega^2 + E\omega^3,\\
&df_{13} - f_{23}\omega_1^2 + (f_{11} - f_{33})\,\omega_1^3 + f_{12}\omega_2^3 = (C + f_3)\,\omega^1 + E\omega^2 + (F + f_1)\,\omega^3,\\
&df_{22} + 2f_{12}\omega_1^2 - 2f_{23}\omega_2^3 = D\omega^1 + G\omega^2 + H\omega^3,\\
&df_{23} + f_{13}\omega_1^2 + f_{12}\omega_1^3 + (f_{22} - f_{33})\,\omega_2^3 = E\omega^1 + (H + f_3)\,\omega^2 + (I + f_2)\,\omega^3,\\
&df_{33} + 2f_{13}\omega_1^3 + 2f_{23}\omega_2^3 = F\omega^1 + I\omega^2 + J\omega^3,
\end{aligned}$$

(5) $$\begin{aligned}
&dA - (3B + 2f_2)\,\omega_1^2 - (3C + 2f_3)\,\omega_1^3 = T_1\omega^1 + T_2\omega^2 + T_3\omega^3,\\
&dB + (A - 2D - 2f_1)\,\omega_1^2 - 2E\omega_1^3 - C\omega_2^3 = (T_2 + 2f_{12})\,\omega^1 + T_4\omega^2 + T_5\omega^3,\\
&dC - 2E\omega_1^2 + (A - 2F - 2f_1)\,\omega_1^3 + B\omega_2^3 = (T_3 + 2f_{13})\,\omega^1 + T_5\omega^2 + T_6\omega^3,\\
&dD + (2B - G + 2f_2)\,\omega_1^2 - H\omega_1^3 - 2E\omega_2^3 = (T_4 - 2f_{11} + 2f_{22})\,\omega^1 +\\
&\qquad + (T_7 + 2f_{12})\,\omega^2 + T_8\omega^3,\\
&dE + (C - H)\,\omega_1^2 + (B - I)\,\omega_1^3 + (D - F)\,\omega_2^3 = (T_5 + 2f_{23})\,\omega^1 +\\
&\qquad + (T_8 + 2f_{13})\,\omega^2 + (T_9 + 2f_{12})\,\omega^3,\\
&dF - I\omega_1^2 + (2C - J + 2f_3)\,\omega_1^3 + 2E\omega_2^3 = (T_6 - 2f_{11} + f_{22} + f_{33})\,\omega^1 +\\
&\qquad + T_9\omega^2 + (T_{10} + 2f_{13})\,\omega^3,\\
&dG + (3D + 2f_1)\,\omega_1^2 - (3H + 2f_3)\,\omega_2^3 = T_7\omega^1 + T_{11}\omega^2 + T_{12}\omega^3,\\
&dH + 2E\omega_1^2 + D\omega_1^3 + (G - 2I - 2f_2)\,\omega_2^3 = T_8\omega^1 +\\
&\qquad + (T_{12} + f_{23})\,\omega^2 + T_{13}\omega^3,\\
&dI + F\omega_1^2 + 2E\omega_1^3 + (2H - J + 2f_3)\,\omega_2^3 = T_9\omega^1 +\\
&\qquad + (T_{13} - 2f_{22} + 2f_{33})\,\omega^2 + (T_{14} + 2f_{23})\,\omega^3,\\
&dJ + (3F + 2f_1)\,\omega_1^2 + (3I + 2f_2)\,\omega_2^3 = T_{10}\omega^1 + T_{14}\omega^2 + T_{15}\omega^3.
\end{aligned}$$

By means of these covariant derivatives, one can introduce the following differential operators Δ, $\mathscr{L}$, $\mathscr{K}$ and $\mathscr{M}$.

$$\begin{aligned}
&\Delta f = \sum_{i=1}^{3} f_{ii},\\
&\mathscr{L}f = \Delta f + 3f,\\
&\mathscr{K}f = \sum_{\substack{i,j=1\\ i<j}}^{3} (f_{ii}f_{jj} - f_{ij}^2) + 2f\,\Delta f + 3f^2,\\
&\mathscr{M}f = \det(f_{ij}) + f\sum_{\substack{i,j=1\\ i<j}}^{3} (f_{ii}f_{jj} - f_{ij}^2) + f^2\sum_{i=1}^{3} f_{ii} + f^3.
\end{aligned}$$

Now, let $\{m, v_k^*\}$ $(k = 1, 2, 3, 4)$ be another field of tangent frames. That means there exists an orthonormal matrix (a_i^j) such that

$$v_j^* = a_j^i v_i, \qquad v_4^* = v_4, \qquad i, j = 1, 2, 3.$$

Let (α_i^j) be the inverse matrix, i.e. $a_i^j\alpha_k^i = \delta_k^j$. Then we have the following relations

$$\omega^i = a_j^i\omega^{j*}, \qquad \omega^{j*} = \alpha_i^j\omega^i,$$
$$f_i = \alpha_i^j f_j^*, \qquad f_j^* = a_j^i f_i,$$
$$\omega_1^{2*} = a_3^3\omega_1^2 - a_3^2\omega_1^3 + a_3^1\omega_2^3,$$
$$\omega_1^{3*} = -a_2^3\omega_1^2 + a_2^2\omega_1^3 - a_2^1\omega_2^3,$$
$$\omega_2^{3*} = a_1^3\omega_1^2 - a_1^2\omega_1^3 + a_1^1\omega_2^3,$$
$$f_{ij}^* = a_i^k f_{kl} a_j^l, \qquad f_{kl} = \alpha_k^i f_{ij}^* \alpha_l^j.$$

Remark. It is easy to see that the functions $\mathscr{L}f$, $\mathscr{K}f$ and $\mathscr{M}f$ satisfy $\mathscr{L}f^* = \mathscr{L}f$, $\mathscr{K}f^* = \mathscr{K}f$, $\mathscr{M}f^* = \mathscr{M}f$, i.e. are invariant with respect to the choice of the field of tangent frames.

3. Now, let us consider the function

$$\mathscr{G}(f) = (\mathscr{L}f)^2 - 3\mathscr{K}f = \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{3} \{(f_{ii} - f_{jj})^2 + 3f_{ij}^2\}.$$

It is invariant with respect to the choice of the field of tangent frames. For $\mathscr{G}(f)$ the following lemma is fulfilled

Lemma 1. If $\mathscr{G}(f) = (\mathscr{L}f)^2 - 3\mathscr{K}f = 0$ on some domain $\bar{D} \subset S^3$, then f is linear on $\bar{D}$.

Proof: The supposition $\mathscr{G}(f) = 1/2 \sum_{\substack{i,j=1\\ i<j}}^{3} \{(f_{ii} - f_{jj})^2 + 3f_{ij}^2\} = 0$ implies $f_{ii} = f_{jj}, f_{ij} = 0, (i < j, i, j = 1, 2, 3)$. From (4), we get $A = D = F = -f_1$, $B = G = I = -f_2$, $C = H = J = -f_3$, $E = 0$ and from this $\mathrm{d}f_{ii} = -\mathrm{d}f$. This implies $f_{ii} = -f + c$, where $c \in R$. Now, let us consider the vector field

$$a = -\sum_{i=1}^{3} f_i v_i + (f - c)\, v_4.$$

Then $\mathrm{d}a = 0$ and hence a is a constant vector. Then

$$f = \langle a, v_4\rangle + c, \qquad \text{QED.}$$

In the following propositions we shall use the maximum principle in the form described in [3].

Proposition 1. Let $D \subset S^3$ be a domain, ∂D its boundary. Let $f: S^3 \to \mathrm{R}$ be a differentiable function. If

(i) $\mathscr{G}(f) = (\mathscr{L}f)^2 - 3\mathscr{K}f = 0$ on ∂D,

(ii) $\mathscr{L}f =$ const. on D.

then f is linear on D.

Proposition 2. Let $D \subset S^3$ be a domain, ∂D its boundary. Let $f : S^3 \to \mathbf{R}$ be a differentiable function. If

(i) $\mathscr{G}(f) = (\mathscr{L}f)^2 - 3\mathscr{K}f = 0$ on ∂D,

(ii) $2\mathscr{L}f\Delta\mathscr{L}f \geqq 3\Delta\mathscr{K}f$ on D,

then f is linear on D.

Proof: First of all we must calculate the covariant derivatives of the functions $\mathscr{L}f$, $\mathscr{K}f$ and $\mathscr{G}(f)$.

$$(6)\quad \begin{aligned} (\mathscr{L}f)_1 &= A + D + F + 3f_1, \\ (\mathscr{L}f)_2 &= B + G + I + 3f_2, \\ (\mathscr{L}f)_3 &= C + H + J + 3f_3, \end{aligned}$$

$$(7)\quad \begin{aligned} (\mathscr{L}f)_{11} &= T_1 + T_4 + T_6 - f_{11} + 3f_{22} + f_{33}, \\ (\mathscr{L}f)_{22} &= T_4 + T_{11} + T_{13} + 2f_{33} + f_{22}, \\ (\mathscr{L}f)_{33} &= T_6 + T_{13} + T_{15} + 3f_{33}, \end{aligned}$$

$$(8)\quad \begin{aligned} (\mathscr{K}f)_1 &= (f_{22} + f_{33} + 2f)\,A + (f_{11} + f_{33} + 2f)\,D + \\ &\quad + (f_{11} + f_{22} + 2f)\,F + 2f_1\mathscr{L}f - 2f_{12}(B + f_2) - \\ &\quad - 2f_{13}(C + f_3) - 2f_{23}E, \\ (\mathscr{K}f)_2 &= (f_{22} + f_{33} + 2f)\,B + (f_{11} + f_{33} + 2f)\,G + \\ &\quad + (f_{11} + f_{22} + 2f)\,I + 2f_2\mathscr{L}f - 2f_{12}(D + f_1) - \\ &\quad - 2f_{13}E - 2f_{23}(H + f_3), \\ (\mathscr{K}f)_3 &= (f_{22} + f_{33} + 2f)\,C + (f_{11} + f_{33} + 2f)\,H + \\ &\quad + (f_{11} + f_{22} + 2f)\,J + 2f_3\mathscr{L}f - 2f_{12}E - \\ &\quad - 2f_{13}(F + f_1) - 2f_{23}(I + f_2), \end{aligned}$$

$$(9)\quad \begin{aligned} (\mathscr{K}f)_{11} &= 2\{(A + f_1)(D + f_1) + (A + f_1)(F + f_1) - (B + f_2)^2 + \\ &\quad + (D + f_1)(F + f_1) - (C + f_3)^2 - E^2\} + 2f_{11}\mathscr{L}f + \\ &\quad + (f_{22} + f_{33} + 2f)\,T_1 + (f_{11} + f_{33} + 2f)(T_4 - 2f_{11} + 2f_{22}) + \\ &\quad + (f_{11} + f_{22} + 2f)(T_6 - 2f_{11} + f_{22} + f_{33}) - \\ &\quad - 2f_{12}(T_2 + 3f_{12}) - 2f_{13}(T_3 + 3f_{13}) - 2f_{23}(T_5 + 2f_{23}), \\ (\mathscr{K}f)_{22} &= 2\{(B + f_2)(G + f_2) + (B + f_2)(I + f_2) - (D + f_1)^2 + \\ &\quad + (G + f_2)(I + f_2) - E^2 - (H + f_3)^2\} + 2f_{22}\mathscr{L}f + \\ &\quad + (f_{22} + f_{33} + 2f)\,T_4 + (f_{11} + f_{33} + 2f)\,T_{11} + \\ &\quad + (f_{11} + f_{22} + 2f)(T_{13} - 2f_{22} + 2f_{33}) - \\ &\quad - 2f_{12}(T_7 + 3f_{12}) - 2f_{13}(T_8 + 2f_{13}) - 2f_{23}(T_{12} + 3f_{23}), \\ (\mathscr{K}f)_{33} &= 2\{(C + f_3)(H + f_3) + (C + f_3)(J + f_3) - 2E^2 + \\ &\quad + (H + f_3)(J + f_3) - 2(F + f_1)^2 - 2(I + f_2)^2\} + \end{aligned}$$

$$+ 2f_{33}\mathscr{L}f + (f_{22} + f_{33} + 2f)\, T_6 + (f_{11} + f_{33} + 2f)\, T_{13} +$$
$$+ (f_{11} + f_{22} + 2f)\, T_{15} - 2f_{12}(T_9 + 2f_{12}) -$$
$$- 2f_{13}(T_{10} + 3f_{13}) - 2f_{23}(T_{14} + 3f_{23}),$$

(10) $$\mathscr{G}_1(f) = 3(f_{11} + f)\, A + 3(f_{22} + f)\, D + 3(f_{33} + f)\, F +$$
$$+ 6f_{12}(B + f_2) + 6f_{13}(C + f_3) + 6f_{23}E - (A + D + F)\, \mathscr{L}f,$$
$$\mathscr{G}_2(f) = 3(f_{11} + f)\, B + 3(f_{22} + f)\, G + 3(f_{33} + f)\, I +$$
$$+ 6f_{12}(D + f_1) + 6f_{13}E + 6f_{23}(H + f_3) - (B + G + I)\, \mathscr{L}f,$$
$$\mathscr{G}_3(f) = 3(f_{11} + f)\, C + 3(f_{22} + f)\, H + 3(f_{33} + f)\, J +$$
$$+ 6f_{12}E + 6f_{13}(F + f_1) + 6f_{23}(I + f_2) - (C + H + J)\, \mathscr{L}f,$$

(11) $$\mathscr{G}_{11}(f) = 3(A + f_1)^2 + 3(D + f_1)^2 + 3(F + f_1)^2 + 6(B + f_2)^2 +$$
$$+ 6(C + f_3)^2 + 6E^2 - (A + D + F + 3f_1)^2 +$$
$$+ 3(f_{11} + f)\, T_1 + 3(f_{22} + f)\,(T_4 - 2f_{11} + 2f_{22}) +$$
$$+ 3(f_{33} + f)\,(T_6 - 2f_{11} + f_{22} + f_{33}) +$$
$$+ 6f_{12}(T_2 + 3f_{12}) + 6f_{13}(T_3 + 3f_{13}) + 6f_{23}(T_5 + 2f_{23}) -$$
$$- (T_1 + T_4 + T_6 - 4f_{11} + 3f_{22} + f_{33})\, \mathscr{L}f,$$
$$\mathscr{G}_{22}(f) = 3(B + f_2)^2 + 3(G + f_2)^2 + 3(I + f_2)^2 + 6(D + f_1)^2 +$$
$$+ 6E^2 + 6(H + f_3)^2 - (B + G + I + 3f_2)^2 +$$
$$+ 3(f_{11} + f)\, T_4 + 3(f_{22} + f)\, T_{11} + 3(f_{33} + f)\,(T_{13} - 2f_{22} + 2f_{33}) +$$
$$+ 6f_{12}(T_7 + 3f_{12}) + 6f_{13}(T_8 + 2f_{13}) + 6f_{23}(T_{12} + 3f_{23}) -$$
$$- (T_4 + T_{11} + T_{13} - 2f_{22} + 2f_{33})\, \mathscr{L}f,$$
$$\mathscr{G}_{33}(f) = 3(C + f_3)^2 + 3(H + f_3)^2 + 3(J + f_3)^2 + 6E^2 +$$
$$+ 6(F + f_1)^2 + 6(I + f_2)^2 - (C + H + J + 3f_3)^2 +$$
$$+ 3(f_{11} + f)\, T_6 + 3(f_{22} + f)\, T_{13} + 3(f_{33} + f)\, T_{15} +$$
$$+ 6f_{12}(T_9 + 2f_{12}) + 6f_{13}(T_{10} + 3f_{13}) + 6f_{23}(T_{14} + 3f_{23}) -$$
$$- (T_6 + T_{13} + T_{15})\, \mathscr{L}f.$$

From assumption (ii) of Proposition 1, (6), (7), we have

(12) $$\Delta\mathscr{G}(f) - 3\mathscr{G}(f) = 3(f_{11} - f_{22})^2 + 9f_{12}^2 + 9f_{13}^2 + 9f_{23}^2 +$$
$$+ 3(A + f_1)^2 + 9(B + f_2)^2 + 9(C + f_3)^2 +$$
$$+ 9(D + f_1)^2 + 18E^2 + 9(F + f_1)^2 + 3(G + f_2)^2 +$$
$$+ 9(H + f_3)^2 + 9(I + f_2)^2 + 3(J + f_3)^2.$$

From (6), (7), (9) and (11), we get

(13) $$\Delta\mathscr{G}(f) = 2\sum_{i=1}^{3} (\mathscr{L}f)_i^2 + 2\mathscr{L}f\Delta\mathscr{L}f - 3\Delta\mathscr{K}f.$$

These equations (12) and (13) satisfy the conditions of the maximum principle if the assumptions of Propositions 1 and 2 hold, respectively. This proves our propositions because of Lemma 1, QED.

Consequence. Replacing (ii) in Proposition 2 by

(ii)′ $\mathscr{K}f = \text{const.}$ on D,

(iii)′ $\mathscr{L}f\Delta\mathscr{L}f \geqq 0$ on D,

then f is linear on D.

4. Now, let us consider a new function

$$\mathscr{G}'(f) = (\mathscr{K}f)^2 - 3\mathscr{L}f\mathscr{M}f.$$

It is easy to prove the following

Lemma 2. Let $f: S^3 \to \mathrm{R}$ be a differentiable function satisfying $f_{ij} = 0$ $(i \neq j)$ on some domain $\bar{D} \subset S^3$ and $\mathscr{G}'(f) = (\mathscr{K}f)^2 - 3\mathscr{L}f \,.\, \mathscr{M}f = 0$ on $\bar{D}$, then f is linear on $\bar{D}$.

Proof: From the assumption $f_{ij} = 0$ $(i \neq j)$ and $\mathscr{G}'(f) = 0$, we obtain either $f_{ii} = -f$ or $f_{ii} = f_{jj}$. Now, our assertion follows from Lemma 1, QED.

Proposition 3. Let $D \subset S^3$ be a domain, ∂D its boundary. Let $f: S^3 \to \mathrm{R}$ be a differentiable function satisfying $f_{ij} = 0$ $(i \neq j)$ on $\bar{D} = D \cup \partial D$. If

(i) $\mathscr{G}'(f) = 0$ on ∂D,

(ii) $2\mathscr{K}f\Delta\mathscr{K}f \geqq 3\mathscr{L}f\Delta\mathscr{M}f + 3\mathscr{M}f\Delta\mathscr{L}f + 6\sum_{i=1}^{3}(\mathscr{L}f)_i \,.\, (\mathscr{M}f)_i$,

then f is linear on $\bar{D}$.

Proof: From $f_{ij} = 0$, we obtain

$$\begin{aligned}
(14)\quad (\mathscr{M}f)_1 &= (f_{22}+f)(f_{33}+f)(A+f_1) + (f_{11}+f)(f_{33}+f)(D+f_1) + \\
&\quad + (f_{11}+f)(f_{22}+f)(F+f_1), \\
(\mathscr{M}f)_2 &= (f_{22}+f)(f_{33}+f)(B+f_2) + (f_{11}+f)(f_{33}+f)(G+f_2) + \\
&\quad + (f_{11}+f)(f_{22}+f)(I+f_2), \\
(\mathscr{M}f)_3 &= (f_{22}+f)(f_{33}+f)((C+f_3) + (f_{11}+f)(f_{33}+f)(H+f_3) + \\
&\quad + (f_{11}+f)(f_{22}+f)(J+f_3), \\
(15)\quad (\mathscr{M}f)_{11} &= 2\{(f_{11}+f)(D+f_1)(F+f_1) + (f_{22}+f)(A+f_1)(F+f_1) + \\
&\quad + (f_{33}+f)(A+f_1)(D+f_1)\} + (f_{22}+f)(f_{33}+f)(T_1+f_{11}) + \\
&\quad + (f_{11}+f)(f_{33}+f)(T_4 - f_{11} + 2f_{22}) + \\
&\quad + (f_{11}+f)(f_{22}+f)(T_6 - f_{11} + f_{22} + f_{33}), \\
(\mathscr{M}f)_{22} &= 2\{(f_{11}+f)(G+f_2)(I+f_2) + (f_{22}+f)(B+f_2)(I+f_2) + \\
&\quad + (f_{33}+f)(B+f_2)(G+f_2)\} + (f_{22}+f)(f_{33}+f)(T_4+f_{22}) +
\end{aligned}$$

$$+ (f_{11} + f)(f_{33} + f)(T_{11} + f_{22}) +$$
$$+ (f_{11} + f)(f_{22} + f)(T_{13} - f_{22} + 2f_{33}),$$

$$(\mathscr{M}f)_{33} = 2\{(f_{11} + f)(H + f_3)(J + f_3) + (f_{22} + f)(C + f_3)(J + f_3) +$$
$$+ (f_{33} + f)(C + f_3)(H + f_3)\} + (f_{22} + f)(f_{33} + f)(T_6 + f_{33}) +$$
$$+ (f_{11} + f)(f_{33} + f)(T_{13} + f_{33}) + (f_{11} + f)(f_{22} + f)(T_{15} + f_{33}),$$

$$\mathscr{G}'_1(f) = (A + f_1)\{(f_{11} + f)(f_{22} - f_{33})^2 +$$
$$+ (f_{22} + f)^2 (f_{11} - f_{33}) + (f_{33} + f)^2 (f_{11} - f_{22})\} +$$
$$+ (D + f_1)\{(f_{22} + f)(f_{11} - f_{33})^2 +$$
$$+ (f_{11} + f)^2 (f_{22} - f_{33}) - (f_{33} + f)^2 (f_{11} - f_{22})\} +$$
$$+ (F + f_1)\{(f_{33} + f)(f_{11} - f_{22})^2 -$$
$$- (f_{22} + f)^2 (f_{11} - f_{33}) - (f_{11} + f)^2 (f_{22} - f_{33})\},$$

$$\mathscr{G}'_2(f) = (B + f_2)\{(f_{11} + f)(f_{22} - f_{33})^2 +$$
$$+ (f_{22} + f)^2 (f_{11} - f_{33}) + (f_{33} + f)^2 (f_{11} - f_{22})\} +$$
$$+ (G + f_2)\{(f_{22} + f)(f_{11} - f_{33})^2 +$$
$$+ (f_{11} + f)^2 (f_{22} - f_{33}) - (f_{33} + f)^2 (f_{11} - f_{22})\} +$$
$$+ (I + f_2)\{(f_{33} + f)(f_{11} - f_{22})^2 -$$
$$- (f_{11} + f)^2 (f_{22} - f_{33}) - (f_{22} + f)^2 (f_{11} - f_{33})\},$$

$$\mathscr{G}'_3(f) = (C + f_3)\{(f_{11} + f)(f_{22} - f_{33})^2 +$$
$$+ (f_{22} + f)^2 (f_{11} - f_{33}) + (f_{33} + f)^2 (f_{11} - f_{22})\} +$$
$$+ (H + f_3)\{(f_{22} + f)(f_{11} - f_{33})^2 +$$
$$+ (f_{11} + f)^2 (f_{22} - f_{33}) - (f_{33} + f)^2 (f_{11} - f_{22})\} +$$
$$+ (J + f_3)\{(f_{33} + f)(f_{11} - f_{22})^2 -$$
$$- (f_{11} + f)^2 (f_{22} - f_{33}) - (f_{22} + f)^2 (f_{11} - f_{33})\},$$

$$\mathscr{G}'_{11}(f) = (A + f_1)^2 \{(f_{22} - f_{33})^2 + (f_{33} + f)^2 + (f_{22} + f)^2\} +$$
$$+ (D + f_1)^2 \{(f_{11} - f_{33})^2 + (f_{11} + f)^2 + (f_{33} + f)^2\} +$$
$$+ (F + f_1)^2 \{(f_{11} - f_{22})^2 + (f_{22} + f)^2 + (f_{11} + f)^2\} +$$
$$+ 2(A + f_1)(D + f_1)\{2(f_{22} + f)(f_{11} - f_{33}) -$$
$$- (f_{33} + f)^2 + 2(f_{11} + f)(f_{22} - f_{33})\} +$$
$$+ 2(A + f_1)(F + f_1)\{2(f_{33} + f)(f_{11} - f_{33}) -$$
$$- 2(f_{11} + f)(f_{22} - f_{33}) - (f_{22} + f)^2\} +$$
$$+ 2(F + f_1)(D + f_1)\{-2(f_{33} + f)(f_{11} - f_{22}) -$$
$$- 2(f_{22} + f)(f_{11} - f_{33}) - (f_{11} + f)^2\} +$$
$$+ \{(f_{11} + f)(f_{22} - f_{33})^2 + (f_{22} + f)^2 (f_{11} - f_{33}) +$$

$$
\begin{aligned}
& + (f_{33} + f)^2 (f_{11} - f_{22})\} (T_1 + f_{11}) + \\
& + \{(f_{22} + f)(f_{11} - f_{33})^2 + (f_{11} + f)^2 (f_{22} - f_{33}) - \\
& + (f_{33} + f)^2 (f_{11} - f_{22})\} (T_4 - f_{11} + 2f_{22}) + \\
& + \{(f_{33} + f)(f_{11} - f_{22})^2 - (f_{22} + f)^2 (f_{11} - f_{33}) - \\
& - (f_{11} + f)^2 (f_{22} - f_{33})\} (T_6 - f_{11} + f_{22} + f_{33}),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}'_{22}(f) = {} & (B + f_2)^2 \{(f_{22} - f_{33})^2 + (f_{22} + f)^2 + (f_{33} + f)^2\} + \\
& + (G + f_2)^2 \{(f_{11} - f_{33})^2 + (f_{11} + f)^2 + (f_{33} + f)^2\} + \\
& + (I + f_2)^2 \{(f_{11} - f_{22})^2 + (f_{11} + f)^2 + (f_{22} + f)^2\} + \\
& + 2(B + f_2)(G + f_2) \{2(f_{11} + f)(f_{22} - f_{33}) + \\
& + 2(f_{22} + f)(f_{11} - f_{33}) - (f_{33} + f)^2\} + \\
& + 2(B + f_2)(I + f_2) \{2(f_{33} + f)(f_{11} - f_{22}) - \\
& - 2(f_{11} + f)(f_{22} - f_{33}) - (f_{22} + f)^2\} + \\
& + 2(G + f_2)(I + f_2) \{-2(f_{22} + f)(f_{11} - f_{33}) - \\
& - 2(f_{33} + f)(f_{11} - f_{22}) - (f_{11} + f)^2\} + \\
& + \{(f_{11} + f)(f_{22} - f_{33})^2 + (f_{22} + f)^2 (f_{11} - f_{33}) + \\
& + (f_{33} + f)^2 (f_{11} - f_{22})\} (T_4 + f_{22}) + \\
& + \{(f_{22} + f)(f_{11} - f_{33})^2 + (f_{11} + f)^2 (f_{22} - f_{33}) - \\
& - (f_{33} + f)^2 (f_{11} - f_{22})\} (T_{11} + f_{22}) + \\
& + \{(f_{33} + f)(f_{11} - f_{22})^2 - (f_{22} + f)^2 (f_{11} - f_{33}) - \\
& - (f_{11} + f)^2 (f_{11} - f_{33})\} (T_{13} - f_{22} + 2f_{33}),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}'_{33}(f) = {} & (C + f_3)^2 \{(f_{22} - f_{33})^2 + (f_{22} + f)^2 + (f_{33} + f)^2\} + \\
& + (H + f_3)^2 \{(f_{11} - f_{33})^2 + (f_{11} + f)^2 + (f_{33} + f)^2\} + \\
& + (J + f_3)^2 \{(f_{11} - f_{22})^2 + (f_{11} + f)^2 + (f_{22} + f)^2\} + \\
& + 2(C + f_3)(H + f_3) \{2(f_{11} + f)(f_{22} - f_{33}) + \\
& + 2(f_{22} + f)(f_{11} - f_{33}) - (f_{33} + f)^2\} + \\
& + 2(C + f_3)(J + f_3) \{2(f_{33} + f)(f_{11} - f_{22}) - \\
& - 2(f_{11} + f)(f_{22} - f_{33}) - (f_{22} + f)^2\} + \\
& + 2(H + f_3)(J + f_3) \{-2(f_{22} + f)(f_{11} - f_{33}) - \\
& - (f_{11} + f)^2 - 2(f_{33} + f)(f_{11} - f_{22})\} + \\
& + \{(f_{11} + f)(f_{22} - f_{33})^2 + (f_{22} + f)^2 (f_{11} - f_{33}) + \\
& + (f_{33} + f)^2 (f_{11} - f_{22})\} (T_6 + f_{33}) + \\
& + \{(f_{22} + f)(f_{11} - f_{33})^3 + (f_{11} + f)^2 (f_{22} - f_{33}) - \\
& - (f_{33} + f)^2 (f_{11} - f_{22})\} (T_{13} + f_{33}) + \\
& + \{(f_{33} + f)(f_{11} - f_{22})^2 - (f_{11} + f)^2 (f_{22} - f_{33}) - \\
& - (f_{22} + f)^2 (f_{11} - f_{33})\} (T_{15} + f_{33}).
\end{aligned}
$$

From (6), (7), (9), (14), (15) and (17), we obtain

$$\Delta\mathscr{G}'(f) = 2\sum_{i=1}^{3} (\mathscr{K}f)_i^2 + 2\mathscr{K}f\Delta\mathscr{K}f - 3\mathscr{L}f\Delta\mathscr{M}f - $$
$$- 3\mathscr{M}f\Delta\mathscr{L}f - 6\sum_{i=1}^{3} (\mathscr{L}f)_i \,.\, (\mathscr{M}f)_i \,.$$

This expression satisfies the conditions of the maximum principle because of (ii) and the assertion follows from Lemma 2, QED.

REFERENCES

[1] J. Janyška: *On linear functions on the sphere S^2*, *Cz. mat. J.* 31 (106) 1981, 75–82.

[2] A. Švec: *A remark on the differential equations on the sphere,* Čas. pro pěst. mat., 101 (1976), 278—282.

[3] A. Švec: *Contributions to the global differential geometry of surfaces,* Rozpravy ČSAV, 87 (1977), No. 1.

[4] M. Afwat, A. Švec: *Global differential geometry of hypersurfaces,* Rozpravy ČSAV, 88 (1978), No. 7.

J. Janyška
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia