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LINEAR DIFFERENTIAL EQUATION OF THE 2nd ORDER WHOSE PRINCIPAL SOLUTION HAS UNBOUNDED LOGARITHMIC DERIVATIVE

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Let

$$x'' = q(t)\,x \tag{1}$$

be a nonoscillatory differential equation on an interval $[t_0, \infty)$. If $q(t) \leqq 0$, then $\frac{x'(t)}{x(t)}$ is a nonincreasing function for every solution x of (1). This follows from the fact that

$$\left(\frac{x'}{x}\right)' = \frac{x''}{x} - \frac{x'^2}{x^2} = q - \left(\frac{x'}{x}\right)^2 \leqq 0.$$

It is also well known that the inequality $q(t) \geqq 0$ implies that (1) is nonoscillatory and $\frac{x'(t)}{x(t)} \leqq 0$ for the principal solution $x(t)$ of (1) (see e. g. [1] p. 355).

In this paper there will be constructed a nonoscillatory differential equation (1) for which the logarithmic derivative of $x(t)$ is unbounded from above for $t \to \infty$.

Theorem. *There exists a continuous function* $q(t)$,

$$\liminf_{t \to \infty} q(t) \geqq 0 \tag{2}$$

such that the equation (1) *is nonoscillatory and its principal solution* $x(t)$ *has the property*

$$\limsup_{t \to \infty} \frac{x'(t)}{x(t)} = \infty. \tag{3}$$

Proof. Let $\{f_n(t)\}$ be an arbitrary sequence of functions $f_n : R \to R$ with the following properties:

i) $f_n(t) \in C^1(R)$;

ii) $f_n(t) \equiv 0$ for $t \leqq 0$ and $t \geqq n + 1$;

iii) $f_n\left(\frac{1}{n}\right) = n,\ 0 \leqq f_n(t) \leqq n$ on R;

iv) $f_n'(t) \geqq -1$.

If we denote $\alpha_n = \int_0^{n+1} f_n(t)\,dt$, then there is evidently

$$\frac{1}{2}n^2 \leqq \alpha_n \leqq n^2, \tag{4}$$

since

$$n + \frac{1}{n} - t \leqq f_n(t) \leqq n + 1 - t \qquad \text{for } t \in \left[\frac{1}{n},\ n+1\right].$$

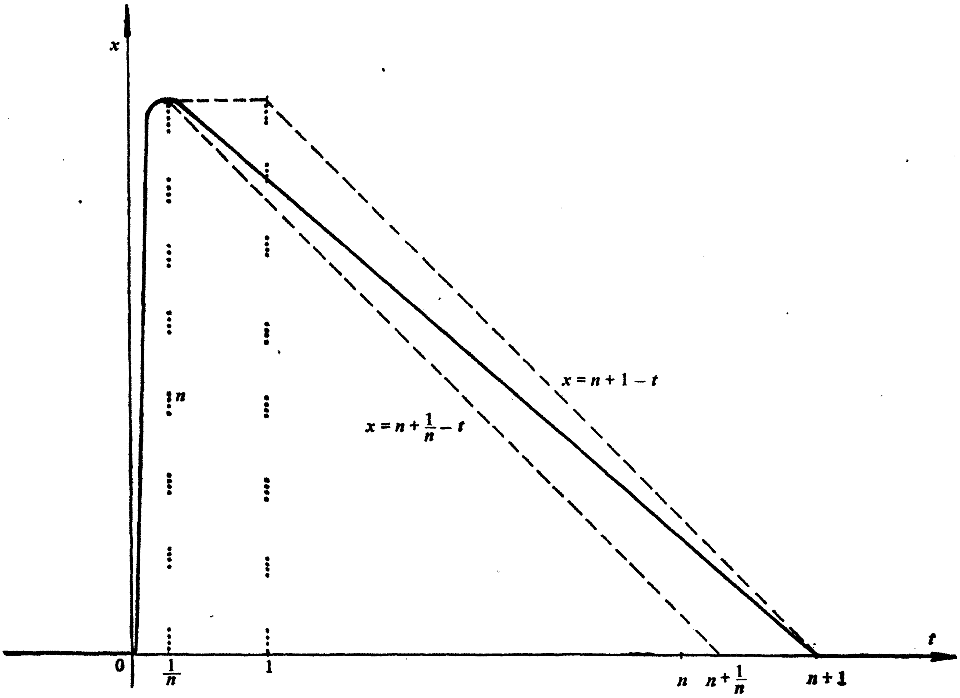

Fig. 1

Define

$$q_1 = 2, \qquad q_{n+1} = n(q_1^2 + \ldots + q_n^2), \tag{5}$$

$$F_n(t) = \int_0^t f_{q_n}(s)\,ds. \tag{6}$$

Then

$$F_n(t) = \alpha_{q_n} \qquad \text{for } t \geqq q_n + 1 \qquad \text{in view of ii)},$$

$$F_n'\left(\frac{1}{q_n}\right) = q_n \qquad \text{in view of iii)},$$

$$F_n''(t) \geqq -1 \qquad \text{for } t \in R \qquad \text{in view of iv).}$$

If we define

$$\sigma_0 = 0, \qquad \sigma_n = \sigma_{n-1} + \alpha_{qn}, \qquad b_n = \sigma_n^2,$$

then there is

$$\sigma_{n+1}^2 - \sigma_n^2 = (\sigma_{n+1} + \sigma_n)(\sigma_{n+1} - \sigma_n) \geqq 2\alpha_{qn+1} \geqq q_{n+1}^2 > q_n + 1,$$

so that

$$b_n + q_n + 1 < b_{n+1}, \qquad n = 1, 2, \ldots .$$

Let $x(t)$ be defined by means of the formula

$$x(t) = \sigma_{n-1} + F_n(t - b_n), \qquad t \in [b_n, b_{n+1}).$$

Then

$$x(t) \leqq \sigma_{n-1} + \alpha_{qn} = \sigma_n = \sqrt{b_n} \leqq \sqrt{t} \qquad \text{for } b_n \leqq t < b_{n+1}$$

and

$$\lim_{t \to b_{n+1}-} x(t) = \sigma_n = x(b_{n+1}).$$

Thus, $x(t)$ is continuous, has a continuous derivative of the second order and $x'(t) = f_{qn}(t - b_n) \geqq 0$ on $[b_n, b_{n+1})$, $n = 1, 2, \ldots$, so that $x(t)$ is nondecreasing. Since

$$x(b_{n+1}) = \sigma_n \geqq \alpha_{qn} \geqq \frac{q_n^2}{2} \to \infty,$$

it is $x(t) \to \infty$ for $t \to \infty$. Let $k_n = b_n + \dfrac{1}{q_n}$. Then $x'(k_n) = f_{qn}\left(\dfrac{1}{q_n}\right) = q_n$,

$$x(k_n) = \sigma_{n-1} + F_n\left(\frac{1}{q_n}\right) < \sigma_{n-1} + \frac{1}{q_n} f_{qn}\left(\frac{1}{q_n}\right) = 1 + \sigma_{n-1}.$$

Consequently

$$\frac{x'(k_n)}{x(k_n)} > \frac{q_n}{1 + \sigma_{n-1}}.$$

Since in view of (5)

$$\sigma_{n-1} = \sum_1^{n-1} \alpha_{qk} \leqq \sum_1^{n-1} q_k^2 = \frac{q_n}{n-1},$$

we have

$$\frac{q_n}{1 + \sigma_{n-1}} \geqq \frac{q_n}{1 + \dfrac{q_n}{n-1}} = \frac{1}{\dfrac{1}{q_n} + \dfrac{1}{n-1}} \to \infty.$$

Thus the solution $x(t)$ satisfies (3). If we denote $q(t) = \dfrac{x''(t)}{x(t)}$, it is with respect to (iv) $q(t) \geqq -\dfrac{1}{x(t)} \to 0$ since $x(t) \to \infty$ for $t \to \infty$. This implies (2). The inequality

$x(t) \leqq \sqrt{t}$ guaranties the divergence of the integral $\int_{b_1} x^{-2}$ for $t \to \infty$ which means that $x(t)$ is a principal solution.

The proof is complete.

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