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Ivan Kolář

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CONNECTIONS IN 2-FIBERED MANIFOLDS

IVAN KOLÁŘ, Brno

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Given an arbitrary fibered manifold $p\colon Y \to X$, a connection on Y means any section $\Gamma\colon Y \to J^1Y$ (= the first jet prolongation of Y), [5], [9]. To underline this general point of view, we shall sometimes say that Γ is a generalized connection. In local fiber coordinates x^i, y^p on Y, the equations of Γ are

$$\Gamma \equiv \mathrm{d}y^p = F_i^p(x, y)\,\mathrm{d}x^i \tag{1}$$

with arbitrary smooth functions F_i^p. At first sight, this approach seems to be too general to get any deeper result. However, our recent research suggests that the theory of generalized connection can be as rich as the classical theory of principal or linear connections.

The basic tool in the theory of generalized connections is the vertical prolongation $V\Gamma$ of Γ. This is a connection on the vertical tangent bundle VY of Y considered as a fibered manifold over X. A natural generalization leads to the concept of a projectable connection on a 2-fibered manifold. We treat some basic operations with such connections and apply the results to the generalized connections on Y. Another important example of a 2-fibered manifold is the first jet prolongation J^1Y of $Y \to X$. We define the torsion of a projectable connection on J^1Y and give its interpretation in terms of the alternation tensor of a special pair of non-holonomic 2-jets. Finally, we study a connection $J^1(\Gamma, \Lambda)$ on J^1Y determined by a generalized connection Γ on Y and a linear connection Λ on TX. – Our consideration is in the category C^∞.

1. A 2-fibered manifold is a quintuple $U \xrightarrow{q} Y \xrightarrow{p} X$, where $q\colon U \to Y$ and $p\colon Y \to X$ are fibered manifolds, [6]. Hence $r = p \circ q\colon U \to X$ is also a fibered manifold. In the sequel, J^1U will always mean the first jet prolongation of $r\colon U \to X$. Let $J^1q\colon J^1U \to J^1Y$ be the induced map $j_x^1\sigma \mapsto j_x^1(q\sigma)$. A connection $\Sigma\colon U \to J^1U$ will be called projectable, if there exists a (unique) connection $\Gamma\colon Y \to J^1Y$ satisfying $\Gamma \circ q = J^1q \circ \Sigma$. In fiber coordinates x^i, y^p, u^α on $U \to Y \to X$, the equations of Σ are

$$\Sigma \equiv \begin{cases} \mathrm{d}y^p = F_i^p(x, y)\,\mathrm{d}x^i, \\ \mathrm{d}u^\alpha = F_i^\alpha(x, y, u)\,\mathrm{d}x^i. \end{cases} \tag{2}$$

In this paper, we shall study projectable connections only.

If $U \xrightarrow{q} Y$ is a vector bundle, $U \xrightarrow{q} Y \xrightarrow{p} X$ or $U \xrightarrow{r} X$ is said to be a semi-vector bundle. Then $J^1U \to J^1Y \to X$ is also a semi-vector bundle. Given a projectable connection $\Sigma\colon U \to J^1U$ over $\Gamma\colon Y \to J^1Y$, we have a map $\Sigma \mid U_y$ of vector space U_y into vector space $(J^1U)_{\Gamma(y)}$ for every $y \in Y$. If all these maps are linear, then Σ is called a semi-linear connection. In linear coordinates u^α, the equations of Σ are

(3)
$$\Sigma \equiv \begin{cases} \mathrm{d}y^p = F^p_i(x, y)\,\mathrm{d}x^i, \\ \mathrm{d}u^\alpha = F^\alpha_{\beta i}(x, y)\,u^\beta\,\mathrm{d}x^i. \end{cases}$$

(The second row is linear in u^α, that is why Σ is said to be semi-linear.)

The simpliest example of a semi-linear connection is the vertical prolongation $V\Gamma$ of a connection $\Gamma\colon Y \to J^1Y$. Given a vector field ξ on X, $\xi \equiv \xi^i(x)\dfrac{\partial}{\partial x^i}$, denote by $\Gamma\xi$ the Γ-lift of ξ,

(4)
$$\Gamma\xi \equiv \xi^i(x)\frac{\partial}{\partial x^i} + F^p_i(x, y)\,\xi^i(x)\frac{\partial}{\partial y^p}.$$

Using flows, [6], we prolong $\Gamma\xi$ into a vector field $V\Gamma\xi$ on VY,

(5)
$$V\Gamma\xi \equiv \xi^i\frac{\partial}{\partial x^i} + F^p_i\xi^i\frac{\partial}{\partial y^p} + \frac{\partial F^p_i}{\partial y^q}\xi^i\eta^q\frac{\partial}{\partial \eta^p},$$

where $\eta^p = \mathrm{d}y^p$ are the induced coordinates on VY. As there are no derivatives of ξ^i in (5), this formula describes lifting with respect to a unique connection $V\Gamma$ on $VY \to X$,

(6)
$$V\Gamma \equiv \begin{cases} \mathrm{d}y^p = F^p_i(x, y)\,\mathrm{d}x^i, \\ \mathrm{d}\eta^p = \dfrac{\partial F^p_i}{\partial y^q}\,\eta^q\,\mathrm{d}x^i. \end{cases}$$

(Another construction of $V\Gamma$ is given in [1].)

Let $W \to Y \to X$ be another semi-vector bundle and Π a semi-linear connection on W over Γ,

(7)
$$\Pi \equiv \begin{cases} \mathrm{d}y^p = F^p_i(x, y)\,\mathrm{d}x^i, \\ \mathrm{d}w^\lambda = G^\lambda_{\mu i}(x, y)\,w^\mu\,\mathrm{d}x^i. \end{cases}$$

The tensor product $U \otimes W$ over Y is a semi-vector bundle $U \otimes W \to Y \to X$. Similarly to the classical case, see e.g. [3], one defines the tensor product $\Sigma \otimes \Pi$. This is a semi-linear connection on $U \otimes W$ over Γ with the following equations

(8)
$$\Sigma\otimes\Pi \equiv \begin{cases} \mathrm{d}y^p = F^p_i(x, y)\,\mathrm{d}x^i, \\ \mathrm{d}v^{\alpha\lambda} = (F^\alpha_{\beta i}v^{\beta\lambda} + G^\lambda_{\mu i}v^{\alpha\mu})\,\mathrm{d}x^i, \end{cases}$$

where $v^{\alpha\lambda}$ are the induced coordinates on $U \otimes W$.

Consider now a 2-fibered manifold $U \to Y \to X$ with connection (2) and a map f: $Z \to X$, $x^i = f^i(z)$. On the induced 2-fibered manifold $f^*U \to f^*Y \to Z$, we get an induced connection $f^*\Sigma$ over $f^*\Gamma$, [7],

$$(9) \qquad f^*\Sigma \equiv \begin{cases} dy^p = F_i^p(f(z), y)\dfrac{\partial f^i}{\partial z^a} dz^a, \\ du^\alpha = F_i^\alpha(f(z), y, u)\dfrac{\partial f^i}{\partial z^a} dz^a. \end{cases}$$

A more interesting situation is if we have another fibered manifold s: $W \to Z$ and a fibered manifold morphism F: $W \to Y$ over f: $Z \to X$. The induced fibered manifold $F^*U \to W$ is 2-fibered over Z. Let Δ: $W \to J^1 W$ be a connection F-related with Γ, [7],

$$(10) \qquad \Delta \equiv dw^\lambda = G_b^\lambda(z, w)\, dz^a.$$

Then we construct an induced connection $F^*(\Sigma, \Delta)$ on F^*U as follows. Let $(w, u) \in F^*U$, $\Sigma(u) = j_x^1\sigma$, $\Delta(w) = j_z^1\varrho$, so that $\Gamma(qu) = j_x^1 s\sigma$ and sections $s\sigma$ and ϱ satisfy $s\sigma \circ f = F \circ \varrho$. Hence $(\varrho(t), \sigma(f(t)))$, $t \in Z$, is a section of F^*U and we set $F^*(\Sigma, \Delta)(w, u) = j_z^1(\varrho(t), \sigma(f(t)))$. In coordinates,

$$(11) \qquad F^*(\Sigma, \Delta) \equiv \begin{cases} dw^\lambda = G_a^\lambda\ \ (z, w)\, dz^a, \\ du^\alpha = F_i^\alpha(f(z), g(z, w), u)\dfrac{\partial f^i}{\partial z^a} dz^a, \end{cases}$$

where $x^i = f^i(z)$, $y^p = g^p(z, w)$ is the coordinate expression of F. Thus, $F^*(\Sigma, \Delta)$ is a projectable connection over Δ. Obviously, if Σ is semi-linear, then $F^*(\Sigma, \Delta)$ is also semi-linear.

2. Given a section σ: $X \to Y$, $y^p = \sigma^p(x)$, the absolute differential $\nabla_\Gamma\sigma(x)$ of σ with respect to connection (1) at $x \in X$ is the projection of the tangent map $T_x\sigma$ into V_yY in the direction $\Gamma(y)$, $y = \sigma(x)$. The coordinate expression of $\nabla_\Gamma\sigma$: $X \to VY \otimes T^*X$ is

$$(12) \qquad \nabla_\Gamma\sigma \equiv \eta^p = \frac{\partial\sigma^p}{\partial x^i} - F_i^p(x, \sigma(x)).$$

Let s: $Z \to X$ be another fibered manifold with a connection Δ,

$$(13) \qquad \Delta \equiv z^a = G_i^a(x, z)\, dx^i,$$

and φ: $Y \to Z$, $z^a = \varphi^a(x, y)$, a base-preserving morphism. We have $\Gamma(y) = j_{x_i}^1\sigma$ for a local section σ of Y and $\varphi \circ \sigma$ is a local section of Z, $y \in Y$. The absolute differential of $\varphi \circ \sigma$ with respect to Δ at x will be denoted by ${}_\Gamma\nabla_\Delta\varphi(y)$. Hence ${}_\Gamma\nabla_\Delta\varphi$: $Y \to VZ \otimes T^*X$,

$$(14) \qquad {}_\Gamma\Delta_\Delta\varphi \equiv \zeta^a = \frac{\partial\varphi^a}{\partial x^i} + \frac{\partial\varphi^a}{\partial y^p}F_i^p - G_i^a(x, \varphi(x, y)).$$

By (14), we deduce

Proposition 1. *${}_{\Gamma}\nabla_{\Delta}\varphi = 0$ iff vector fields $\Gamma\xi$ and $\Delta\xi$ are φ-related for any vector field ξ on X.*

To express (14) in a more concise form, we recall that every function $\psi\colon Y \to R$ determines the formal differential $D\psi\colon J^1 Y \to T^*X$, $D\psi \equiv \left(\frac{\partial\psi}{\partial x^i} + \frac{\partial\psi}{\partial y^p} y_i^p\right) dx^i$, where y_i^p are the induced coordinates on $J^1 Y$, [4]. The function $D_i\psi\colon = \frac{\partial\psi}{\partial x^i} + \frac{\partial\psi}{\partial y^p} y_i^p$ is called the i-th formal derivative of ψ. Given $\Gamma\colon Y \to J^1 Y$, the composition $D_{\Gamma i}\psi := := (D_i\psi) \circ \Gamma$,

$$D_{\Gamma i}\psi = \frac{\partial\psi}{\partial x^i} + \frac{\partial\psi}{\partial y^p} F_i^p \tag{15}$$

will be said to be the i-th derivative of ψ with respect to Γ. We can now rewrite (14) as

$${}_{\Gamma}\nabla_{\Delta}\varphi \equiv \zeta^a = D_{\Gamma i}\varphi^a - G_i^a(x, \varphi). \tag{16}$$

Consider a semi-vector bundle $U \to Y \to X$ with a semi-linear connection (3) and a section

$$\varphi\colon Y \to U \otimes \wedge^k T^*X,\ u^\alpha = \varphi^\alpha_{i_1 \dots i_k}(x, y)\, dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

We shall show that φ determines a section $d_\Sigma\varphi\colon Y \to U \otimes \wedge^{k+1} T^*X$ (in the classical case of a linear connection on a vector bundle this concept is due to Koszul). The shortest way how to define $d_\Sigma\varphi$ is as follows. Take an auxiliary linear symmetric connection Λ on TX that determines a connection $\wedge^k\Lambda^*$ on $\wedge^k T^*X$. Then $\Sigma \otimes \wedge^k\Lambda^*$ is a semi-linear connection on $U \otimes \wedge^k T^*X$ and we can construct ${}_{\Gamma}\nabla_{\Sigma\otimes\wedge^k\Lambda^*}\varphi\colon Y \to \to V(U \otimes \wedge^k T^*X) \otimes T^*X$. Applying antisymmetrization and natural identifications, we obtain a section $d_\Sigma\varphi\colon Y \to U \otimes \wedge^{k+1} T^*X$ that does not depend on the choice of Λ. In coordinates,

$$d_\Sigma\varphi \equiv u^\alpha = (D_{\Gamma i}\varphi^\alpha_{i_1 \dots i_k} - F^\alpha_{\beta i}\varphi^\beta_{i_1 \dots i_k})\, dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}. \tag{17}$$

We are going to use this operation in the theory of generalized connections. We recall, [5], that the curvature of Γ is a section $\Omega_\Gamma\colon Y \to VY \otimes \wedge^2 T^*X$,

$$\Omega_\Gamma \equiv \eta^p = (D_{\Gamma j} F_i^p)\, dx^i \wedge dx^j =: \Omega^p_{ij}\, dx^i \wedge dx^j. \tag{18}$$

Using the vertical prolongation $V\Gamma$, we can construct $d_{V\Gamma}\Omega_\Gamma\colon Y \to VY \otimes \wedge^3 TX^*$. By direct evaluation, we deduce

Proposition 2. (*Bianchi identity.*) *It holds*

$$d_{V\Gamma}\Omega_\Gamma = 0. \tag{19}$$

Since $J^1 Y \to Y$ is an affine bundle associated with $VY \otimes T^*X$, two connections (1) and Δ,

(20) $$\Delta \equiv \mathrm{d}y^p = G_i^p(x, y)\,\mathrm{d}x^i,$$

determine a section $\overrightarrow{\Gamma\Delta}\colon Y \to VY \otimes T^*X$,

(21) $$\overrightarrow{\Gamma\Delta} \equiv \eta^p = (G_i^p - F_i^p)\,\mathrm{d}x^i.$$

The exterior differential $\varkappa(\Gamma, \Delta) := \mathrm{d}_{V\Gamma}\overrightarrow{\Gamma\Delta}\colon Y \to VY \otimes \wedge^2 T^*X$ will be called the mixed curvature of the ordered pair (Γ, Δ). In coordinates,

(22) $$\varkappa(\Gamma, \Delta) \equiv \eta^p = \left[(D_{\Gamma j}G_i^p - D_{\Gamma j}F_i^p - \frac{\partial F_j^p}{\partial y^q}(G_i^q - F_i^q)\right]\mathrm{d}x^j \wedge \mathrm{d}x^i.$$

Taking into account $\varkappa(\Delta, \Gamma) = \mathrm{d}_{V\Delta}\overrightarrow{\Delta\Gamma}$, we verify directly the following relation

(23) $$\varkappa(\Gamma, \Delta) - \varkappa(\Delta, \Gamma) = 2\Omega_\Gamma - 2\Omega_\Delta.$$

The connections $\Gamma = \Gamma_0$ and $\Delta = \Gamma_1$ determine a pencil $\Gamma_t = t\Gamma_1 + (1 - t)\Gamma_0$, $t \in \mathrm{R}$. We find easily

Proposition 3. *The curvature Ω_t of Γ_t is*

(24) $$\Omega_t = \Omega_0 - t\varkappa(\Gamma_0, \Gamma_1) + t^2[\varkappa(\Gamma_0, \Gamma_1) - \Omega_0 + \Omega_1].$$

In particular, all connections of a pencil are integrable if any three connections of the pencil are integrable.

3. Consider a projectable connection Σ over Γ on 2-fibered manifold $J^1Y \xrightarrow{\beta} Y \xrightarrow{p} X$,

(25) $$\Sigma \equiv \begin{cases} \mathrm{d}y^p = F_i^p(x, y)\,\mathrm{d}x^i, \\ \mathrm{d}y_i^p = F_{ij}^p(x, y, \dot{y})\,\mathrm{d}x^j, \end{cases}$$

where $\dot{y}$ denotes the collection y_i^p. There is a canonical morphism $\psi\colon TJ^1Y \to VY$ (called the structure 1-form in [2]),

(26) $$\psi \equiv \eta^p = \mathrm{d}y^p - y_i^p\,\mathrm{d}x^i.$$

Combining this morphism with lifting with respect to Γ, we get a section $(\psi, \Gamma)\colon J^1Y \to \beta^*VY \otimes T^*X$,

(27) $$(\psi, \Gamma) \equiv \eta^p = (F_i^p(x, y) - y_i^p)\,\mathrm{d}x^i.$$

By (11), we derive from $V\Gamma$ and Σ an induced connection $\beta^*(V\Gamma, \Sigma)$ on β^*VY,

(28) $$\beta^*(V\Gamma, \Sigma) \equiv \begin{cases} \mathrm{d}y^p = F_i^p(x, y)\,\mathrm{d}x^i, \\ \mathrm{d}y_i^p = F_{ij}^p(x, y, \dot{y})\,\mathrm{d}x^j, \\ \mathrm{d}\eta^p = \dfrac{\partial F_i^p}{\partial y^q}\,\eta^q\,\mathrm{d}x^i. \end{cases}$$

We define the torsion $\tau\Sigma$ of Σ by

(29) $$\tau\Sigma := \mathrm{d}_{\beta^*(V\Gamma,\Sigma)}(\psi, \Gamma)\colon J^1Y \to \beta^*VY \otimes \wedge^2 T^*X.$$

In coordinates,

(30) $$\tau\Sigma \equiv \eta^p = \left[D_{\Gamma i}F^p_j + F^p_{ij} - \frac{\partial F^p_i}{\partial y^q}(F^q_j - y^q_j)\right]\mathrm{d}x^i \wedge \mathrm{d}x^j.$$

To find an interpretation of $\tau\Sigma$, we first introduce a general concept related with non-holonomic 2-jets. Given two manifolds M, N, there are two natural projections p_1, p_2 of the space of all non-holonomic 2-jets $\tilde{J}^2(M, N)$ into $J^1(M, N)$, namely $p_1(j^1_x\sigma) = \sigma(x)$ and $p_2(j^1_x\sigma) = j^1_x(\beta\sigma)$. Let $A, B \in \tilde{J}^2_x(M, N)_n$ satisfy

(31) $$p_1A = p_2B, \qquad p_2A = p_1B.$$

In coordinates, let $A = (a^p_i, b^p_i, a^p_{ij})$ and $B = (b^p_i, a^p_i, b^p_{ij})$. By direct evaluation, we deduce that

(32) $$a^p_{[ij]} + b^p_{[ij]}$$

are coordinates of a tensor, which will be denoted by $\varrho(A, B) \in T_yN \otimes \wedge^2 T^*_xM$ and called the alterpation tensor of A and B. In particular, if A is a semi-holonomic 2-jet and B any holonomic 2-jet over the same first order jet, then $\varrho(A, B)$ coincides with the difference tensor of A.

In our situation, $\Gamma\colon Y \to J^1Y$ is prolonged into $J^1\Gamma\colon J^1Y \to \tilde{J}^2Y$. (This is not a connection, as $J^1\Gamma$ is a section of p_2 and not of p_1.) However, Σ and $J^1\Gamma$ satisfy (31) at every point, so that we obtain a map $\varrho(\Sigma, J^1\Gamma)\colon J^1Y \to \beta^*VY \otimes \wedge^2T^*X$. By (30) and (32), we deduce

Proposition 4. *It holds*

(33) $$\tau\Sigma = -2\beta^*\Omega_\Gamma + \varrho(\Sigma, J^1\Gamma).$$

4. Using flows, we prolong vector field (4) into a vector field $J^1\Gamma\xi$ on J^1Y, [8],

(34) $$J^1\Gamma\xi \equiv \Gamma\xi + \left[(D_iF^p_j)\,\xi^j + \frac{\partial\xi^j}{\partial x^i}(F^p_j - y^p_j)\right]\frac{\partial}{\partial y^p_i}.$$

This formula represents a map of the fiber product $J^1Y \oplus J^1TX$ into TJ^1Y. Consider further a linear connection $\Lambda\colon TX \to J^1TX$,

(35) $$\Lambda \equiv \mathrm{d}\xi^i = \Gamma^i_{jk}(x)\,\xi^j\,\mathrm{d}x^k.$$

If we compose Λ with (34), we obtain a mapping $J^1Y \oplus TX \to TJ^1Y$. This is lifting with respect to a connection $J^1(\Gamma, \Lambda)$ on J^1Y called the prolongation of Γ with respect to Λ. In coordinates,

(36) $$J^1(\Gamma, \Lambda) \equiv \begin{cases}\mathrm{d}y^p = F^p_i(x, y)\,\mathrm{d}x^i,\\ \mathrm{d}y^p_i = [D_iF^p_j + \Gamma^k_{ji}(F^p_k - y^p_k)]\,\mathrm{d}x^j.\end{cases}$$

We now describe another construction of $J^1(\Gamma, \Lambda)$. Let T^1_n be the functor of first order n-dimensional velocities, $n = \dim X$. Hence $p\colon Y \to X$ is prolonged into $T^1_n p\colon T^1_n Y \to T^1_n X$. According to [5], Γ determines a connection $T^1_n \Gamma$ on the latter fibered manifold,

$$(37) \qquad T^1_n \Gamma \equiv \begin{cases} dy^p = F^p_i(x, y)\, dx^i, \\ d\eta^p_i = \left(\xi^k_i \dfrac{\partial F^p_j}{\partial x^k} + \eta^q_i \dfrac{\partial F^p_j}{\partial y^q}\right) dx^j + F^p_j\, d\xi^j_i, \end{cases}$$

where ξ^j_i, η^p_i are the induced coordinates on $T^1_n Y$. The elements of $T^1_n X$ being n-tuples of tangent vectors on X, denote by $\hat{T}^1_n X$ the subspace of linearly independent n-tuples. Obviously, Λ induces the following connection on $\hat{T}^1_n X$

$$(38) \qquad d\xi^k_i = \Gamma^k_{lj}\xi^l_i\, dx^j.$$

Let $\hat{T}^1_n Y$ be the restriction of $T^1_n Y$ over $\hat{T}^1_n X$. The composition of (37) and (38) is a connection Π on $\hat{T}^1_n Y \to X$, whose equations are (1), (38) and

$$(39) \qquad d\eta^p_i = \left(\xi^k_i \frac{\partial F^p_j}{\partial x^k} + \eta^q_i \frac{\partial F^p_j}{\partial y^q} + F^p_k \Gamma^k_{lj}\xi^l_i\right) dx^j.$$

There is a canonical projection $\lambda\colon \hat{T}^1_n Y \to J^1 Y$ transforming a vector n-tuple into its linear span, $\lambda \equiv y^p_i = \eta^p_j \bar{\xi}^j_i$ with $\bar{\xi}^j_i \xi^k_j = \delta^k_i$. Denote by $\bar{\Lambda}$ the conjugate connection

$$(40) \qquad \bar{\Lambda} \equiv d\xi^i = \Gamma^i_{kj}(x)\, \xi^j\, dx^k.$$

By direct evaluation, we deduce

Proposition 5. *Π is λ-projectable and $J^1(\Gamma, \bar{\Lambda})$ is the underlying connection on $J^1 Y$, i.e. $J^1(\Gamma, \bar{\Lambda}) \circ \lambda = J^1\lambda \circ \Pi$.*

By (30), we find the following coordinate formula for the torsion of $J^1(\Gamma, \Lambda)$

$$(41) \qquad \tau J^1(\Gamma, \Lambda) \equiv \eta^p = [-2\Omega^p_{ij} + \Gamma^k_{ji}(F^p_k - y^p_k)]\, dx^i \wedge dx^j.$$

To interpret (41) geometrically, we recall that the classical torsion $t\Lambda$ of Λ is a mapping $t\Lambda\colon X \to TX \otimes \wedge^2 T^*X$, $\xi^k = \Gamma_{ij}\, dx^i \wedge dx^j$. Consider further $(\psi, \Gamma)\colon J^1 Y \to \beta^* VY \otimes T^*X$ and construct tensor contraction $\langle(\psi, \Gamma), t\Lambda\rangle\colon J^1 Y \to \beta^* VY \otimes \wedge^2 T^*X$. By (27) and (41), we deduce

Proposition 6. *It is*

$$(42) \qquad \tau p(\Gamma, \Lambda) = -2\beta^*\Omega_\Gamma - \langle(\psi, \Gamma), t\Lambda\rangle.$$

In particular, $J^1(\Gamma, \Lambda)$ is without torsion iff Γ is integrable and Λ is symmetric.

To determine the curvature $\Omega_{J^1(\Gamma,\Lambda)}\colon J^1 Y \to V(J^1 Y) \otimes \wedge^2 T^*X$, we first recall an exact sequence

$$(43) \qquad 0 \to \beta^* VY \otimes T^*X \to VJ^1 Y \xrightarrow{V\beta} VY \to 0,$$

see e.g. [4]. The classical curvature $\Omega_{\bar{\Lambda}}$ of $\bar{\Lambda}$ can be considered as a map $X \to TX \otimes \otimes T^*X \otimes \wedge^2 T^*X$. Applying tensor contraction, we obtain $\langle(\psi, \Gamma), \Omega_{\bar{\Lambda}}\rangle$: $J^1Y \to \to \beta^* VY \otimes T^*X \otimes \wedge^2 T^*X$. By (43), the values of $\langle(\psi, \Gamma), \Omega_{\bar{\Lambda}}\rangle$ lie in $VJ^1Y \otimes \wedge^2 T^*X$. Hence

$$J^1_\Lambda \Omega_\Gamma := \Omega_{J^1(\Gamma, \Lambda)} - \langle(\psi, \Gamma), \Omega_{\bar{\Lambda}}\rangle \tag{44}$$

is a well-defined map $J^1Y \to VJ^1Y \otimes \wedge^2 T^*X$ with the following coordinate expression

$$\eta^p = \Omega^p_{ij}\, dx^i \wedge dx^j,$$
$$\eta^p_k = (D_k \Omega^p_{ij} + \Gamma^l_{ki}\Omega^p_{lj} + \Gamma^l_{kj}\Omega^p_{il})\, dx^i \wedge dx^j. \tag{45}$$

(We shall give a direct construction of $J^1_\Lambda \Omega_\Gamma$ in a next paper.) In particular, (44) and (45) imply that $J^1(\Gamma, \Lambda)$ is integrable iff both Γ and Λ are integrable.

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I. Kolář
Institute of Mathematics of the ČSAV,
branch Brno,
662 95 Brno, Janáčkovo nám. 2a,
Czechoslovakia