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ON EXISTENCE OF OSCILLATORY SOLUTION OF THE SYSTEM OF DIFFERENTIAL EQUATIONS

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Consider the system of differential equations

$$y_i' = f_i(t, y_1, \dots, y_n), \qquad i \in N_n \tag{1}$$

where $f_i : D \to R$, $D = [0, \infty) \times R^n$ are continuous and there exist numbers $\nu_i \in \{0, 1\}$, $i \in N_n$ such that

$$(-1)^{\nu_i} f_i(t, x_1, \dots, x_n) \operatorname{sgn} x_{i+1} > 0 \qquad \text{for } x_{i+1} \neq 0, \tag{2}$$
$$t \in [0, \infty), x_i \in R, i \in N_n; \qquad N_n = \{1, 2, \dots, n\}, R = (-\infty, \infty).$$

Denote $R_+ = [0, \infty)$.

Let $(y_i)_1^n$ be a non-trivial solution of (1), defined on $[a, b)$, $0 \leqq a < b \leqq \infty$. Then it is said to be proper if $b = \infty$ and

$$\sup \{ \sum_{i=1}^{n} y_i(s) : t \leqq s < b \} > 0 \qquad \text{for } t \in [a, \infty).$$

A proper solution is called oscillatory if every y_i has a sequence of zeros tending to ∞.

$(y_i)_1^n$ is called singular if one of the two following relations holds

(i) $b = \infty$ and there exists a number $b_1 \in [a, \infty)$ such that $y_i(t) = 0$ for $t \in [b_1, \infty)$, $i \in N_n$.

(ii) $b < \infty$ and

$$\limsup_{t \to b-} \sum_{i=1}^{\infty} | y_i(t) | = \infty \tag{3}$$

(denote $b_1 = b$ in this case).

A singular solution is called oscillatory if every y_i has a sequence of zeros tending to b_1.

The system (1) is said to satisfy Property A (Property A^s) if each proper (singular) solution $(y_i)_1^n$ of (1) for n even is oscillatory and for n odd is either oscillatory or there exist numbers $\delta_i \in \{0, 1\}$ and $a_1 \in [a, b)$ such that

$$\lim_{t \to b-} | y_i(t) | = 0, \qquad (-1)^{\delta_i} y_i(t) \geqq 0, \qquad i \in N_n, \ t \in [a_1, b). \tag{4}$$

The system (1) is said to satisfy Property B (Property B^s) if each proper (singular) solution of (1) for n even is either oscillatory or satisfies (4) or satisfies

$$\lim_{t \to b-} |y_i(t)| = \infty, \qquad i \in N_n \tag{5}$$

and for n odd is either oscillatory or satisfies (5).

In this paper some sufficient conditions will be given under which (1) satisfies Property A or B. With regard to differential equations of higher order this problem was studied in [2] and [3]. The paper [1] contains some comparison theorems for the system (1).

Theorem 1. *Let there exist continuous functions* $g_i : R_+^2 \to R_+$, $i \in N_n$ *such that* $g_i(x_i, 0) \equiv 0$, $g_i(x_i, x_{i+1}) > 0$ *for* $x_{i+1} > 0$, *and* $x_i \geqq 0$, g_i *are non-decreasing with respect to the second argument and*

$$|f_i(t, x_1, \dots, x_n)| \geqq g_i(|x_i|, |x_{i+1}|), \qquad i \in N_n, x_{n+1} = x_1 \tag{6}$$

on D. If $\sum_{i=1}^{n} v_i$ *is odd (even), then the system* (1) *has Property A (Property B).*

Proof. Let $(y_i)_1^n$ be an arbitrary proper solution of (1) and not oscillatory. Then, according to (1) and (2) there exists a number $a_1 \in [a, \infty)$ such that

$$y_i(t) \neq 0, \qquad y_i'(t) \neq 0 \qquad \text{for } t \in [a_1, \infty) \text{ and } i \in N_n.$$

Thus, according to (2) there exist numbers α_i, $\alpha_i \in \{0, 1\}$, $i \in N_n$ such that for $t \in [a_1, \infty)$ there holds

$$\operatorname{sgn} y_i(t) = (-1)^{\alpha_i}, \quad \operatorname{sgn} y_i'(t) = (-1)^{\alpha_{i+1} + v_i}, \qquad i \in N_n, \alpha_{n+1} = \alpha_1. \tag{7}$$

First, we prove the following relation

$$j \in N_n, \quad \lim_{t \to \infty} y_j'(t) = 0, \quad |y(t)| \leqq M < \infty, \quad t \in [a_1, \infty) \Rightarrow \lim_{t \to \infty} y_{j+1}(t) = 0 \tag{8}$$

($y_{n+1} = y_1$). Suppose on the contrary that there exists a sequence of numbers $(\tau_k)_1$, $\tau_k \in [a_1, \infty)$ such that

$$\lim_{k \to \infty} \tau_k = \infty, \qquad |y_{j+1}(\tau_k)| \geqq M_1 > 0, \qquad k \in N.$$

Then

$$|y_j'(\tau_k)| = |f_j(\tau_k, y_1(\tau_k), \dots, y_n(\tau_k))| \geqq g_j(|y_j(\tau_k)|, |y_{j+1}(\tau_k)|) \geqq$$
$$\geqq g_j(|y_j(\tau_k)|, M_1) \geqq \min_{0 \leqq s \leqq M} g_j(s, M_1) > 0$$

which contradicts $\lim_{t \to \infty} y_j'(t) = 0$. Thus (8) is valid.

Suppose that there exists an integer $j \in N_n$ such that

$$y_j(t)\, y_j'(t) < 0, \qquad y_{j+1}(t)\, y_{j+1}'(t) > 0, \qquad t \in [a_1, \infty). \tag{9}$$

Then, according to (8) $\lim_{t\to\infty} y_{j+1}(t) = 0$ which contradicts the second inequality of (9). It follows from (7) that (9) is not valid only in the two following cases

(10) $$Z_i = \alpha_i + \alpha_{i+1} + \nu_i \quad \text{is odd}, \qquad i \in N_n$$

(11) $$Z_i = \alpha_i + \alpha_{i+1} + \nu_i \quad \text{is even}, \qquad i \in N_n.$$

If (10) is valid, then

$$y_i(t)\, y_i'(t) < 0 \qquad \text{for } t \in [a_1, \infty),\ i \in N_n$$

and according to (8)

$$(10) \text{ is valid} \Rightarrow \lim_{t\to\infty} y_i(t) = 0, \qquad i \in N_n.$$

Suppose that (11) is valid. Then

(12) $$y_i(t)\, y_i'(t) > 0 \qquad \text{for } t \in [a_1, \infty),\ i \in N_n$$

and $|y_i|$ is increasing. Suppose that there exists an integer $j \in N_n$ such that $|y_j|$ does not tend to infinity. Then

$$\lim_{t\to\infty} |y_j(t)| = M_2 < \infty, \qquad \lim_{t\to\infty} y_j'(t) = 0$$

and according to (8) $\lim_{t\to\infty} y_{j+1}(t) = 0$, which contradicts (12). Thus the following relation holds

(13) $$(11) \textit{ is valid} \Rightarrow \lim_{t\to\infty} |y_i(t)| = \infty, \qquad i \in N_n.$$

Let $\sum_{i=1}^{n} \nu_i$ be odd. We have

(14) $$Z = \sum_{i=1}^{n} Z_i = 2\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \nu_i, \qquad Z \text{ is odd.}$$

If (11) is valid or (10) is valid and n is odd, then according to (10) and (11) Z is even which contradicts (14). If $\sum_{i=1}^{n} \nu_i$ is even, then according to the definition of Z, Z is even, too. Thus (10) is inadmissible for n odd. The theorem is proved.

Theorem 2. *Let $(y_i)_1^n$ be a singular solution of* (1) *and b_1 the number defined in its definition. Let there exist continuous functions $g_i : R_+ \times R \to (0, \infty)$, $i \in N_n$ and $G_i : R_+ \to (0, \infty)$, $i \in N_n$ such that $\int_0^\infty \frac{ds}{G_i(s)} = \infty$ and*

(15) $$|f_i(t, x_1, \ldots, x_n)| \leqq g_i(t, |x_{i+1}|)\, G_i(|x_i|), \qquad i \in N_n,\ x_{n+1} = x_1$$

on D. If $\sum_{i=1}^{n} \nu_i$ is odd (even), then the system (1) *has Property A^s (Property B^s).*

Proof. Let $(y_i)_1^n$ be not oscillatory. Then similarly to the proof of Theorem 1 there exist $a_1 \in [a, b_1)$, $\alpha_i, \alpha_i \in \{0, 1\}$ such that for $t \in [a_1, b_1)$ (7) holds. It follows from the estimation (see (15))

$$\left| \int_{|y_i(a_1)|}^{|y_i(t)|} \frac{ds}{G_i(s)} \right| = \int_{a_1}^{t} \frac{|y_i'(s)|\,ds}{G_i(|y_i(s)|)} = \int_{a_1}^{t} \frac{|f_i(s, y_1(s), \ldots, y_n(s))|\,ds}{G_i(|y_i(s)|)} \leqq$$

$$\leqq \int_{a_1}^{b_1} g_i(s, |y_{i+1}(s)|)\,ds, \qquad i \in N_n,\ y_{n+1} = y_1,\ t \in [a_1, b_1)$$

that there holds the relation

(16) $\qquad y_{i+1}$ *is bounded on* $[a_1, b_1) \Rightarrow y_i$ *is bounded on* $[a_1, b_1)$

Suppose that there exists an integer $j \in N_n$ such that (9) is valid for $t \in [a_1, b_1)$. Then y_j is bounded on $[a_1, b_1)$ and according to (16) y_i are bounded for all $i \in N_n$, too. But with respect to y_{j+1} which is positive and increasing we get the contradiction to (3).

The rest of the proof is the same as in Theorem 1, only (13) must be proved in another way. If (11) is valid, then (12) holds and all $|y_i|$ are non-decreasing on $[a_1, b_1)$ and positive. If there exists an index $j \in N_n$ such that

$$|y_j(t)| \leqq M < \infty, \qquad t \in [a_1, b_1),$$

then according to (16) all y_i are also bounded which contradicts (3).

The theorem is proved.

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