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# A REMARK ON ALGEBRAIC IDENTITIES FOR THE COVARIANT DERIVATIVE OF THE CURVATURE TENSOR

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In the author's paper [1] lectured in a shortened form at the Colloquium on Differential Geometry in Budapest, September 3–7, 1979, there are discussed, among others, the algebraic identities for the curvature tensor of a linear connection and its first covariant derivative. These well-known identities involve the so called first- and second Bianchi identities, and an identity expressing the standard antisymmetry property of both tensors with respect to two covariant indices. The components of the curvature tensor and its covariant derivative are considered, roughly speaking, as functions defined on a certain space of jets, $T_n^2Q$. Free unknowns of these identities are found, and it is shown that a system of free unknowns is a subsystem of a global coordinate system on $T_n^2Q$.

At the same time the "jet origin" of the algebraic identities for the curvature tensor and its covariant derivative is clarified, and there is given a proof of the assertion that these tensors do not satisfy any other non-trivial identities.

All these considerations concerning the identities for the covariant derivative of the curvature tensor are based on an algebraic lemma given in [1] without proof. Since I was asked at the Budapest Colloquium by some mathematicians for the details, I consider this remark to be a complement to the paper [1].

Let $n$ be a positive integer, and consider the system of $3n^4$ homogeneous linear equations

(1a) $$P_{kljm} + P_{lkjm} = 0,$$

(1b) $$P_{kljm} + P_{jklm} + P_{ljkm} = 0,$$

(1c) $$P_{kljm} + P_{mkjl} + P_{lmjk} = 0$$

for $n^4$ unknowns $P_{kljm}$, where $1 \leq k, l, j, m \leq n$. Obviously, these equations are identical with the algebraic identities for the components of the covariant derivative of the curvature tensor.

**Lemma.** (1) *The rank of the system* (1a–c) *is equal to*

(2) $$\frac{1}{24}n(19n^3 - 6n^2 + 5n + 6),$$

*and each solution of this system is a linear combination of*

(3) $$\frac{1}{24}n(n-1)(5n^2 + 11n + 6)$$

*free unknowns.*

(2) (a) *Let* $n = 1$. *Then the unique solution of the system* (1a–c) *is the trivial solution.*

(2) (b) *Let* $n = 2$. *Then the free unknowns of the system* (1a–c) *may be taken as*

(4) $$P_{1212}, P_{1221}, P_{1211}, P_{1222}.$$

(2) (c) *Let* $n = 3$. *Then the free unknowns of the system* (1a–c) *may be taken as*

(5) $$P_{1212}, P_{1221}, P_{1211}, P_{1222}, P_{1313}, P_{1331}, P_{1311}, P_{1333},$$
$$P_{2323}, P_{2332}, P_{2322}, P_{2333},$$
$$P_{1231}, P_{1232}, P_{1312}, P_{1321}, P_{1323}, P_{2312}, P_{2321}, P_{2313}, P_{2331}.$$

(2d) *Let* $n \geqq 4$. *Then the free unknowns of the system* (1a–c) *may be taken as*

(6) $$P_{klkl}, P_{kllk}, P_{klkk}, P_{klll}, k < l,$$
$$P_{klmk}, P_{klml}, P_{kmkl}, P_{kmlk}, P_{kmlm}, P_{lmkl},$$
$$P_{lmlk}, P_{lmkm}, P_{lmmk}, k < l < m,$$
$$P_{kjlm}, P_{mjlk}, P_{lmkj}, P_{mjkl}, P_{ljmk}, k < l < m < j.$$

Proof. 1. Firstly, we shall show that the system (1a–c) is equivalent to the following system of linear equations:

(7a) $$P_{kljm} + P_{lkjm} = 0,$$

(7b) $$P_{kljm} + P_{jklm} + P_{ljkm} = 0,$$

(7c) $$P_{kljm} - P_{klmj} - P_{jmkl} + P_{jmlk} = 0.$$

Consider the system (1a–c). (1c) implies

$$P_{jklm} + P_{mjlk} + P_{kmlj} = 0.$$

Using this equation and (1b) we obtain

$$-P_{ljkm} + P_{mkjl} + P_{kmlj} - P_{jlmk} = 0.$$

(7c) now follows from (1a) if we change the subscripts $j$ and $k$. Conversely, consider the system (7a–c). Beside (7c), write

$$P_{jklm} + P_{kjml} + P_{mljk} + P_{lmkj} = 0,$$
$$P_{ljkm} + P_{jlmk} + P_{mklj} + P_{kmjl} = 0.$$

Adding these equations to (7c) and using (7b) and (7a), we get

$$P_{klijm} + P_{jklm} + P_{ljkm} + P_{lkmj} + P_{lmkj} + P_{mklj} + \\ + P_{mjkl} + P_{kjml} + P_{kmjl} + P_{jmlk} + P_{mljk} + P_{jlmk} = 0,$$

i.e.,

$$2P_{lkmj} + 2P_{kjml} + 2P_{jlmk} = 0,$$

which gives (1c).

2. Secondly, consider the system (7a–c) instead of the equivalent system (1a–c). In this part of the proof we assume that $n \geqq 4$. To examine (7a–c), we shall use a method applied in a different context by Rashevskij [2, p. 544]. We shall divide the system (7a–c) into four subsystems in such a way that each of these subsystems contains the unknowns which are not present in the remaining three subsystems. This will enable us to solve each of these systems separately. The first subsystem will be formed by all equations of (7a–c) containing the unknowns of the form $P_{ijkl}$ with $i = j = k = 1$. The second (the third or the fourth, resp.) subsystem will be formed by the equations containing the unknowns $P_{ijkl}$ in which the subscripts $i, j, k, l$ take only two (three or four, resp.) different values. We shall determine the free unknowns of each of these four subsystems. Accordingly, this part of the proof is divided into four steps.

(a) Consider the first subsystem of the system (7a–c). This subsystem defines no free unknowns $P_{iiii}$, since by (7a), $P_{iiii} = 0$ for each $i$.

(b) Consider the second subsystem of the system (7a–c). Let us first examine the equations involving the unknowns indexed by $k, k, l, l$ where $k \neq l$, and then the equations involving the unknowns indexed by $k, l, l, l$, $k \neq l$.

Consider, for example, the indices 1, 2. Then we have the following unknowns: $P_{1122}$, $P_{1212}$, $P_{1221}$, $P_{2112}$, $P_{2121}$, $P_{2211}$. By (7a), $P_{1122} = 0$, $P_{2211} = 0$. Applying (7a) again, we can see that the free unknowns may be found among the unknowns $P_{1212}$, $P_{1221}$. The conditions (7b, c) do not give any new independent equations, which implies that these two unknowns are precisely the free unknowns for the choice of the indices $k, l$ considered. Repeating this consideration for all pairs of indices $k, l$, $k \neq l$ we obtain altogether

$$2\binom{n}{2},$$

free unknowns, and we can see that these free unknowns may be chosen as $P_{klkl}$, $P_{kllk}$, $k < l$.

Consider the case of unknowns indexed by $k, l, l, l$, where $k \neq l$, and examine, for example, the case of the indices 1,2. To this choice of indices there correspond the unknowns $P_{1222}$, $P_{2122}$, $P_{2111}$, $P_{1211}$, and the unknowns $P_{2212}$, $P_{2221}$, $P_{1121}$, $P_{1112}$ which are, however, equal to 0, by (7a). Using (7a) again, we get

$$P_{1211} + P_{2111} = 0, \qquad P_{1222} + P_{2122} = 0,$$

which implies that the free unknowns of the system of equations considered may be found among the unknowns $P_{1211}, P_{1222}$. Consider (7b) for $m = 1$. Then $P_{1211} + P_{2111} = 0$, and we have no new independent equation. Analogously, $P_{1222} + P_{2122} = 0$. These two equations represent the system (7b), since $P_{2221} = 0$ and $P_{1112} = 0$. It remains to examine whether the system (7c) gives a restriction of the two free unknowns. We obtain the relations

$$\begin{aligned} P_{2111} - P_{2111} - P_{1121} + P_{1112} &= 0, \\ P_{1211} - P_{1211} - P_{1112} + P_{1121} &= 0, \\ P_{1121} - P_{1112} - P_{2111} + P_{2111} &= 0, \\ P_{1112} - P_{1121} - P_{2111} + P_{2111} &= 0, \end{aligned}$$

which are identically satisfied. In the case considered one can therefore take $P_{1211}$ and $P_{1222}$ for the free unknowns. Repeating this consideration for all possible choices of $k, l, k \neq l$, we obtain

$$2\binom{n}{2}$$

free unknowns of the second subsystem of the system (7a–c) in the form $P_{klkk}$, $P_{klll}$, $k < l$.

(c) Consider the third subsystem of the system (7a–c), and assume that the three mutually different indices are equal to 1, 2, 3. By (7a), the free unknowns may then be found in the collection

(8)
$$\begin{aligned} &P_{1213}, P_{1231}, P_{1223}, P_{1232}, P_{1233}, \\ &P_{1312}, P_{1321}, P_{1322}, P_{1323}, P_{1332}, \\ &P_{2311}, P_{2312}, P_{2321}, P_{2313}, P_{2331}. \end{aligned}$$

Equations (7b) give

(9)
$$\begin{aligned} P_{1231} + P_{3121} + P_{2312} &= 0, \\ P_{1231} + P_{3122} + P_{2312} &= 0, \\ P_{1233} + P_{3123} + P_{2313} &= 0, \end{aligned}$$

where, by (7a), $P_{3121} = -P_{1321}$, $P_{3122} = -P_{1322}$, $P_{3123} = -P_{1323}$. Each of the remaining equations (7b) arises by a cyclic permutation of the first three indices, or by a change of the first two indices followed by a cyclic permutation of the first three ones. It is directly verified that all these equations are reduced to the system (9). The equations (9) allow to compute $P_{2311}, P_{1322}, P_{1233}$ as functions of the remaining unknowns. Consider the equations (7c). Putting in these equations $j = m$ or $k = l$ we obtain identities. Independent conditions for the considered unknowns (8) thus arise only if the index, having the same value as another one, stands on the first or the second place. We therefore obtain

$$
(10) \qquad \begin{aligned}
P_{1213} - P_{1231} - P_{1312} + P_{1321} &= 0,\\
P_{2123} - P_{2132} - P_{2321} + P_{2312} &= 0,\\
P_{3132} - P_{3123} - P_{3231} + P_{3213} &= 0.
\end{aligned}
$$

Using (7a), we may substitute

$$
\begin{aligned}
P_{2123} &= -P_{1223}, P_{2132} = -P_{1232}, P_{3132} = -P_{1332},\\
P_{3123} &= -P_{1323}, P_{3231} = -P_{2331}, P_{3213} = -P_{2313}.
\end{aligned}
$$

For example, the equation arising from the first of the equations (10) by the change $2 \leftrightarrow 3$, is dependent. The systems (10) and (9) are independent; it follows from the fact that (10) contains the unknowns $P_{1213}$, $P_{1223}$, $P_{1332}$ which are not contained in (9). One can compute these unknowns from (10). Altogether, we excluded the unknowns $P_{2311}$, $P_{1322}$, $P_{1233}$, $P_{1213}$, $P_{1223}$, $P_{1332}$ from (8) and the remaining 9 unknowns $P_{1231}$, $P_{1232}$, $P_{1312}$, $P_{1321}$, $P_{1323}$, $P_{2312}$, $P_{2321}$, $P_{2313}$, $P_{2331}$ are the free unknowns of the system considered.

Taking into consideration all possibilities of the choice of the indices in the third subsystem of (7c–c), we can see that there exist precisely

$$
9\binom{n}{3}
$$

free unknowns of this subsystem. For the free unknowns one may take the unknowns $P_{klmk}$, $P_{klml}$, $P_{kmkl}$, $P_{kmlk}$, $P_{kmlm}$, $P_{lmkl}$, $P_{lmlk}$, $P_{lmkm}$, $P_{lmmk}$, $k < l < m$.

(d) Consider the fourth subsystem of the system (7a–c). Let us first examine equations in which the indices of $P_{ijkl}$ take the values 1, 2, 3 and 4. Then (7a) implies that the free unknowns of these equations can be found in the system

$$
\begin{aligned}
&P_{1234}, P_{1243}, P_{1324}, P_{1342}, P_{1423}, P_{1432},\\
&P_{2314}, P_{2341}, P_{2413}, P_{2431}, P_{3412}, P_{3421}.
\end{aligned}
$$

The equations (7b) give four independent equations

$$
(11) \qquad \begin{aligned}
P_{1234} + P_{3124} + P_{2314} &= 0,\\
P_{1243} + P_{4123} + P_{2413} &= 0,\\
P_{1324} + P_{4232} + P_{3412} &= 0,\\
P_{2341} + P_{4231} + P_{3421} &= 0
\end{aligned}
$$

where, by (7a), $P_{3124} = -P_{1324}$, $P_{4123} = -P_{1423}$, $P_{4232} = -P_{2432}$, $P_{4231} = -P_{2431}$. The independence of these equations is obvious. They can be used to compute, for instance, $P_{1234}$, $P_{1243}$, $P_{1342}$, and $P_{2341}$. Consider the system (7c). (7a) implies that independent equations may be found among the equations with $k < l$. Furthermore, if $k < l$, then the equations with $j \geq m$ are dependent, which means that the independent equations are contained in the subsystem defined by the inequalities $k < l, j < m$. In the case considered we have six different pairs of indices 1, 2, 3, 4: (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4). Under the change $k \leftrightarrow j$, $l \leftrightarrow m$ the expression (7c) changes the sign, hence the equation obtained is dependent. Consequently, a system of independent equations in (7c) may be chosen in the form

(12)
$$\begin{aligned} P_{1234} - P_{1243} - P_{3412} + P_{3421} &= 0, \\ P_{1324} - P_{1342} - P_{2413} + P_{2431} &= 0, \\ P_{1423} - P_{1432} - P_{2314} + P_{2341} &= 0. \end{aligned}$$

We shall find an explicit solution of the system (12), (11). (11) implies

$$\begin{aligned} P_{1234} &= P_{1324} - P_{2314}, \\ P_{1243} &= P_{1423} - P_{2413}, \\ P_{1324} &= P_{1432} - P_{3412}, \\ P_{2341} &= P_{2431} - P_{3421}. \end{aligned}$$

Substituting into (12), one gets

(13)
$$\begin{aligned} P_{1324} - P_{2314} - P_{1423} + P_{2413} - P_{3412} + P_{3421} &= 0, \\ P_{1324} - P_{1432} + P_{3412} - P_{2413} + P_{2431} &= 0, \\ P_{1423} - P_{1432} - P_{2314} + P_{2431} - P_{3421} &= 0. \end{aligned}$$

Subtracting the third equation from the second one, we obtain

$$P_{1324} + P_{3412} - P_{2413} - P_{1423} + P_{2314} + P_{3421} = 0,$$

and adding this equation to the first one,

(14)
$$P_{1324} - P_{1423} + P_{3421} = 0;$$

subtracting (14) from the first equation (13), we get

$$-P_{2314} + P_{2413} - P_{3412} = 0.$$

Hence

$$\begin{aligned} P_{1324} &= P_{1423} - P_{3421}, \\ P_{2413} &= P_{3412} + P_{2314}. \end{aligned}$$

The second equation (13) gives

$$P_{1432} = P_{1423} - P_{3412} - P_{2314} + P_{2431}.$$

The result of our explicit calculation can be summarized as follows:

$$\begin{aligned} P_{1234} &= P_{1423} - P_{3421} - P_{2314}, \\ P_{1243} &= P_{1423} - P_{3412} - P_{2314}, \\ P_{1324} &= P_{1423} - P_{2314} + P_{2431} - 2P_{3412}, \end{aligned}$$

$$P_{2341} = P_{2431} - P_{3421},$$
$$P_{1324} = P_{1423} - P_{3421},$$
$$P_{2413} = P_{3412} + P_{2314},$$
$$P_{1432} = P_{1423} - P_{3412} - P_{2314} + P_{2431}.$$

In particular, these formulae show that the free unknowns for the considered equations indexed by 1, 2, 3, 4 may be taken as $P_{1423}, P_{3421}, P_{3412}, P_{2431}, P_{2314}$.

Now it is easy to determine the total number of free unknowns for the fourth subsystem of the system (7a–c). Since there are $\binom{n}{4}$ quadruples of different indices $i, j, k, l$ in the set $1, 2, 3, \ldots, n$ of indices, there are precisely

$$5\binom{n}{4}$$

free unknowns of this subsystem. One can take $P_{kjlm}, P_{mjlk}, P_{lmkj}, P_{mjkl}, P_{ljmk}$, $k < l < n < j$, for these free unknowns.

Let us summarize the results of the steps (a)–(d) for $n \geqq 4$. The total number of the free unknowns of the system (7a–c) is given by

$$4\binom{n}{2} + 9\binom{n}{3} + 5\binom{n}{4} = \frac{1}{24} n(n-1)(5n^2 + 11n + 6).$$

This proves (3) for $n \geqq 4$. Subtracting the total number of the free unknowns of the system (7a–c) from the number of all unknowns, we obtain the rank of this system:

$$n^4 - \frac{1}{24} n(n-1)(5n^2 + 11n + 6) = \frac{1}{24} n(19n^3 - 6n^2 + 5n + 6),$$

which proves (2). Finally, collecting the free unknowns of (a)–(d), we obtain exactly the system (6). This proves our lemma for $n \geqq 4$.

3. Thirdly, consider the cases $n = 1, 2, 3$. Let $n = 1$. Then (7a–c) is a system of equations for a single unknown $P_{1111}$, which is, however, equal to 0, by (7a). The rank of (7a–c) is equal to 1, and we can see that there hold all assertions of the Lemma.

Let $n = 2$. Then the total number of the free unknowns of (7a–c) is, by (a) and (b), equal to $4\binom{2}{2} = 4$. The relation (3) gives

$$\frac{1}{12}(5.4 + 11.2 + 6) = \frac{48}{12} = 4,$$

and again the Lemma holds.

Let $n = 3$. Then by (a), (b), and (c), the total number of the free unknowns of (7a – c) is equal to

$$4\binom{3}{2} + 9 = 12 + 9 = 21,$$

which coincides with (3). Clearly, this implies that (2) must also hold. The considerations (a), (b), (c) of the second part of the proof imply directly that (5) is a system of free unknowns of the system (7a – c).

This completes the proof of the Lemma in all its parts.

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