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Archivum Mathematicum, Vol. 16 (1980), No. 4, 189--198

Persistent URL: http://dml.cz/dmlcz/107073

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVI: 189—198, 1980

LINEAR DIFFERENTIAL TRANSFORMATIONS OF THE 2nd ORDER AS A REPRESENTATION OF AN ABSTRACT MODEL

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(Received June 9, 1979)

INTRODUCTION

We start with the notation and the terminology. By the carriers q, Q, ... we mean the continuous real functions $q(t)$, $Q(t)$, ... in open intervals. We deal with 2nd order differential equations

$$(q) \qquad y'' = q(t)\,y$$

and

$$(Q) \qquad Y'' = Q(t)\,Y$$

provided the coefficients q, Q are continuous in convenient open intervals.

For any two equations (q), (Q) there are considered transformations of the form $Y(t) = m(t)\,y(\alpha(t))$ with convenient $m(t)$ and $\alpha(t)$, where y and Y are solutions of (q) and (Q), respectively.

Solutions of the present differential equations are considered in open intervals only. By the term integral, we mean a non-continuable solution which is, moreover, for the differential equations (q), (Q), ... a non-trivial one.

Recall that for any map $f: M \twoheadrightarrow N$ the symbols $M = \operatorname{Dom} f$ and $N = \operatorname{Im} f$ are used.

It is proved [1] that

1° $m(t) = \text{const}/\sqrt{|\alpha'(t)|}$ and thus the transformation is of the form

$$(*) \qquad Y(t) = \frac{y(\alpha(t))}{\sqrt{|\alpha'(t)|}},$$

2° if the last formula holds in some open interval J, then α is a solution in J of the 3rd order non-linear differential equation

$$(q, Q) \qquad -\{\alpha, t\} + q(\alpha)\,\alpha'^2 = Q(t),$$

where

$$\{\alpha, t\} = \frac{1}{2}\frac{\alpha'''}{\alpha'} - \frac{3}{4}\frac{\alpha''^2}{\alpha'^2} = -\sqrt{|\alpha'|}\left(\frac{1}{\sqrt{|\alpha'|}}\right)'' = \frac{1}{2}\left(\frac{\alpha''}{\alpha'}\right)' - \frac{1}{4}\left(\frac{\alpha''}{\alpha'}\right)^2$$

is Schwarz's derivative,

3° for any integral y of (q) and α of (q, Q) the function (*) is a solution of (Q) in Dom α and the formula

$$(**) \qquad y = \frac{Y(A)}{\sqrt{|A'|}}$$

holds in Im α, where $A = \alpha^{-1}$ means the inverse function.

4° for the arbitrary initial conditions $\alpha_0 \in \text{Dom } q$, $\alpha_0' \neq 0$, $\alpha_0'' \in \mathbf{R}$ at $t_0 \in \text{Dom } Q$ the equation (q, Q) has the unique integral α. Thus $\alpha \in C^3_{\text{Dom } Q}$, $\alpha' \neq 0$ and α approaches the boundary of Dom $Q \times$ Dom q,

5° for integrals β of (q, Q) and α of $(Q, \tilde{q})$ the composition $\beta \circ \alpha$ – if it exists, i.e. iff Dom $\beta \cap$ Im α is an open interval – is a solution of $(q, \tilde{q})$,

6° for any integral α of (q, Q) the inverse function α^{-1} is an integral of (Q, q).

Note that the equation (q, Q) splits in two equations: one of them is

$$\sqrt{\alpha'}\left(\frac{1}{\sqrt{\alpha'}}\right)'' + q(\alpha)\,\alpha'^2 = Q(t)$$

and admits only increasing solutions, the other is

$$\sqrt{-\alpha'}\left(\frac{1}{\sqrt{-\alpha'}}\right)'' + q(\alpha)\,\alpha'^2 = Q(t)$$

and has only decreasing solutions.

Let us borrow the symbol $[y, z]$ for denoting the ordered couple of linearly independent integrals of the equation (q) and call it a basis of (q). Putting any basis $[y, z]$ of (q) to the form $y = \pm r \sin \alpha$, $z = \pm r \cos \alpha$, $r > 0$ we get $\frac{y}{z} = \operatorname{tg} \alpha$ and $r = \sqrt{y^2 + z^2} = \frac{\text{const}}{\sqrt{|\alpha'|}}$.

Every continuous solution α in Dom q of the functional equation $\operatorname{tg} \alpha = \frac{y}{z}$ is called a phase of the ordered couple $[y, z]$.

There holds [1]

7° every phase α is an integral of the differential equation

$$(-1, q) \qquad -\{\alpha, t\} - \alpha'^2 = q(t)$$

in Dom q and, on the contrary, each integral α of $(-1, q)$ exists in Dom q and is a phase of (q), i.e. of some convenient basis $[y, z]$ of (q).

Consequently every integral w of (q) is expressible in the form

$$w(t) = \frac{a}{\sqrt{|\alpha'(t)|}} \sin(\alpha(t) - b),$$

where $a, b \in \mathbf{R}$.

1. BOTH-SIDED OSCILLATORY CARRIERS AND PHASES

Henceforth only both-sided oscillatory equations (q), (Q), ... in $\mathbf{R}$ are considered. Without any loss of generality we limit ourselves to increasing phases and put

$$\mathfrak{P} = \{\alpha \in C^3_{\mathbf{R}} \mid \alpha' > 0,\ \operatorname{Im} \alpha = \mathbf{R}\}.$$

Evidently $\mathfrak{P}$ is a group with respect to the composition of functions. Every $\alpha \in \mathfrak{P}$ is the phase of the basis $\left[\frac{\sin \alpha}{\sqrt{\alpha'}}, \frac{\cos \alpha}{\sqrt{\alpha'}}\right]$ of the both-sided oscillatory equation (q) in $\mathbf{R}$, where $q(t) = -\{\alpha, t\} - \alpha'^2$.

On the contrary, if α is any increasing phase of the basis $[y, z]$ of some both-sided oscillatory equation (q) in $\mathbf{R}$, then α is an integral of $(-1, q)$ in $\mathbf{R}$ and according to the property 3° the function $w = \frac{\sin \alpha}{\sqrt{\alpha'}}$ is a solution of (q) in $\mathbf{R}$. Since w has infinitely many zeros at $-\infty$ and $+\infty$ the phase α fulfils $\operatorname{Im} \alpha = \mathbf{R}$ and thus $\alpha \in \mathfrak{P}$.

This proves that $\mathfrak{P}$ is in fact the group of (increasing) phases and can be written as $\mathfrak{P} = \bigcup \langle -1, q \rangle$ where q ranges over all both-sided oscillatory carriers in $\mathbf{R}$, the union being disjoint and provided that $\langle -1, q \rangle$ means the set of all increasing phases of the equation (q).

Let us denote the subgroup $\langle -1, -1 \rangle$ by $\mathfrak{E}$, i.e. the set of all increasing integrals of the equation

$$(-1, -1) \qquad -\{\alpha, t\} - \alpha'^2 = -1.$$

By the same arguments as in [2] it can be proved that ${}^{\alpha}\mathfrak{E} = \mathfrak{E}$. In comparison with the basic model we have put here -1 instead of e. We denote here by $\langle q, Q \rangle$ the set of all increasing integrals of the differential equation (q, Q). The map Γ is here tg t and $\mathscr{M}$ is the set of all functions tg $\alpha(t)$, where $\alpha(t)$ ranges over $\mathfrak{P}$.

The group $\mathscr{H}$ is here the group of all real homographies $h(t) = \frac{at + b}{ct + d}$ with the positive determinant. The multiplication $\mathscr{M} \circ \mathfrak{P} \subseteq \mathscr{M}$ is here the composition of functions.

We can see that the subgroup $\mathfrak{Z} = \Gamma^{-1}(\Gamma\iota)$ is here the set of all $\alpha \in \mathfrak{P}$ such that $\operatorname{tg} \alpha(t) = \operatorname{tg} t$, i.e. $\mathfrak{Z} = \{\varepsilon^\nu\}_{\nu \in \mathbf{Z}}$ where $\mathbf{Z}$ denotes the set of all integers and $\varepsilon^\nu(t) = t + \nu\pi$. In other words $\mathfrak{Z}$ is the infinite cyclic group generated by the function $\varepsilon(t) = t + \pi$.

The basic properties, known from the basic model, of the decomposition $\mathfrak{P}/_n\mathfrak{E}$ and the map $\Gamma : \mathfrak{P} \twoheadrightarrow \mathscr{M}$ are here consequences of the following statements

1° $\{\iota, t\} = 0$, where ι denotes the identity on $\mathbf{R}$,

2° $\{\operatorname{tg} t, t\} = 1$,

3° $\{\alpha, t\} = \{\beta, t\}$ iff there exists a homography $h \in \tilde{\mathscr{H}}$ such that $\beta = h \circ \alpha$, ($\tilde{\mathscr{H}}$ are homographies with det $\neq 0$)

4° for the composed functions $\beta \circ \alpha$ there holds

$$\{\beta \circ \alpha, t\} = \{\beta(\alpha), \alpha\}\, \alpha'^2 + \{\alpha, t\}.$$

2. BOTH-SIDES OSCILLATORY BASES AND DISPERSIONS

For every both-sided oscillatory carrier $q(t)$ on $\mathbf{R}$ let us consider the corresponding 2-dimensional real vector space $\mathscr{V}_q$ consisting of the zero function on $\mathbf{R}$ and all integrals of the equation (q).

If $u = [y, z]$ is a basis of (q), then the formula $\operatorname{tg} \alpha = \dfrac{y}{z}$ implies $\alpha'/\cos^2 \alpha = -W(u)/z^2$, where $W(u)$ means the Wronskian of the basis u. Hence here the constant value $W(u)$ has always the opposite sign than α'.

Since we consider the increasing phases only, we must limit ourselves to the bases $u = [y, z]$ with negative Wronskians. Let $\langle q \rangle$ denote the set of all bases of (q) the Wronskians of which are negative. Then we put $\mathscr{B} = \bigcup \langle q \rangle$, where q ranges over all both-sided oscillatory carriers on $\mathbf{R}$.

To obtain the realization of the map $\Delta : \mathscr{B} \twoheadrightarrow \mathscr{M}$, known from the basic model, let us put $\Delta u = \dfrac{y}{z}$ for every basis $u = [y, z] \in \mathscr{B}$. All needed properties of this map Δ follow from the statements 1°–4° sub 1. about Schwarz's derivative.

If $u = [y, z]$ is a basis of (q), then all bases $U = [Y, Z]$ of (q) are given by the formula

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},$$

where $k = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ranges over all real non-singular matrices.

Owing to the formula $W(U) = W(u) \,.\, \det k$ we must choose $\mathscr{K}$ as the set of all 2nd order real matrices with positive determinants. Evidently this group $\mathscr{K}$ with multiplication of matrices as the group operation works as a group of permutations on $\mathscr{B}$.

The kernel $\mathfrak{A}$ of the homomorphism $\Theta : \mathscr{K} \twoheadrightarrow \mathscr{H}$ is the set $\{\lambda I\}_{\lambda \neq 0}$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and λ ranges over all real numbers different from zero.

For every $\alpha \in \mathfrak{P}$ and every $u = [y, z] \in \mathfrak{B}$ the product $u \square \alpha$ is defined by the formula

$$(_*{}^*{}_*) \qquad u \square \alpha = \left[\frac{y(\alpha(t))}{\sqrt{\alpha'(t)}}, \frac{z(\alpha(t))}{\sqrt{\alpha'(t)}}\right]$$

according to the introduction. Hence the multiplication $\mathfrak{B} \square \mathfrak{P} = \mathfrak{B}$ is well defined and we can see that it is associative with respect to Δ, $\mathscr{K}$ and $\mathfrak{P}$ and fulfils all other needed properties supposed in the basic model.

A phase $\varphi \in \mathfrak{P}$ will be called a dispersion of the carrier q if it satisfies the differential equation (q, q). The set $\langle q, q\rangle$ of all dispersions of the carrier q is a subgroup in $\mathfrak{P}$, conjugated with $\mathfrak{C} = \langle -1, -1\rangle$ by the formula $\langle q, q\rangle = \alpha^{-1} \circ \mathfrak{C} \circ \alpha$ for any phase $\alpha \in \langle -1, q\rangle$.

The nucleus $\mathfrak{Z}$ of $\mathfrak{C}$ is the kernel of the homomorphism $\vartheta : \mathfrak{C} \twoheadrightarrow \mathscr{H}$ which assigns the homography $h \in \mathscr{H}$ to the dispersion $\eta \in \mathfrak{C}$ according to the formula $\operatorname{tg} \eta = h \circ \operatorname{tg} t$. This nucleus $\mathfrak{Z}$ generates the nucleus $\mathfrak{Z}_q$ of $\langle q, q\rangle$ by the formula $\mathfrak{Z}_q = \alpha^{-1} \circ \mathfrak{Z} \circ \alpha$ for each $\alpha \in \langle -1, q\rangle$ owing to the normality of $\mathfrak{Z}$ in $\mathfrak{C}$.

Two things are here particularly important, first that $\mathfrak{Z}_q$ are infinite cyclic groups and secondly that we deal with increasing phases only. If the invertor of the group $\mathfrak{Z}_q$ in $\mathfrak{P}$ is defined as ${}^i\mathfrak{Z}_q = \{\alpha \in \mathfrak{P} \mid \alpha \circ \gamma = \gamma^{-1} \circ \alpha \;\forall \gamma \in \mathfrak{Z}_q\}$ and the centralizator as ${}^z\mathfrak{Z}_q = \{\alpha \in \mathfrak{P} \mid \alpha \circ \gamma = \gamma \circ \alpha \;\forall \gamma \in \mathfrak{Z}_q\}$, then owing to the first property of $\mathfrak{Z}$ we have $\alpha \in {}^i\mathfrak{Z}$ iff $\alpha \circ \varepsilon = \varepsilon^{-1} \circ \alpha$ and $\alpha \in {}^z\mathfrak{Z}$ iff $\alpha \circ \varepsilon = \varepsilon \circ \alpha$. The second property implies that ${}^i\mathfrak{Z} = \emptyset$ since for $\alpha \in {}^i\mathfrak{Z}$ it is $\alpha \circ \varepsilon = \varepsilon^{-1} \circ \alpha$ or $\alpha(t + \pi) = \alpha(t) - \pi$ which contradicts the increasing of α. The first property ensures that ${}^n\mathfrak{Z} = {}^z\mathfrak{Z} \cup {}^i\mathfrak{Z}$ where ${}^n\mathfrak{Z} = \{\alpha \in \mathfrak{P} \mid \alpha \circ \mathfrak{Z} = \mathfrak{Z} \circ \alpha\}$ is the normalizator of $\mathfrak{Z}$. Thus we have here ${}^n\mathfrak{Z} = {}^z\mathfrak{Z}$ and consequently the isomorphism ${}^*\alpha : \mathfrak{Z} \twoheadrightarrow \mathfrak{Z}_q$ such that $\gamma \mapsto \alpha^{-1} \circ \gamma \circ \alpha$ for each $\gamma \in \mathfrak{Z}$ is really independent on $\alpha \in \langle -1, q\rangle$ owing to the inclusion $\mathfrak{C} \subseteq {}^n\mathfrak{Z} = {}^z\mathfrak{Z}$. Hence the generator $\varphi = \alpha^{-1} \circ \varepsilon \circ \alpha$ of $\mathfrak{Z}_q$ is determined univocally and independently on $\alpha \in \langle -1, q\rangle$.

In other words, for the dispersions $\varepsilon^\nu \in \mathfrak{Z}$, $\varphi^\nu \in \mathfrak{Z}_q$ and the phases $\alpha \in \langle -1, q\rangle$ we have the Abelian relations $\alpha \circ \varphi^\nu = \varepsilon^\nu \circ \alpha$, $\nu \in \mathbb{Z}$.

Recall that the notion of Abelian relations for the group $\mathfrak{Z}$ according to the basic model implies that for every $\gamma \in \mathfrak{Z}$ the mapping $\gamma \mapsto \alpha^{-1} \circ \gamma \circ \alpha$ is an automorphism of $\mathfrak{Z}$ independent on α taken from the same class of the decomposition $\mathfrak{C}/_n({}^z\mathfrak{Z}) \cap \mathfrak{C}$.

Since here $\mathfrak{C} \subseteq {}^z\mathfrak{Z}$, we have only one automorphism for all $\alpha \in \mathfrak{C}$, which is necessarily the identity. Hence $\mathfrak{Z} \subseteq {}^{\mathfrak{S}}\mathfrak{C}$ where ${}^{\mathfrak{S}}\mathfrak{C}$ means the centre of $\mathfrak{C}$. Certainly, there holds $\mathfrak{Z}_q \subseteq {}^{\mathfrak{S}}\langle q, q\rangle$ for every carrier q.

Note that in this 2^{nd} order realization we have ${}^{8}\mathscr{H} = \{i\}$ and ${}^{8}\mathscr{K} = \mathfrak{R} = \{\lambda I\}_{\lambda \neq 0}$. This implies ${}^{8}\langle q, q\rangle \subseteq \mathfrak{Z}_q = {}^{*}\langle q, q\rangle$ and hence we have here a very particular realization of the basic model, namely that fulfilling ${}^{8}\langle q, q\rangle = \mathfrak{Z}_q = {}^{*}\langle q, q\rangle$ and, moreover, ${}^{8}\mathscr{K} = \mathfrak{R}$.

3. HOMOMORPHISMS AND PSEUDONORMS

Certainly, we still suppose that all carriers are both-sides oscillatory on $\mathbf{R}$.

We deal with three decompositions $\mathfrak{P}/_r\mathfrak{E}$, $\mathscr{M}/_r\mathscr{H}$ and $\mathscr{B}/_r\mathscr{K}$ the first two being in one-to-one correspondence under the map Γ and the last two under Δ. Their classes can be written as $\langle -1, q\rangle$, $\langle 0, q\rangle$ and $\langle q\rangle$, respectively. Note that the notation $\langle 0, q\rangle$ has here a real sense since $\zeta \in \langle 0, q\rangle$ are all solutions on $\mathbf{R}$ with increasing branches of the differential equation

$$(0, q) \qquad -\{\zeta, t\} = q(t).$$

For every $\zeta \in \langle 0, q\rangle$ the homomorphism ${}^{*}\zeta : \langle q, q\rangle \twoheadrightarrow \mathscr{H}$ is defined by the equation $\zeta \circ \alpha = h\zeta$. Its kernel is $\mathfrak{Z}_q$.

In this realization of the basic model the group $\mathscr{K}$ has the special property similarly as $\mathscr{H}$, namely that the implication $(\boldsymbol{k}_1\boldsymbol{u} = \boldsymbol{k}_2\boldsymbol{u}) \Rightarrow (\boldsymbol{k}_1 = \boldsymbol{k}_2)$ holds for any fixed $\boldsymbol{u} \in \mathscr{B}$ and $\boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathscr{K}$.

For every $\boldsymbol{u} \in \langle q\rangle$ the homomorphism ${}^{*}u : \langle q, q\rangle \rightarrow \mathscr{K}$ is defined by the equation $\boldsymbol{u} \,\square\, \alpha = \boldsymbol{k}\boldsymbol{u}$, i.e. by the equation

$$\begin{bmatrix} \dfrac{y(\alpha)}{\sqrt{\alpha'}} \\ \dfrac{z(\alpha)}{\sqrt{\alpha'}} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix},$$

where $\boldsymbol{u} = [y, z]$ and $\boldsymbol{k} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Passing to the Wronskians, we get $W(\boldsymbol{u}(\alpha)) = W(\boldsymbol{u}) \,.\, \det \boldsymbol{k}$, where on the left it is the value of $W(\boldsymbol{u})$ at the number $\alpha(t)$. Hence we have necessarily $\det \boldsymbol{k} = 1$.

The most important thing now is to prove that the group $\mathscr{K}'$ of all unimodular 2^{nd} order real matrices with the determinant equal to 1 is the image of the homomorphism ${}^{*}u$. We can prove a slightly more general

Lemma 1. Let $\boldsymbol{u} = [y, z]$ be a basis of an arbitrary (not necessarily both-sided oscillatory) differential equation (q) defined in some open interval $\operatorname{Dom} q \subseteq \mathbf{R}$ and similarly $\boldsymbol{U} = [Y, Z]$ a basis of (Q) defined in $\operatorname{Dom} Q \subseteq \mathbf{R}$. Let $\zeta = \dfrac{y}{z}$ and $Z = \dfrac{Y}{Z}$ be the corresponding semi-phases.

There exists a solution α of the differential equation (q, Q) such that

$$(\begin{smallmatrix}* & *\\ * & *\end{smallmatrix}) \qquad \frac{y(\alpha(t))}{\sqrt{|\alpha'(t)|}} = Y(t) \quad \text{and} \quad \frac{z(\alpha(t))}{\sqrt{|\alpha'(t)|}} = Z(t)$$

holds if and only if the absolute values of Wronskians are equal, i.e. $|W(U)| = |W(u)|$, and an open interval $J \subseteq \operatorname{Im}\zeta \cap \operatorname{Im} Z$ exists such that $\operatorname{sgn} y(\alpha_0) = \operatorname{sgn} Y(t_0)$ (or $\operatorname{sgn} z(\alpha_0) = \operatorname{sgn} Z(t_0)$) is fulfilled for some $\alpha_0 \in \zeta^{-1}(J)$ and $t_0 \in Z^{-1}(J)$.

Then α can be determined by the initial conditions $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha'_0$ and $\alpha''_0(t_0) = \alpha''_0$, where α'_0 is uniquely calculated from the relations

$$|\alpha'_0| = \frac{y^2(\alpha_0)}{Y^2(t_0)} \left(= \frac{z^2(\alpha_0)}{Z^2(t_0)}\right), \qquad W(U) = W(u) \, . \operatorname{sgn} \alpha'_0$$

and α''_0 is uniquely calculated from the equation

$$\begin{bmatrix} Y'(t_0) \\ Z'(t_0) \end{bmatrix} = \begin{bmatrix} y'(\alpha_0) \\ z'(\alpha_0) \end{bmatrix} \sqrt{|\alpha'_0|} \operatorname{sgn} \alpha'_0 - \frac{1}{2} \begin{bmatrix} Y(t_0) \\ Z(t_0) \end{bmatrix} \frac{\alpha''_0}{\alpha'_0}.$$

Moreover, the formulae $(\begin{smallmatrix}* & *\\ * & *\end{smallmatrix})$ hold in the $\operatorname{Dom}\alpha$ of the whole integral α of (q, Q) which is given by those initial conditions.

Now, in case of both-sided oscillatory carriers on $\mathbf{R}$ it is clear that the condition of lemma is satisfied. We can even put t_0 equal to an arbitrary zero of Y and α_0 equal to one of any two consecutive zeros of y (for one of them the condition is fulfilled and for the other not, since the sign of z changes in any two consecutive zeros of y).

Certainly, we deal here in fact with a basis $u = [y, z] \in \langle q \rangle$ and another basis $ku = U = [Y, Z]$ where $k \in \mathscr{K}$ and $\det k = 1$. Thus $U \in \langle q \rangle$ and $W(U) = W(u)$ so that the transformation $(\begin{smallmatrix}* & *\\ * & *\end{smallmatrix})$ is realized by means of some increasing integral α of the equation (q, q).

The question of $\operatorname{Dom}\alpha$ and $\operatorname{Im}\alpha$ for integrals of (q, q) is now topical. We shall prove.

Lemma 2. If and only if the carrier q in $\operatorname{Dom} q \subseteq \mathbf{R}$ is both-sided oscillatory, then for every carrier Q it is $\operatorname{Dom}\alpha = \operatorname{Dom} Q$ for each integral α of the equation (q, Q).

Proof. Let $\operatorname{Dom}\alpha = \operatorname{Dom} Q$ for every Q and each integral α of (q, Q). Then for each integral A of (Q, q) the inverse function A^{-1} is an integral of (q, Q) and thus $\operatorname{Im} A = \operatorname{Dom} A^{-1} = \operatorname{Dom} Q$. Particularly if Q is both-sided oscillatory in $\operatorname{Dom} Q$, then for each integral Y of (Q) and each integral A of (Q, q) the solution of (q), $y = Y(A)/\sqrt{|A'|}$, has infinitely many roots at both ends of its interval of existence $\operatorname{Dom} A$. Hence y is an integral of both-sided oscillatory equation (q) and it is by the way $\operatorname{Dom} A = \operatorname{Dom} q$.

On the contrary, let (q) be both-sided oscillatory in Dom q. Let us admit that there exists a carrier Q and an integral α of (q, Q) such that Dom $\alpha \neq$ Dom Q.

Without loss of generality we can suppose that α is increasing, Dom $q =]a, b[$, Dom $\alpha =]c, d[$, where $a < c$.

Let us denote the restriction $Q/_{]c,d[}$ by Q^*. Then α is an integral of (q, Q) and for every integral y of (q) the solution of (Q^*), $Y = y(\alpha)/\sqrt{|\alpha'|}$, has infinitely many roots at c. Hence Q^* is a left-sided oscillatory carrier and no left-sided continuous prolongation of Q^* exists. This contradicts the existence of Q, which is such a prolongation. Hence we find that for every carrier Q and each integral α of (q, Q) it holds Dom α = Dom Q. Lemma 2 is proved.

Consequently the dual affirmation holds: if and only if the carrier Q is both-sided oscillatory in Dom $Q \subseteq \mathbf{R}$ then for every carrier q and each integral α of (q, Q) it holds Im α = Dom q.

Particularly if and only if both carriers q and Q are both-sided oscillatory, then each integral α of (q, Q) is defined on Dom Q and maps this interval onto Dom q.

Now we can finish the exposition that the group $\mathscr{K}'$ of all 2nd order real matrices $\boldsymbol{k}$ with det $\boldsymbol{k} = 1$ is the image of any homomorphism $^*u$.

In fact, only the both-sided oscillatory carriers q on $\mathbf{R}$ are considered and we have seen that for any $\boldsymbol{u} \in \langle q \rangle$ and $\boldsymbol{k} \in \mathscr{K}'$ there exists an increasing integral α of (q, q) such that $\binom{*\ *}{*\ *}$ holds. Now it is clear that Dom α = Im $\alpha = \mathbf{R}$ and thus $\alpha \in \mathfrak{P}$ or even $\alpha \in \langle q, q \rangle$ and thus to every $\boldsymbol{k} \in \mathscr{K}'$ there exists $\alpha \in \langle q, q \rangle$ such that $\boldsymbol{u} \,\square\, \alpha = \boldsymbol{k}\boldsymbol{u}$ holds.

Evidently the group $\mathscr{K}'$ is invariant in $\mathscr{K}$. Moreover, there exists a minimal subgroup $\mathscr{L} \subseteq \mathscr{K}$ such that $\mathscr{L} \cap \mathscr{K}' = \{\boldsymbol{I}\}$ and $\mathscr{L}\mathscr{K}' = \mathscr{K}$; namely $\mathscr{L} = \{\lambda \boldsymbol{I}\}_{\lambda > 0}$ where λ ranges over all positive real numbers.

Note that $\mathscr{R} = \{\lambda \boldsymbol{I}\}_{\lambda \neq 0}$ and that $\mathscr{R} \cap \mathscr{K}' = \pm \boldsymbol{I}$ and $\mathscr{R}\mathscr{K}' = \mathscr{K}$.

The same arguments as in the preceding paragraph ensure that the set $\mathscr{B}'(\boldsymbol{u}) = \{\boldsymbol{u} \,\square\, \alpha \mid \alpha \in \mathfrak{P}\}$ consists of all bases $\boldsymbol{v} \in \mathscr{B}$ which have the same (negative) value of Wronskians as $\boldsymbol{u}$.

According to the absolute values of Wronskians the set $\mathscr{B}$ of all bases decomposes into classes $\mathscr{B}'(\boldsymbol{u})$.

It is natural to keep the multiplicative group $G =]0, \infty[$ and assign to every basis $\boldsymbol{u} \in \mathscr{B}$ the pseudonorm $|\boldsymbol{u}| = |W(\boldsymbol{u})|$.

Note that the groups $G, \mathscr{L}$ and the factorgroup $\mathscr{K}/\mathscr{K}'$ are isomorph. The pseudonorm of $\boldsymbol{k} \in \mathscr{K}$ is defined here as the (positive) value det $\boldsymbol{k}$.

The other decomposition of $\mathscr{B}$ consists of the classes $\langle q \rangle$. We can see that any class $\langle q \rangle$ of the latter decomposition intersects with any class $\mathscr{B}'(\boldsymbol{u})$ of the former decomposition in the set of all bases $\boldsymbol{u} \,\square\, \langle q, q \rangle = \mathscr{K}'\boldsymbol{u}$.

In comparison with the basic model here the unimodular bases are exactly those with the Wronskian equal to -1. One of them is the basis $[\sin t, \cos t]$.

Now the following test for the transformation $v = u \,\square\, \alpha$ between two given bases $u, v \in \mathscr{B}$ becomes more visible: such a transformation holds iff the (negative) values of Wronskians of u and v are equal.

4. APPLICATION TO THE DISTRIBUTION OF ZEROS

A real function $\alpha(t)$ of real variable t will describe the distribution of zeros of integrals of the differential equation (q) if for every integral w of (q) some non-zero constant λ_w exists such that

$$((*)) \qquad \frac{w(\alpha(t))}{\sqrt{|\alpha'(t)|}} = \lambda_w w(t)$$

holds. Certainly, then α is a solution of the differential equation (q, q). Moreover, $\alpha(t)$ is increasing owing to the Sturm theorem.

Let $u = [y, z]$ be a basis of (q). Then for every integral $w = ay + bz$ of (q) the formula ((*)) gives $a\lambda_y y + b\lambda_z z = \lambda_w(ay + bz)$ and hence $\lambda_w = \lambda_y = \lambda_z$. Thus in the formula ((*)), if it holds for all integrals of (q), the constants λ_w do not depend on w, say $\lambda_w = \lambda$ for all w.

If q is any both-sided oscillatory carrier on $\mathbf{R}$ and if $\alpha : \mathbf{R} \to \mathbf{R}$ describes the distribution of zeros of (q), then $\alpha \in \mathfrak{P}$ and for every basis $u \in \langle q \rangle$ we have $u \,\square\, \alpha = \lambda u$ for some fixed real number $\lambda \neq 0$.

In conformity with [4] it is $\lambda I \in \mathscr{K}' \cap {}^{8}\mathscr{K}$ and since here is ${}^{8}\mathscr{K} = \mathscr{R}$ we find $\lambda I = \pm I$.

Hence the unique dispersions which can describe the distribution of zeros are the nuclear dispersions φ^{ν} where $\varphi = \alpha^{-1} \circ \varepsilon \circ \alpha$ for $\alpha \in \langle -1, q \rangle$ and $\varepsilon(t) = t + \pi$.

According to [4] it is $\mathfrak{Z}_q/\mathrm{Ker}^* u = \mathscr{K}' \cap \mathscr{R} = \pm I$. Since $\mathfrak{Z}_q$ is an infinite cyclic group, the unique subgroup $\mathrm{Ker}^* u$ having the index 2, is $\{\varphi^{2\nu}\}_{\nu \in \mathbf{Z}}$. Hence the equation $u \,\square\, \alpha = u$ holds iff $\alpha = \varphi^{2\nu}$ and the other equation $u \,\square\, \alpha = -u$ holds iff $\alpha = \varphi^{2\nu+1}$.

The constructive meaning, describing the distribution of zeros, of the dispersion φ^{ν} is evident: for any $t \in \mathbf{R}$ the number $\varphi^{\nu}(t)$ means the ν-th zero of w with respect to t of any integral w of (q) which vanishes at t. For positive ν the roots are counted to the right side and for ν negative to the left side.

For any carrier q the fundamental central dispersion $\varphi = \alpha^{-1} \circ \varepsilon \circ \alpha$, where $\alpha \in \langle -1, q \rangle$, is most important. This one describes completely the distribution of zeros for all integrals of the differential equation (q). Evidently, $\varphi \in C^3_{\mathbf{R}}$, $\varphi' > 0$, $\varphi(t) > t$ and $\mathrm{Im}\, \varphi = \mathbf{R}$.

Unfortunately we do not know $\varphi(t)$ for every given equation (q). That's why the backward procedure was needed. It is based on the fact proved by the author in 1961 [3]:

Let $\varphi \in C^3_{\mathbf{R}}$ be such that $\varphi'(t) > 0$, $\varphi(t) > t$ and $\operatorname{Im} \varphi = \mathbf{R}$.

Then there exist both-sided oscillatory (in $\mathbf{R}$) differential equations (q) for which $\varphi(t)$ is the fundamental central dispersion. All such carriers q are given by the formula $q(t) = -\{\alpha, t\} - \alpha'^2$ where α ranges over all solutions of the Abelian functional equation

1° $$\alpha(\varphi(t)) = \alpha(t) + \pi,$$

such that $\alpha \in \mathfrak{P}$.

By the method of the proof of this statement it is evident that there exists the continuum of carriers having the same fundamental central dispersion.

Every solution α of 1° depends namely on an arbitrary function $\gamma \in C^3_{[t, \varphi(t)[}$ with $\gamma' > 0$ and fulfilling some boundary conditions. For some fixed t_0 inside $[t, \varphi(t)]$ and for arbitrary initial conditions $\alpha_0, \alpha'_0 > 0, \alpha''_0$ at t_0 there are $\aleph$ functions γ fulfilling these initial conditions.

Now, every q corresponds to many solutions α of the Abelian functional equation, but only one α of them has the initial conditions $\alpha_0, \alpha'_0 > 0, \alpha''_0$ at t_0. Hence there is one-to-one correspondence between the carriers q and these solutions of 1° and thus there exists the continuum of carriers q with the same fundamental central dispersion φ.

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