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ASYMPTOTIC AND OSCILLATION PROPERTIES OF THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

We investigate a linear differential equation of the third order of the form

$$y''' + p(t)\,y'' + 2A(t)\,y' + (A'(t) + b(t))\,y = 0, \tag{S}$$

where $p(t)$, $A(t)$, $A'(t) + b(t)$ are continuous on interval of definition $[a, \infty)$. Some new results for this equation in the case that $A(t) \geqq 0$ were obtained by Regenda [3] and Šoltés [6].

A new canonical form was derived by F. Neuman [1], [2] for linear differential equations of the n-th order of the form

$$y^{(n)} + a_1(t)\,y^{(n-1)} + \dots + a_n(t)\,y = 0, \tag{T}$$

$a_i \in C^0(I)$ for $i = 1, 2, \dots, n$; I is an open interval (bounded or unbounded). Here $C^n(I)$ denotes for $n \geqq 0$ the class of all continuous functions on I having here continuous derivative up and including the n-th order. This canonical form is global, i.e. each linear differential equation of the n-th order can be transformed into the form on the whole interval of definition, on the contrary to local canonical forms due to Laguerre—Forsyth characterized by $a_1 \equiv 0$ and $a_2 \equiv 0$.

This general canonical form depends on an interval of definition and $n - 2$ positive functions $\alpha_i \in C^{n-i}(J)$, $i = 1, 2, \dots, n - 1$.

For $n = 3$ the canonical form (see [1]) is

$$u''' - \alpha'(x)/\alpha(x)\,u'' + (1 + \alpha^2(x))\,u' - \alpha'(x)/\alpha(x)\,u = 0, \tag{U}$$

$\alpha \in C^1(J)$ and $\alpha(x) > 0$ for all $x \in J$.

In this paper oscillation properties and boundedness of solutions of the linear differential equation of the form (S) or (U) are studied as a continuation of [7].

We use the same methods as that by ŠVEC, SINGH [4], [5], ŠOLTÉS [6] and REGENDA [3].

2. BASIC RELATIONS

It can be verified through differentiation that for (S) on $J = [a, \infty)$ the following identity is satisfied. If we denote $L(t, a) = \exp\{\int_a^t p(s)\,ds\}$ and $F(y(t)) = y'^2(t) - 2y(t)y''(t) - 2A(t)y^2(t)$ then

$$\text{(F)}\qquad F(y)L(t, a) = F(y(a)) + \int_a^t (py'^2 + 2(b - Ap)y^2)L(s, a)\,ds.$$

In the proofs of some theorems in the papers [3], [5], [6], [7] there is used the procedure given in the form of the following

Lemma 1. *Let $u_i(t) \in C^r[a, \infty)$ be functions, c_{in} constants, $i = 1, 2, \dots, s$. Let the sequence $\{y_n\}$ be defined by the relations*

$$y_n = \sum_{i=1}^{s} c_{in}u_i, \qquad \sum_{i=1}^{s} c_{in}^2 = 1.$$

Then there exists a subsequence $\{n_j\}$ such that $c_{in_j} \to c_i$ and $\{y_{n_j}\}$ converges on every finite subinterval of $[a, \infty)$ uniformly to the function

$$y = \sum_{i=1}^{s} c_iu_i, \qquad \sum_{i=1}^{s} c_i^2 = 1,$$

as $n_j \to \infty$ such that

$$y^{(z)} = \sum_{i=1}^{s} c_iu_i^{(z)}, \qquad z = 0, 1, 2, \dots, m \leqq r.$$

In this paper we use the following results given in [3] and [5].

Lemma 2. ([5]) *Let a function $y = y(t)$ be a solution of the equation $y^{(n)} + p_1y^{(n-1)} + \dots + p_ny = p_0$ with bounded continuous coefficients $p_k(t)$, $k = 0, 1, \dots, n$, on $[a, \infty)$. If the solution y is bounded on $[a, \infty)$, then the derivatives $y^{(s)}(t)$, $s = 1, 2, \dots, n$ of the solution y are bounded on $[a, \infty)$.*

Lemma 3. ([5]) *If a function y has a finite limit as $t \to \infty$ and $y^{(n)}(t)$ is bounded for all $t \geqq t_0$, then $y^{(k)}(t) \to 0$ as $t \to \infty$ for $0 < k < n$.*

Lemma 4. ([5]) *Let $f(t) \in C^1[a, \infty)$. If $\int_a^\infty f^2(t)\,dt < \infty$ and f' is bounded on $[a, \infty)$, then $f(t) \to 0$ as $t \to \infty$.*

Lemma 5. ([3]) *If $p(t) \geqq 0$ and $b(t) - A(t)p(t) \geqq 0$ being not identically zero in any interval, and* (S) *has one oscillatory solution, then a necessary and sufficient condition for a solution $y \not\equiv 0$ to be nonoscillatory is that $F(y(t)) < 0$ for all $t \in [a, \infty)$.*

Lemma 6. ([3]) *If $p(t) \geqq 0$ and $b(t) - A(t)p(t) \geqq 0$ being not identically zero in any interval, then* (S) *has a solution for which $F(y(t))$ is always negative. Consequently $y(t)$ is nonoscillatory.*

Lemma 7. ([7]) *Let $A(t) \geqq 0$, $p(t) \leqq 0$, $A'(t) + b(t) \leqq 0$ not identically zero on any subinterval of $[a, \infty)$ and $y(t) \not\equiv 0$ be nonoscillatory solution of* (S) *satisfying the inequality $F(y(t)) > 0$ for all $t \geqq a$. Then $c \in [a, \infty)$ exists such that for all $t \geqq c$ there holds $y(t)\,y'(t) > 0$.*

3. FURTHER RELATIONS

Theorem 1. *Let $A(t) \geqq 0$, $p(t) \leqq 0$ and $b(t) - A(t)p(t) \leqq 0$ be not identically zero on any subinterval of $[a, \infty)$. Then the equation* (S) *has two linearly independent nontrivial solutions $v(t)$, $w(t)$ with the property that $F(y(t))$, $F(w(t))$ are positive for all $t \geqq a$.*

Proof: Let the solutions y_1, y_2, y_3 of the equation (S) be determined by the initial conditions

$$y_i^{(j)}(a) = \delta_{i,j+1} = \begin{cases} 0 & i \neq j+1 \\ 1 & i = j+1 \end{cases} \qquad \begin{array}{l} i = 1, 2, 3, \\ j = 0, 1, 2. \end{array}$$

Let $n > a$ be positive integers, b_{1n}, b_{3n} and c_{2n}, c_{3n} constants such that the solutions v_n and w_n of the equation (S) defined by

$$\begin{aligned} v_n(t) &= b_{1n}y_1(t) + b_{3n}y_3(t), \qquad b_{1n}^2 + b_{3n}^2 = 1, \\ w_n(t) &= c_{2n}y_2(t) + c_{3n}y_3(t), \qquad c_{2n}^2 + c_{3n}^2 = 1, \end{aligned}$$

satisfy $v_n(n) = w_n(n) = 0$. Then $F(v_n(n)) \geqq 0$, $F(w_n(n)) \geqq 0$ and since $F(y(t))\,L(t, a)$ is a decreasing function, there holds

$$\text{(1)} \qquad F(v_n(t)) > 0, \ F(w_n(t)) > 0 \qquad \text{on} \qquad [a, n) \qquad \text{for } L(t, a) > 0.$$

By Lemma 1 the sequence $\{n_k\}$ exists such that $\{v_{n_k}(t)\}$ converges for $n_k \to \infty$ on every finite subinterval from $[a, \infty)$ uniformly to a function $v(t)$ and there holds

$$v^{(s)}(t) = b_1 y_1^{(s)}(t) + b_3 y_3^{(s)}(t), \qquad s = 0, 1, 2 \qquad \text{and} \qquad b_1^2 + b_3^2 = 1.$$

From (1) it follows that $F(v(t)) \geqq 0$ on $[a, \infty)$. As $F(y(t))\,L(t, s)$ is a decreasing function, there must be $F(v(t)) > 0$ on $[a, \infty)$. Otherwise $F(v)$ obtains negative values which is a contradiction. We can prove similarly that $F(w(t)) > 0$ and $c_2^2 + c_3^2 = 1$ on $[a, \infty)$. Let the solutions $v(t)$, $w(t)$ be dependent. As $b_1^2 + b_3^2 = c_2^2 + c_3^2 = 1$

is satisfied, there holds $v(t) = Ky_3(t)$ for some $K \neq 0$. Then $F(v(a)) = F(y_3(a)) = 0$ by the definition of y_3, which is a contradiction to $F(v(t)) > 0$ on $[a, \infty)$. We have proved that $v(t)$, $w(t)$ are linearly independent solutions. This completes the proof.

Lemma 8. *Let $A(t) \geqq 0$, $p(t) \leqq 0$, $A'(t) + b(t) \leqq 0$ not identically zero on any subinterval of $[a, \infty)$ and $y(t)$ be a nontrivial solution of* (S) *satisfying the inequality $F(y(t)) > 0$ for all $t \geqq a$. If $\int_a^\infty A(t)\,dt = \infty$, then $y(t)$ is oscillatory.*

Proof: For $y(t) \not\equiv 0$ nonoscillatory solution of the equation (S) there exists $c \in [a, \infty)$ such that for all $t \geqq c$ there holds $y(t)\,y'(t) > 0$ by Lemma 7. If the inequality $F(y(t)) > 0$ on $[c, \infty)$ is satisfied then $F(y) = y'^2 - 2yy'' - 2Ay^2 > 0$ i and only if $(y'(t)/y(t))' < -A(t)$ on this interval. By integration of the last inequality from c to t we obtain

$$y'(t)/y(t) < y'(c)/y(c) - \int_a^t A(s)\,ds \to -\infty \qquad \text{as} \qquad t \to \infty,$$

which is a contradiction to $y(t)\,y'(t) > 0$ on $[c, \infty)$ and $y(t)$ cannot be nonoscillatory.

Theorem 2. *Let $A(t) \geqq 0, p(t) \leqq 0$ and $A'(t) + b(t) \leqq 0$ and $b(t) - A(t)p(t) \leqq 0$, being not identically zero on any subinterval of $[a, \infty)$. If $\int_a^\infty A(t)\,dt = \infty$ then the equation* (S) *has two linearly independent oscillatory solutions.*

Proof: Under our suppositions the equation (S) has two nontrivial linearly independent solutions $v(t)$, $w(t)$ with the property $F(v(t)) > 0$ and $F(w(t)) > 0$ for all $t \geqq a$ by Theorem 1. Solutions $v(t)$, $w(t)$ are oscillatory by Lemma 8.

Lemma 9. *Let $A(t) \geqq 0$, $p(t) \leqq 0$ and $A'(t) + b(t) \leqq 0$ and $b(t) - A(t)p(t) \leqq 0$, being not identically zero on any subinterval of $[a, \infty)$. If $\int_a^\infty A(t)\,dt = \infty$ then a nontrivial solution of the equation* (S) *is nonoscillatory if and only if $c \in [a, \infty)$ exists such that $F(y(c)) \leqq 0$.*

Proof: The necessity follows from Lemma 8. Under the given suppositions the function $F(y(t))\,L(t, a)$ is strictly decreasing, thus $F(y(t)) < 0$ on $[d, \infty)$, $d \geqq c$. Let $y(t_0) = 0$ for $t_0 \in [d, \infty)$. Then $F(y(t_0)) = y'^2(t_0) \geqq 0$ which is a contradiction and the solution $y(t)$ must be nonoscillatory.

Theorem 3. *Let $p(t) \geqq 0$, $A(t) \geqq 0$, $b(t) - A(t)\,p(t) \geqq m > 0$ and coefficients of the equation* (S) *are bounded. If*

$$\int_a^\infty A(t)\,dt = \infty \qquad \text{and} \qquad \int_a^\infty p(t)\,dt = \infty,$$

then for a nontrivial nonoscillatory solution $y(t)$ of the equation (S) *there holds $y^{(k)}(t) \to 0$ as $t \to \infty$, $k = 0, 1, 2, 3$.*

Proof: Let $y(t)$ be a nontrivial nonoscillatory solution of the equation (S). We can suppose without loss of generality that $y(t) > 0$ for all $t \geqq t_0 \geqq a$. The function $F(y(t))\,L(t, a)$ is increasing, thus $F(y(t)) < 0$ on $[t_0, \infty)$ or there exists $t_1 \in [t_0, \infty)$ such that $F(y(t_1)) \geqq 0$ and $F(y(t)) > 0$ for all $t \geqq t_1$.

In the first case

$$0 > F(y(t))\,L(t, t_0) = F(y(t_0)) + \int_{t_0}^{t} py'^2 L(s, t_0)\,ds + + 2\int_{t_0}^{t} (b - Ap)\,y^2 L(s, t_0)\,ds > F(y(t_0)) + + 2m\int_{t_0}^{t} y^2(s)\,ds \quad \text{because} \quad L(t, t_0) \geqq 1.$$

We have $\int_{t_0}^{\infty} y^2(s)\,ds < -F(y(t_0))/2m$ and $F(y(t_0)) < 0$, thus $\int_{t_0}^{\infty} y^2(t)\,dt < \infty$. We assert that $y'(t)$ is a bounded function on $[a, \infty)$. Indeed if there exists a constant $K_1 > 0$ such that $|y'| \geqq K_1$ on some interval $[t_2, \infty)$, $t_2 \geqq t_0 \geqq a$, then from identity (F) we have for $L(t, t_2) \geqq 1$

$$F(y(t))\,L(t, t_2) > F(y(t_2)) + K_1^2 \int_{t_2}^{t} p(s)\,ds \to \infty,$$

as $t \to \infty$ which is a contradiction to $F(y(t)) < 0$ on $[t_0, \infty)$. Since $\int_{t_0}^{\infty} y^2(y)\,dt < \infty$ and y' is a bounded function, thus $y(t) \to 0$ as $t \to \infty$ by Lemma 4.

In the second case $F(y(t)) > 0$ on (t_1, ∞) and $\int_{t_0}^{\infty} A(t)\,dt = \infty$ and $y(t) > 0$ on (t_1, ∞). Hence $y'^2 - 2yy'' - 2Ay^2 > 0$ if and only if $(y'/y)' < -A$ on $[d, \infty)$, $d > t_1$. By integration of this inequality from d to t we obtain

$$y'(t)/y(t) < y'(d)/y(d) - \int_{d}^{t} A(s)\,ds \to -\infty \qquad \text{as} \qquad t \to \infty.$$

There exists a positive constants K_2 such that $y'(t) < -K_2 y(t)$ on $[d, \infty)$ and $\lim y(t) = k \geqq 0$ as $t \to \infty$. If $k > 0$ then $y' < -K_2 k$ which is a contradiction to $y > 0$ on $[d, \infty)$. We have $\lim y(t) = 0$ as $t \to \infty$.

The function y''' is bounded by Lemma 2, $y' \to 0$ and $y'' \to 0$ as $t \to \infty$ by Lemma 3 and $y''' = -py'' - 2Ay' - (A' + b)\,y \to 0$ as $t \to \infty$ under our suppositions. The assertion is proved.

Remark 1. Under the suppositions of Theorem 3 there exists a nontrivial solution for which $F(y(t))$ is always negative by Lemma 6. This solution $y(t)$ is nonoscillatory. Otherwise $F(y)$ obtains positive values which is a contradiction.

Remark 2. In the oscillation criterion of Šoltés [6] there is the supposition $\int_{a}^{\infty} p(t)\,dt < \infty$, whereas we have $\int_{a}^{\infty} p(t)\,dt = \infty$.

4. APPLICATIONS TO THE CANONICAL FORM

Now we consider a global canonical form (U) on $J = [a, \infty)$

$$u''' - \alpha'(t)/\alpha(t)\, u'' + (1 + \alpha^2(t))\, u' - \alpha'(t)/\alpha(t)\, u = 0,$$

$\alpha \in C^1(J)$ and $\alpha(t) > 0$ for all $t \in J$.

Remark 3. Let $f(t) \in C^1(J)$ and $0 < k \leqq f'(t) \leqq K$ be satisfied for some positive constants k, K. If we put $\alpha(t) = \exp\{-f(t)\}$ then the coefficients of the equation (U) are bounded, the function $\alpha'(t)$ is negative and bounded and it is evident that the following three conditions are equivalent

1° $-\alpha'(t)/\alpha(t) = f'(t) \geqq k$;

2° $f(t) \to \infty$ as $t \to \infty$;

3° $\alpha(t) \to 0$ for $t \to \infty$.

For example functions f of the form

$f(t) = a \sin^m(bt + c) + nt$ on $(0, \infty)$, where $m > 0$ and $n > |\, mab \,| > 0$;
$f(t) = \log_z (t + c) + kt$ on $(-c, \infty)$ where $k > 0$;
$f(t) = t(k \pm \operatorname{arctg} t) - \ln(1 + t^2)/2$ on $(0, \infty)$ where $k > \mathrm{p}/2$;
$f(t) = t^3/(1 + t^2)$ on $(0, \infty)$;
$f(t) = \exp\{-t\}$ on $[a, \infty]$, a be arbitrary;
e.t.c.

can be considered.

Theorem 4. *Let* $\alpha(t) = \exp\{-f(t)\}$, $f(t) \in C^1(J)$ *and* $0 < k \leqq f' \leqq K$ *be satisfied for some positive constants* k, K *on* J. *If* $y(t)$ *is a nontrivial nonoscillatory solution of the equation* (U) *then* $y^{(s)}(t) \to 0$ *as* $t \to \infty$, $s = 0, 1, 2, 3$.

Proof: According to Remark 3 we have $p(t) = -\alpha'(t)/\alpha(t) = f'(t) \geqq k$ and $A(t) = (1 + \alpha^2(t))/2 > 1/2$, thus $\int_a^\infty A(t)\,\mathrm{d}t = \infty$ and $\int_a^\infty p(t)\,\mathrm{d}t = \infty$ and $b(t) - A(t)p(t) = A(t)p(t) > k/2 > 0$ if and only if $\alpha(t) \to 0$ as $t \to \infty$. The assertion follows from Theorem 3.

Theorem 5. *If* $\alpha'(t) \geqq 0$ *being not identically zero on any interval, then*

(i) *a nontrivial solution of the equation* (U) *is nonoscillatory if and only if* $c \in [a, \infty)$ *exists such that* $F(y(c)) \leqq 0$;

(ii) *the equation* (U) *has two linearly independent oscillatory solutions.*

Proof: (i) follows from Lemma 9 and (ii) from Theorem 2.

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