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IDEAL SYSTEMS OF INTERSECTION AND PRODUCT TYPE

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INTRODUCTION

Let $S \neq \emptyset$ be a set. Then a mapping $^-: \exp S \to \exp S$ with properties

I. $A \subseteq S \Rightarrow A \subseteq \bar{A}$,
II. $A, B \subseteq S, A \subseteq \bar{B} \Rightarrow \bar{A} \subseteq \bar{B}$,

is usually called *a closure operator*. A set $A \subseteq S$ such that $\bar{A} = A$ is a closed set. A system $\Omega \subseteq \exp S$ is a closure system if to every set $A \subseteq S$ one closed set $\bar{A} \in \Omega$ co-ordinates. A set S with a closure system Ω is called *a closure space* (S, Ω). We shall denote by $\bar{s} = \{\bar{s}\}$, for $s \in S$.

Let $(S, .)$ be a grupoid. Then a closure operator $^-: \exp S \to \exp S$ with properties

III. $A \subseteq S \Rightarrow S . \bar{A} \cup \bar{A} . S \subseteq \bar{A}$,
IV. $A, B \subseteq S \Rightarrow A . \bar{B} \cup \bar{A} . B \subseteq \overline{A . B}$,

is called *an ideal operator*. A set $A \subseteq S$ such that $\bar{A} = A$ is *an ideal.* A system $\Omega \subseteq \exp S$ is *an ideal system* for an ideal operator $^-$ on S, if to every set $A \subseteq S$ one ideal $\bar{A} \in \Omega$ co-ordinates. A set S with an ideal system Ω is called *an ideal space* $(S, ., \Omega)$.

This conception of ideals is taken over [1]. Associativity and commutativity of operation . on S, that are usually supposed, are not necessary in this paper. The ideals defined above are a generalization of many systems of ideals in algebraic structures, for example ideals in rings, semigroups, distributive lattices, normal subgroups in groups, convex subgroups in lattice-ordered groups.

Of course, it depends on a suitable choise of operation . on corresponding algebraic structures.

The following problem is investigated in the paper: Let Ω be a closure system on a non-empty set S. What conditions has an operation . on S to fulfil so that Ω is an ideal system on a grupoid $(S, .)$?

Results of the paper are concerned with that problem and special cases of ideals fulfilling condition $\overline{a . b} = \bar{a} \cap \bar{b}$ ($\overline{a . b} = \bar{a} . \bar{b}$, resp.), for $a, b \in S$, so called ideals of intersection (product, resp.) type.

In § 1. there are some conditions equivalent to III. and IV. from definition of ideal system. § 2. contains results about ideals of intersection and product types. Most results are concerned with ideals of intersection type—for instance uniqueness of operation ., distributivity of Ω.

§ 1. IDEAL SYSTEMS

Proposition 1.1. *Let* $(S, .)$ *be a grupoid and* $^{-}$: exp $S \to$ exp S *be a closure operator on* S. *Then the following assertions are equivalent:*

1. $A \subseteq S \Rightarrow S . \bar{A} \cup \bar{A} . S \subseteq \bar{A}$,
2. $a, b \in S \Rightarrow \overline{a . b} \subseteq \bar{a} \cap \bar{b}$,
3. $A, B \subseteq S \Rightarrow \overline{A . B} \subseteq \bar{A} \cap \bar{B}$,
4. $A, B \subseteq S \Rightarrow \overline{A . \bar{B}} \cup \overline{\bar{A} . B} \subseteq \bar{A} \cap \bar{B}$,
5. $A, B \subseteq S \Rightarrow \overline{\bar{A} . \bar{B}} \subseteq \bar{A} \cap \bar{B}$.

Proposition 1.2. *Let* $(S, .)$ *be a grupoid and* $^{-}$: exp $S \to$ exp S *be a closure operator on* S. *Then the following assertions are equivalent:*

1. $A, B \subseteq S \Rightarrow A . \bar{B} \cup \bar{A} . B \subseteq \overline{A . B}$,
2. $A, B \subseteq S \Rightarrow \bar{A} . \bar{B} \subseteq \overline{A . B}$,
3. $A, B \subseteq S \Rightarrow \overline{A . \bar{B}} = \overline{\bar{A} . B} = \overline{A . B}$,
4. $A, B \subseteq S \Rightarrow \overline{\bar{A} . \bar{B}} = \overline{A . B}$.

Further, if we denote

5. $a, b, c \in S \Rightarrow \bar{a} . \bar{b} \subseteq \overline{a . b}$, $\bar{a} . (\overline{b \cup c}) \subseteq \overline{\bar{a} . \bar{b} \cup \bar{a} . \bar{c}}$, *then* 1. *implies* 5. *In the closure system* Ω *defined by a closure operator* $^{-}$ *is a closure system of finite character, then also* 5. *implies 1.*

Remark. A closure system of finite character is in the sense of [2], i.e., $A \subseteq S \Rightarrow \Rightarrow \bar{A} = \bigcup \{\bar{N} : N \subseteq A, \text{card } N < \aleph_0\}$.

Proof of 1.2. Implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. $\Rightarrow$ 1. are clear. The implication 1. $\Rightarrow$ 5. follows from [1], Th. 1., $A \Leftrightarrow C : A . \bar{B} \subseteq \overline{A . B} \Leftrightarrow A . (B \vee C) \subseteq A . B \vee \vee A . C$, where $A \vee B = A \cup B$. That equivalence can be proved by the method of Aubert's proof without associativity and commutativity of the operation ..

If Ω is a closure system of finite character, then for $\psi = \{N \subseteq B : \text{card}\, N < \aleph_0\}$ we deduce from 5.:

$$A \,.\, \bar{B} = A \,.\, \bigcup \{\bar{N} : N \in \psi\} = \bigcup \{a \,.\, \bigcup \overline{\{n_{1N}, \ldots, n_{kN}\}} : N =$$
$$= \{n_{1N}, \ldots, n_{kN}\}, N \in \psi, a \in A\} \subseteq \bigcup \{\bar{a} \,.\, \overline{(\{n_{1N}\} \cup \ldots \cup \{n_{kN}\})} : N =$$
$$= \{n_{1N}, \ldots, n_{kN}\}, N \in \psi, a \in A\} \subseteq \bigcup \{\bar{a} \,.\, \overline{(\bar{n}_{1N} \cup \ldots \cup \bar{n}_{kN})} : N =$$
$$= \{n_{1N}, \ldots, n_{kN}\}, N \in \psi, a \in A\} \subseteq \bigcup \{\overline{\bar{a} \,.\, \bar{n}_{1N} \cup \ldots \cup \bar{a} \,.\, \bar{n}_{kN}}\} : N =$$
$$= \{n_{1N}, \ldots, n_{kN}\}, N \in \psi, a \in A\} \subseteq \bigcup \{\overline{\overline{a \,.\, n_{1N}} \cup \ldots \cup \overline{a \,.\, n_{kN}}}\} : N =$$
$$= \{n_{1N}, \ldots, n_{kN}\}, N \in \psi, a \in A\} = \bigcup \{\overline{a \,.\, n_{1N}, \ldots, a \,.\, n_{kN}}\} : N =$$
$$= \{n_{1N}, \ldots, n_{kN}\}, N \in \psi, a \in A\} = \bigcup \{\overline{a \,.\, \{n_{1N}, \ldots, n_{kN}\}}\} : N =$$
$$= \{n_{1N}, \ldots, n_{kN}\}, N \in \psi, a \in A\} = \bigcup \{\overline{A \,.\, N} : N \in \psi\} \subseteq \overline{A \,.\, B}.$$

Corollary 1.3. *If Ω is a closure system of finite character defined by a closure operator $^{-}$ on a grupoid $(S, .)$, then Ω is an ideal system defined by an ideal operator $^{-}$ iff it holds:*

$$a, b, c \in S \Rightarrow \bar{a} \,.\, \bar{b} \subseteq \overline{a \,.\, b} \subseteq \bar{a} \cap \bar{b}, \bar{a} \,.\, \overline{(\bar{b} \cup \bar{c})} \subseteq \overline{\bar{a}\,, \bar{b} \cup \bar{a} \,.\, \bar{c}}.$$

Proposition 1.4. *Let (S, Ω) be a closure space and . be an operation on S with the property $\bar{A} \,.\, S \cup S \,.\, \bar{A} \subseteq \bar{A}$, for $A \subseteq S$. Then it holds:*

1. *If $0 \in S$ is a zero, then $0 \in \bigcup \Omega$.*
2. *$\bigcup \Omega = \{s\}$ iff there exists an element $s \in S$ such that $\bar{s} = \{s\}$.*

Further, an element $s \in S$ with the property $\bar{s} = \{s\}$ is unique and it is a zero in $(S, .)$.

Proof. 1. $0 = a \,.\, 0 \in \bar{A}$, for every $A \subseteq S$ and $a \in A$. 2. $\bigcup \Omega = \{s\} \Rightarrow \bar{s} \subseteq \bigcup\Omega \Rightarrow \Rightarrow \bar{s} = \bigcup\Omega = \{s\}$ and on the other hand $\bar{s} = \{s\} \Rightarrow g \,.\, \bar{s} \subseteq \bar{s} = \{s\}$, for every $g \in \in S \Rightarrow g \,.\, s = s$ (and $s \,.\, g = s$, similarly), for every $g \in S \Rightarrow s$ is a zero in $S \Rightarrow s = = s \,.\, a \in s \,.\, \bar{A} \subseteq \bar{A}$, for every $A \subseteq S$ and $a \in A \Rightarrow s \in \bigcup \Omega \Rightarrow \bigcup \Omega = \bar{s} = \{s\}$.

§ 2. IDEALS OF INTERSECTION AND PRODUCT TYPE

Definition 2.1. Let $(S, ., \Omega)$ be an ideal space. If for every $a, b \in S$ it holds

(*I*) $\overline{a \,.\, b} = \bar{a} \cap \bar{b}$,

(*P*) $\overline{a \,.\, b} = \bar{a} \,.\, \bar{b}$, respectively,

then ideals from Ω are called *ideals of intersection type (I-ideals)*, *ideals of product type (P-ideals)*, respectively.

If an ideal from Ω is an ideal of intersection type and product type, then it is called *an IP-ideal.*

Proposition 2.2. *Let $(S, ., \Omega)$ be an ideal space. Then the following assertions are equivalent:*

1. *Ideals from Ω are IP-ideals.*
2. $s \in S \Rightarrow s \in \bar{s} . \bar{s}$.
3. $s \in S \Rightarrow \bar{s} = \bar{s} . \bar{s}$.
4. $A \subseteq S \Rightarrow \bar{A} = \bar{A} . \bar{A}$.
5. $A, B \subseteq S \Rightarrow \bar{A} . \bar{B} = \bar{A} \cap \bar{B}$.

Proposition 2.3. *Let $(S, ., \Omega)$ be an ideal space. Then it holds:*

1. *If every ideal from Ω is an IP-ideal, then $S . \bar{A} = \bar{A}$, for $A \subseteq S$.*
2. *If $\bar{A} = S . A \cup A$, for $A \subseteq s$, then every ideal from Ω is a P-ideal.*

Proof.

1. $x \in \bar{A} \Rightarrow x \in \bar{x} = \bar{x} \cap \bar{x} = \bar{x} . \bar{x} \subseteq S . \bar{A} \Rightarrow \bar{A} \subseteq S . \bar{A}$.
2. $\bar{A} = S . A \cup A \Rightarrow \bar{a} . \bar{b} = (S . a \cup \{a\}) . (S . b \cup \{b\}) = S . a . S . b \cup$
   $\cup\, a . S . b \cup S . a . b \cup \{a . b\} \supseteq S . a . b \cup \{a . b\} = \overline{a . b}$.

Proposition 2.4. *Let $(S, ., \Omega)$ be an ideal space. Then the following assertions are equivalent:*

1. *Ideals from Ω are I-ideals.*
2. $s \in S \Rightarrow s \in \overline{s . s}$.
3. $s \in S \Rightarrow \bar{s} = \overline{s . s}$.
4. $A \subseteq S \Rightarrow \bar{A} = \overline{A . A}$.
5. $A, B \subseteq S \Rightarrow \overline{A . B} = \bar{A} \cap \bar{B}$.

Examples.

1. Ideals in a commutative ring are *P*-ideals and are not *I*-ideals with regard to ring's multiplication.

2. Ideals in a distributative lattice are *IP*-ideals with regard to the infimum.

3. Normal subgroups in a group $(G, +)$ are neither *I*-ideals nor *P*-ideals with regard to the operation

$$a . b = -a - b + a + b, \qquad a, b \in G.$$

4. Convex 1-subgroups in a lattice-ordered group $(G, +, \vee, \wedge)$ are I-ideals and are not *P*-ideals with regard to the operation

$$a . b = |a| \wedge |b|, \qquad a, b \in G.$$

5. Polars in a lattice-ordered group are *I*-ideals and are not *P*-ideals with regard to the same operation as in the example 4.

6. The following proposition is proved in the paper [3]: Let G be a lattice-ordered group, A_l be a convex l-subgroup in G generated by a set $A \subseteq G$. Then $l : \exp G \to \exp G$ is an ideal operator on G with regard to the operation $a . b = |a| \wedge |b|$, for $a, b \in G$. Further, as far as B_l is an ideal in G with regard to the operation ., that is a subgroup in G, then B_l is a convex l-subgroup in G.

Proposition 2.5. *Let $(G, +, \vee, \wedge)$ be a lattice-ordered group and . be an operation on G defined in the following way:*

$$a . b = |a| \wedge |b|, \qquad \text{for } a, b \in G.$$

Then it holds: A closure operator $s : \exp G \to \exp G$ is an ideal operator with regard to the operation . iff the inclusion $A_s \supseteq A \cup \{g \in G : 0 \leqq g \leqq |a|, \text{for some } a \in A\}$ holds, for every $A \subseteq G$.

Proof. $\Rightarrow$: If $a \in A, g \in G, 0 \leqq g \leqq |a|$, then $g = |g| \wedge |a| = g . a \in G . A_s \subseteq A_s$.

$\Leftarrow$: We prove the conditions III. and IV. from definition of an ideal operator: III. If $g \in G$, $a \in A_s$, then $0 \leqq g . a = |g| \wedge |a| \leqq |a|$ and $G . A_s \subseteq A_s$. Similarly $A_s . G \subseteq A_s$.

IV. If $x \in A . B_s$, then $x = |a| \wedge |c|$, where $a \in A$, $c \in B_s$. Further, $c \in B$ or there exists an element $b \in B$ such that $0 \leqq |c| \leqq |b|$. It means that $0 \leqq x \leqq |a| \wedge |b| = a . b$, for a suitable element $b \in B$, i.e., $x \in (A . B)_s$. Similarly $A_s . B \subseteq (A . B)_s$.

Proposition 2.6. *Let (S, Ω) be a closure space, $0 \in S$. Then (S, Ω) is an ideal space with regard to the operation . defined in the following way:*

$$a . b = 0, \qquad \textit{for every } a, b \in S.$$

Further it holds:

a) *Ideals from Ω are I-ideals iff $\bar{A} = S$, for every $A \subseteq S$.*

b) *Ideals from Ω are P-ideals iff $\bar{0} = \{0\}$.*

c) *Ideals from Ω are IP-ideals iff $S = \{0\}$.*

Proposition 2.7. *If S is a non-empty set and $\bar{A} = S$, for every $A \subseteq S$, then $^-$ is an ideal operator and ideals belonging to that operator are I-ideals with regard to each operation on S. Those ideals are P-ideals with regard to an operation . iff $S = S . S$.*

Remark. *If $(S, .)$ is a commutative semigroup, then a mapping $m : \exp S \to \exp S$ such that $A_m = S . A \cup A$, for every $A \subseteq S$, is the smallest ideal operator on S (i.e., for every ideal operator $^-$ on S it is $A_m \subseteq \bar{A}$, for every $A \subseteq S$). Ideals belonging to m are P-ideals and $S . A = S . A_m$, for every $A \subseteq S$.*

Further, ideals belonging to m are I-ideals iff for every $s \in S$ it holds $s = s . s$ or there exists $l = S$ such that $s = l . s . s$.

These facts follows from [4], Proposition 4.5 and definition of I-ideals and P-ideals.

Proposition 2.8. *Let $(S, \Omega, .)$ be a closure space of finite character formed by I-ideals. Then it holds:*

1. *Operation . is unique iff for every $a, b \in S$ it is:*

$$\bar{a} \subseteq \bar{b} \Rightarrow a . b = a, \qquad b . a = a.$$

2. *Operation . is commutative and unique iff for every $a, b \in S$ it is:*

$$\bar{a} = \bar{b} \Leftrightarrow a = b.$$

Proof. 1. $\Rightarrow$: If operations . and $*$ on S fulfil suppositions, then $\overline{a . b} = \bar{a} \cap \bar{b} = \\ = \overline{a * b}$ and $a . b = (a . b) . (a * b) = a * b$, for $a, b \in S$.

$\Rightarrow$: If elements $a, b \in S$ exist such that $\bar{a} \subseteq \bar{b}$, $a . b \neq a$ or $b . a \neq a$, then we define a binary operation $*$ on S in the following way: As far as $\bar{a} \subseteq \bar{b}$ and $a . b \neq a$ or $b . a \neq a$ we define $a * b = a$ or $b * a = a$, respectively, otherwise $a . b = a * b$, for $a, b \in S$. To get a contradiction it is sufficient to prove that $(S, \Omega, *)$ is an ideal space formed by I-ideals: It is $\bar{a}$ non $\subseteq \bar{b} \Rightarrow \overline{a * b} = \overline{a . b} = \bar{a} \cap \bar{b}$ and $\bar{a} \subseteq \bar{b} \Rightarrow$ $\Rightarrow \overline{a * b} = \bar{a} = \bar{a} \cap \bar{b} = \overline{a . b}$. That fact and Proposition 1.1 (2. $\Leftrightarrow$ 5.) imply $\overline{a * b} = \\ = \overline{a . b} = \bar{a} \cap \bar{b} \supseteq \overline{\bar{a} * \bar{b}} \supseteq \bar{a} * \bar{b}$. According to Corollary 1.3 we have to prove $\bar{a} * \overline{(\bar{b} \cup \bar{c})} \subseteq \overline{\bar{a} * \bar{b} \cup \bar{a} * \bar{c}}$, for every $a, b, c \in S$.

If $x \in \bar{a} * \overline{(\bar{b} \cup \bar{c})}$, then $x = y * z$, for suitable $y \in \bar{a}$, $z \in \overline{\bar{b} \cup \bar{c}}$. If $\bar{y}$ non $\subseteq \bar{z}$, then $x = y * z = y . z \in \bar{a} . \overline{\bar{b} \cup \bar{c}} \subseteq \overline{\bar{a} . \bar{b} \cup \bar{a} . \bar{c}}$ and if $\bar{y} \subseteq \bar{z}$, then $x = y * z = y \in \bar{a} \cap \\ \cap \bar{z} \subseteq \bar{a} \cap \overline{(\bar{b} \cup \bar{c})} = \overline{\overline{\bar{a} . (\bar{b} \cup \bar{c})}} \subseteq \overline{\overline{\bar{a} . \bar{b} \cup \bar{a} . \bar{c}}} = \overline{\bar{a} . \bar{b} \cup \bar{a} . \bar{c}}$. Finally, we have $\bar{a} * \overline{(\bar{b} \cup \bar{c})} \subseteq \overline{\bar{a} . \bar{b} \cup \bar{a} . \bar{c}}$ and now we prove $\overline{\bar{a} . \bar{b} \cup \bar{a} . \bar{c}} \subseteq \overline{\bar{a} * \bar{b} \cup \bar{a} * \bar{c}} : \bar{a} \subseteq \bar{b}$ (the case $\bar{a} \subseteq \bar{c}$, similarly) $\Rightarrow \overline{\bar{a} * \bar{b} \cup \bar{a} * \bar{c}} \supseteq \overline{\{a * b\} \cup \{a * c\}} = \bar{a} \supseteq \overline{\bar{a} . \bar{b} \cup \bar{a} . \bar{c}}$ and $\bar{a}$ non $\subseteq \bar{b}$, $\bar{a}$ non $\subseteq \bar{c} \Rightarrow \overline{\bar{a} * \bar{b} \cup \bar{a} * \bar{c}} \supseteq \overline{\{a * b\} \cup \{a * c\}} = \overline{\{a . b\} \cup \{a . c\}} = \\ = \overline{\overline{a . b} \cup \overline{a . c}} \supseteq \overline{\bar{a} . \bar{b} \bar{a} . \bar{c}}$.

2. $\Rightarrow$: $\bar{a} = \bar{b} \Rightarrow a = a . b = b . a = b$.

$\Rightarrow$: If operations . and $*$ on S fulfil suppositions, then $\overline{a . b} = \bar{a} \cap \bar{b} = \overline{a * b}$ and $a . b = a * b$, for every $a, b \in S$. Finally, $\overline{a . b} = \bar{a} \cap \bar{b} = \bar{b} \cap \bar{a} = \overline{b . a}$, i.e., $a . b = \\ = b . a$, for every $a, b \in S$.

Corollary 2.9. *Let (S, Ω) be a closure space and . be the unique operation on S such that $(S, \Omega, .)$ is an ideal space formed by I-ideals. Then it holds:*

1. *Ideals from Ω are P-ideals.*
2. *If . is a commutative operation, then a relation $\leqq$ on S defined in the following way:*

$$a \leqq b \Leftrightarrow \bar{a} \subseteq \bar{b}, \qquad \textit{for } a, b \in S,$$

is a partially order on S and $a . b = a \wedge b$, for $a, b \in S$ in $(S, \leqq)$.

Proof. 1. From 2.8 we have $a = a . a$ and ideals from Ω are P-ideals – see 2.1.

2. $\bar{a} \subseteq \bar{a} \Rightarrow a \leqq a$; $a \leqq b$, $b \leqq a \Rightarrow \bar{a} \subseteq \bar{b}$, $\bar{b} \subseteq \bar{a} \Rightarrow \bar{a} = \bar{b} \Rightarrow a = b$ (see 2.8); $a \leqq b, b \leqq c \Rightarrow \bar{a} \subseteq \bar{b}, \bar{b} \subseteq \bar{c} \Rightarrow \bar{a} \subseteq \bar{c} \Rightarrow a \leqq c$. Further, $a . b \in \overline{a . b} = \bar{a} \cap \bar{b} \Rightarrow$ $\Rightarrow a . b \leqq a$, $a . b \leqq b$. If $c \in S$ exists such that $c \leqq a, c \leqq b$, then $\bar{c} \cap \overline{a . b} =$ $= \bar{c} \cap (\bar{a} \cap \bar{b}) = \bar{c} \Rightarrow \bar{c} \subseteq \overline{a . b} \Rightarrow c \leqq a . b$. Finally, $a . b = a \wedge b$ in $(S, \leqq)$.

Proposition 2.10. *If $(S, \Omega, .)$ is an ideal space formed by I-ideals, then Ω is a distributative lattice with regard to the set-inclusion.*

Proof. From [1], Theorem 1 it follows $A . \overline{B \cup C} = \overline{\overline{A . B} \cup \overline{A . C}}$, for $A, B, C \subseteq$ $\subseteq S$. It implies $\bar{A} \wedge (\bar{B} \vee \bar{C}) = \bar{A} \cap \overline{(\bar{B} \cup \bar{C})} = \bar{A} \cap \overline{(B \cup C)} = \bar{\bar{A}} \cap \overline{\overline{(B \cup C)}} =$ $= \overline{\bar{A} . \overline{B \cup C}} = \overline{\overline{A . B} \cup \overline{A . C}} = \overline{(\bar{A} \cap \bar{B}) \cup (\bar{A} \cap \bar{C})} = (\bar{A} \wedge \bar{B}) \vee (\bar{A} \wedge \bar{C})$ – see 1.1.. Further, $\bar{A} \vee (\bar{B} \wedge \bar{C}) = \overline{\bar{A} \cup (\bar{B} \cap \bar{C})} = \overline{(\bar{A} \cap \bar{B}) \cup (\bar{A} \cap \bar{C})} = \overline{(\bar{A} \wedge \bar{B}) \vee (\bar{A} \wedge \bar{C})}$.

Proposition 2.11. *Let $(G, +)$ be a group, $(G, \Omega, .)$ be an ideal space such that g is a subgroup in $(G, +)$, for every $g \in G$. Then it holds: Ideals from Ω are I-ideals iff $x + (a . b) \in (x + \bar{a}) . (x + \bar{b})$, for every $a, b, x \in G$.*

Proof. $\Leftarrow$: $x \in \bar{a} \cap \bar{b} \Rightarrow x + \bar{a} \subseteq \bar{a}$, $x + \bar{b} \subseteq \bar{b} \Rightarrow \overline{(x + \bar{a}) . (x + \bar{b})} \subseteq \overline{\bar{a} . \bar{b}} =$ $= \overline{a . b} \Rightarrow x + (a . b) \in \overline{a . b} \Rightarrow x \in \overline{a . b} - a . b \subseteq \overline{a . b} - \overline{a . b} \subseteq \overline{a . b} \Rightarrow \bar{a} \cap \bar{b} \subseteq$ $\subseteq \overline{a . b} \Rightarrow \bar{a} \cap \bar{b} = \overline{a . b}$, for every $a, b \in S$, i.e., ideals from Ω are I-ideals.

$\Rightarrow$: $x + (a . b) \in x + \overline{a . b} = x + (\bar{a} \cap \bar{b}) \subseteq (x + \bar{a}) \cap (x + \bar{b}) \subseteq \overline{x + \bar{a}} \cap$ $\cap \overline{x + \bar{b}} = \overline{(x + \bar{a}) . (x + \bar{b})}$, see 2.4.

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