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PERIODIC SOLUTION OF CERTAIN SECOND ORDER DIFFERENTIAL EQUATION

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INTRODUCTION

Consider the equation

$$x'' + [f_1(x)\,x' + f_0(x)]\,f(x') + g(x) = \mu p(t). \tag{1}$$

We assume that $f_0(x), f_1(x), f(x')$, $g(x)$ and $p(t)$ are continuous and that they are such that the initial value problem for (1) has a unique solution. Furthermore $p(t)$ a periodic function of t with the least period ω, $|p(t)| \leqq 1$, and μ is a nonnegative constant. In the first part of this note it will be shown that if the functions involved in the equation (1) satisfy some local conditions given as below, then there will exist at least one periodic solution of period ω. In the second part, we shall consider the equation (1) with $\mu = 0$ and prove the existence of at least one stable limit cycle. The method which is used here is similar to [1] and [2]. The result obtained is in fact a generalization of the result obtained in [2]. Furthermore we prove the existence of periodic solutions for the nonautonomous case ($\mu \neq 0$) which is not included in [2]. It is interesting to note the class of differential equations of the form (1) includes generalization of such well known differential equations as Lienard and Rayleih equations

§ 1: In the sequel we use the following notations:

$$G(x) = \int_0^x g(s)\,\mathrm{d}s, \qquad F_1(x) = \exp\left(\int_0^x f_1(s)\,\mathrm{d}s\right),$$

$$F(x) = \int_0^x F_1(s)\,f_0(s), \qquad L_i(x) = \int_0^x \lambda_i(s)\,\mathrm{d}s,$$

$$r_i(x) = F(x) - L_i(x), \qquad R_i(x) = \frac{r_i(x) - r_i(\alpha_i)}{F_1^2(x)}\,\lambda_i(x),$$

$$V_i(x) = \frac{r_i(x) - r_i(\alpha_i)}{F_1(x)} f_1(s) + f_0(x) : x \in [\alpha_1, \alpha_2] \qquad (i = 1, 2),$$

$$H_1(y) = -f(y) + y, \qquad y \in \left[\frac{r_1(\alpha_1) - r_1(\alpha_2)}{F_1(\alpha_2)},\ 0\right],$$

$$H_2(y) = f(y) - y, \qquad y \in \left[0,\ \frac{r_2(\alpha_2) - r_2(\alpha_1)}{F_1(\alpha_1)}\right].$$

Here α_i and $\lambda_i(x)$ $(i = 1, 2)$ are respectively, real numbers and piece-wise continuous functions occuring in the conditions of the following theorem.

Theorem 1. *Assume that there are numbers* α_1 *and* α_2, $\alpha_1 < 0 < \alpha_2$, *and functions* $\lambda_i(x) \geqq 0$ $(i = 1, 2)$ *such that*

i) $xg(x) > 0$

ii) $R_2(x) \leqq g(x) - \mu < 0,\ R_1(x) \geqq g(x) + \mu;$
$x \in [\alpha_1, 0),\ x \in (0, \alpha_2]$

iii) $r_1(x) > r_1(\alpha_1);\ x \in (\alpha_1, 0],\ r_2(x) < r_2(\alpha_2);\ x \in [0, \alpha_2)$

iv) $\operatorname{sign} V_i(x) \,.\, \operatorname{sign} H_i(y) \geqq 0 \ (i = 1, 2).$

Then (1) *has at least one periodic solution of period* ω.

Proof. Equation (1) is equivalent to the following system

$$\begin{aligned} x^{\cdot} &= y \\ y^{\cdot} &= p(t) - [f_1(x)\, y + f_0(x)]\, f(y) - g(x). \end{aligned} \tag{2}$$

In order to prove the existence of at least one periodic solution, we shall construct certain region in the phase plane appropriate to the application of Brouwer's fixed point theorem.

The outer boundary Γ_1 of the region will consist of four simple arcs joining the points

$$M_1 \left| \begin{array}{l} \alpha_2 \\ \dfrac{r_1(\alpha_1) - r_1(\alpha_2)}{F_1(\alpha_2)} \end{array} \right., \qquad M_2 \left| \begin{array}{l} \alpha_1 \\ 0 \end{array} \right.$$

$$M_3 \left| \begin{array}{l} \alpha_1 \\ \dfrac{r_2(\alpha_2) - r_2(\alpha_1)}{F_1(\alpha_1)} \end{array} \right., \qquad M_4 \left| \begin{array}{l} \alpha_2 \\ 0 \end{array} \right.$$

Consider the following arcs (see Fig. 1)

$$\begin{aligned} &\overset{\frown}{M_1M_2} : yF_1(x) + r_1(x) = r_1(\alpha_1), && (\alpha_1 \leqq x \leqq \alpha_2) \\ &\overset{\frown}{M_2M_3} : x = \alpha_1, && (y \geqq 0) \\ &\overset{\frown}{M_3M_4} : yF_1(x) + r_2(x) = r_2(\alpha_2), && (\alpha_1 \leqq x \leqq \alpha_2) \\ &\overset{\frown}{M_4M_1} : x = \alpha_2, && (y \leqq 0) \end{aligned}$$

Calculation shows that the phase trajectories of (2) intersect the closed curve $\Gamma_1 : M_1M_2M_3M_4M_1$ [which surrounds the origin by virtue of (ii)], crossing it from the outside inward. In fact, the total derivative with respect to the time of the function

$$s_1(x, y) = yF_1(x) + r_1(x) = c \qquad (c < 0)$$

is

$$\frac{d}{dt} s_1(x, y) = F_1(x) \frac{dy}{dt} + (yf_1(x) . F_1(x) + F_1(x) f_0(x) - \lambda_1(x)) \frac{dx}{dt} =$$
$$= F_1(x) [y - f(y)] [f_0(x) + yf_1(x)] + F_1(x) \{\mu p(t) - g(x)\} - \lambda_1(x) y.$$

Since

$$y . F_1(x) + r_1(x) = r_1(\alpha_1)$$
$$\frac{d}{dt} s_1(x, y) = F_1(x) . H_1(y) v_1(x) + F_1(x) . \{\mu p(t) - g(x)\} - \lambda_1(x) . \frac{r_1(\alpha_1) - r_1(x)}{F_1(x)}$$

which implies

$$\frac{1}{F_1(x)} . \frac{d}{dt} s_1(x, y) = H_1(y) . v_1(x) + R_1(x) - \{g(x) - \mu p(t)\}$$

which by conditions (i) and (iii) is nonnegative. But since increasing C corresponds to the passage from exterior curves of the given family to interior curves, this means that the phase trajectories cross the arc $\overset{\frown}{M_1M_2}$ of Γ_1 inward. Moreover, the fact the trajectories of the differential system pass inside Γ_1 across the arc $\overset{\frown}{M_2M_3}$ follows directly from the first equation (2), since $x' > 0$ in the upper half-plane in this case. The behavior of the trajectories on $\overset{\frown}{M_3M_4}$ and $\overset{\frown}{M_4M_1}$ is investigated similarly. Finally, if Ω denotes the region of the xy-plane enclosed by Γ_1, then with every point $p(x_0, y_0) \in \Omega$ we can associate the solution of (2) which satisfies the initial conditions $x(0) = x_0, y(0) = y_0$. In conjunction with this solution, let $p'(x_1, y_1)$ be point defined by $x_1 = x(\omega)$, $y_1 = y(\omega)$, with ω the least period t of $p(t)$. The transformation T mapping $p \in \Omega$ in to p' is defined and continuous in Ω. In addition, it satisfies $T(\Omega) \subset \Omega$. Hence by Brouwer's fixed-point theorem, there is at least one point $(x, y) \in \Omega$ such that for the corresponding solution $[x^*(t), y^*(t)]$, we can write $x^*(\omega) = x^*(0) = x$, $y^*(\omega) = y^*(0) = y$. Furthermore, this solution must for $t > \omega$ trace the same path as for $0 \leq t \leq \omega$, since the ω-periodicity in t of $p(t)$ implies that (1) is invariant under the translation $t \to t + \omega$. The solution $[x^*(t), y^*(t)]$ is therefore, ω periodic, which concludes the proof.

(A_1): Consider the Vander-Pole's equation

$$x'' + (1 - x^2) x' + \frac{1}{9} x = \varepsilon \cos t.$$

Here we assume $f_1(x) = 0$, $f_0(x) = 1 - x^2$, $f(x') = x'$, then we have $H_1(y) = H_2(y) = 0$, $F(x) = x - \frac{1}{3}x^3$, let $\lambda_1(x) = \lambda_2(c) = \frac{2}{3}$ and $\alpha_1 = -\frac{3}{3}$, $\alpha_2 = \frac{3}{3}$, $r_1(x) = r_2(x) = \frac{1}{3}(x - x^3)$, then $R_1(x) = \frac{2}{9}\left[x - x^3 + \frac{2\sqrt{3}}{9}\right]$ and $R_2(x) = \frac{2}{9}\left[x - x^3 - \frac{2\sqrt{3}}{9}\right]$, it follows that for appropriate values of ε, say $0 < \varepsilon < \frac{4\sqrt{3}}{81}$, we have for all $x \in \left(0, \frac{3}{3}\right]$, $\frac{1}{9}x + \varepsilon < \frac{2}{9}\left(x - x^3 + \frac{2\sqrt{3}}{9}\right)$, and for all $x \in \left(-\frac{\sqrt{3}}{3}, 0\right]$, $\frac{2}{9}\left(x - x^3 - \frac{2\sqrt{3}}{9}\right) \leqq \frac{1}{9}x - \varepsilon$ that is the conditions of Theorem 1 are satisfied. Hence equation (3) possesses at least one periodic solution of period 2π.

(A_2): Consider the equation

$$x'' + x' + x^3 = \varepsilon \cos t. \tag{4}$$

Assuming $f_0(x) = 0, f_1(x) = 1, f(x') = x'$, we have $H_1(y) = H_2(y) = 0$, $F_0(x) = 1$, $F(x) = x$, let $\lambda_1(x) = \lambda_2(x) = \frac{1}{2}$ and $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = \frac{1}{2}$ we have $r_1(x) = r_2(x) = \frac{1}{2}x$ and $R_1(x) = \frac{1}{2}\left(x + \frac{1}{2}\right)$, $R_2(x) = \frac{1}{2}\left(x - \frac{1}{2}\right)$, it follows that for appropriate values of ε say $0 < \varepsilon < \frac{1}{4.2}$, we have $\frac{1}{4}\left(x - \frac{1}{2}\right) < x^2 - \varepsilon$ for all $x \in \left[-\frac{1}{2}, 0\right)$ and $x^3 + \varepsilon < \frac{1}{4}\left(x + \frac{1}{2}\right)$ for all $x \in \left[0, \frac{1}{2}\right)$ i.e. the conditions of Theorem 1 are satisfied. Hence equation (4) possesses at least one periodic solution of period 2π.

(A_3): Consider the equation

$$x'' + \left[\frac{4x^3}{x^4 + 1}x' + x^4 - 1\right]x' + \frac{1}{9}x = \varepsilon \cos t. \tag{5}$$

Assuming $\lambda_1(x) = \lambda_2(x) = 1$, and $\alpha_1 = -2$, $\alpha_2 = 2$, with a simple calculation, we obtain $F_1(x) = x^4 + 1$, $F(x) = \frac{1}{9}x^9 - x$, $r_1(x) = r_2(x) = \frac{1}{9}x^9 - 2x$, $R_1(x) = \frac{\frac{1}{9}(x^9 + 2^9) - 2(x + 2)}{(x^4 + 1)^2}$ and $R_2(x) = \frac{\frac{1}{9}(x^9 + 2^9) - 2(x - 2)}{(x^4 + 1)^2}$, it follows that for appropriate values of ε say $0 < \varepsilon < \frac{2^9}{9} - 4$, the conditions of Theorem 1 are satisfied. Hence equation (5) possesses at least one periodic solution of period 2π.

(A_4): Consider the equation

$$x'' + (x^2 + 1)(1 - x'^2) + x = \varepsilon \cos t. \tag{6}$$

Assuming $\lambda_1(x) = \lambda_2(x) = \frac{2}{3}$, and $\alpha_1 = -1$, $\alpha_2 = 1$, $f = 0$ $f_0(x) = 1 + x^2$. The $F_1(x) = 1$, $F(x) = x + \frac{1}{3}x^3$, we obtain $r_1(x) = r_2(x) = \frac{1}{3}(x^3 + x)$, $H_1(y) = -y^3$, $H_2(y) = y^3$, $r_1(x) = r_2(x) = (x^2 + 1)$, $R_1(x) = \frac{2}{9}(x^3 + x + 2)$ and $R_2(x) = = \frac{2}{9}(x^3 + x - 2)$. Obviously $\operatorname{sign}(v_1(x))$. $\operatorname{Sign}(H_1(y)) > 0$ for $y < 0$ and $\operatorname{sign}(v_2(x))$. $\operatorname{sign}(H_2(y)) > 0$ for $y > 0$ also for appropriate values of ε, say $0 < < \varepsilon < \frac{22}{33}$, the conditions of Theorem 1 are satisfied.

§ 2. In this section we assume $\mu = 0$, and then we have the following theorem.

Theorem 2. *If, in addition to* (i)–(iv), *we assume that*

(v) $yf(y) > 0$

(vi) $f_0(0) < 0$,

then (1) *has at least one stable isolated periodic solution in the strip* $[\alpha_1, \alpha_2]$.

Proof. We define

$$v(x, y) = \frac{1}{2}y^2 + G(x).$$

Then in view of condition (i), $v(x, y)$ is locally positive definite at $(0, 0)$. Hence the curves $v(x, y) = C$, with $C > 0$ sufficiently small are closed, enclose the origin, and are completely contained in the neighborhood U of the origin where $f_0(0) < 0$. Moreover, the curve $v(x, y) = C_2$ encloses the curve $v(x, y) = C_1$ if and only if $C_2 > C_1$. Differentiating (7) and using (2), ($\mu = 0$), we obtain

$$\frac{dv}{dt} = -yf(y)\left[yf_1(x) + f_0(x)\right]. \tag{7}$$

It follows from Condition (VI) that for points in the neighborhood U of the origin with boundary Γ_2, we have $\frac{dv}{dt} > 0$. Hence trajectories of (2) cut Γ_2 from the outside. Thus, by virtue of Poincare'–Bendixon theorem, the annular region bounded by Γ_1 and Γ_2 contains at least one stable limit cycle of (2), and the theorem follows.

(A_5): Consider the equation

$$x'' + \left(3x^2 - \frac{1}{2}\right)x' + x = 0, \tag{8}$$

taking $f_0(x) = 0$, $f_1(x) = 3x^2 - \frac{1}{2}$ and $f(x') = x'$, we obtain $F_0(x) = 1$, $F(x) = x^3 - \frac{x}{2}$, let $\lambda_1(x) = \lambda_2(x) = \frac{1}{2}$ and $\alpha_1 = -2$, $\alpha_2 = 2$ we obtain $R_1(x) = \frac{1}{2}(x^3 - x + 6)$ and $R_2(x) = \frac{1}{2}(x^3 - x - 6)$, that is the conditions of theorem 2 are satisfied hence equation (7) possesses at least one stable limit cycle.

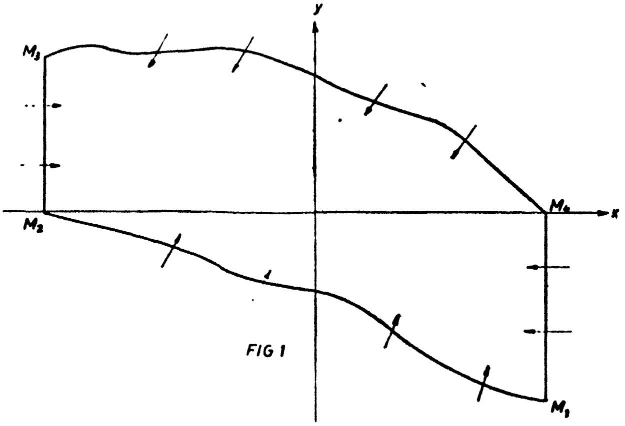

FIG 1

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