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Pavel Horák

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NOTE ON CATEGORY OF SPACES OF PARALLELISM

PAVEL HORÁK, Brno

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The notion of the space of parallelism was introduced and a special case of parallelism on the set of non-negative integers was investigated in [4] and [5]. A proximity space (see e.g. [6]) is also another special case of a space of parallelism. The aim of this note is to study spaces of parallelism from the categorical point of view and to give characterisation of some basic categorical notions for this case. The definitions of basic notions of the theory of categories are taken from [3]. Moreover, following [1] and [2], if f, g are morphisms in a category, we put $f \uparrow g$ iff from $f \circ u = f \circ v$ it follows $g \circ u = g \circ v$. Dual notion to $\uparrow$ will be denoted by $\downarrow$. Particularly, if $f\colon X \to Y$ then f is a monomorphism iff $f \uparrow 1_X$ (f is an epimorphism iff $f \downarrow 1_Y$).

A morphism f is called a *strict monomorphism* iff f is a monomorphism and if every morphism g for which $f \downarrow g$ factors through f, i.e. there exists a morphism h such that $g = f \circ h$. An object in a category will be called *strictly injective* iff it satisfies the usual injectivity condition wiht respect to strict monomorphism. Dual notion to strict monomorphism is that of a *strict epimorphism*. It is easy to show that a morphism f in a category is an isomorphism iff it is both an epimorphism and a strict monomorphism.

A morphism f is called an *essential monomorphism* iff f is a strict monomorphism, and if any morphism g such that $g \circ f$ is a strict monomorphism is itself a strict monomorphism.

1. FUNDAMENTAL CONCEPTS

1.1. Let R be an arbitrary set, ϱ be a binary relation in the system 2^R. The pair (R, ϱ) is then called a *space of parallelism* or briefly a *space*. If $\varrho = 2^R \times 2^R$ then (R, ϱ) is called a *total space of parallelism*, if $\varrho = \emptyset$ then (R, ϱ) is called a *discrete space of parallelism.*

Let (R, ϱ), (P, π) be spaces of parallelism, f be a mapping of the set R into the set P such that for any $X, Y \subseteq R$, $X\varrho Y$ it holds $f(X)\,\pi f(Y)$. The triple (f, ϱ, π) is called a *homomorphism of the space* (R, ϱ) *into the space* (P, π). We write $f\colon (R, \varrho) \to (P, \pi)$ and usually speak briefly about a *homomorphism* f.

Let $f\colon (R, \varrho) \to (P, \pi)$ be a one-to-one mapping such that for $X, Y \subseteq R$, $f(X)\,\pi f(Y)$ it holds $X\varrho Y$. Then f is called an *embedding* of (R, ϱ) into (P, π). If an embedding is an onto mapping it is called an *isomorphism*.

1.2. Let (R, ϱ), (P, π) be spaces of parallelism, let $R \subseteq P$ and ϱ be the restriction of π on 2^R, i.e. $X\varrho Y$ iff $X\pi Y$ for $X, Y \subseteq R$. Then (P, π) is called an *extension* of the space (R, ϱ). If $R \subset P$ then the extension is called a *proper extension*. The mapping $j\colon R \to P$ defined by $j(r) = r$ for all $r \in R$ is obviously an embedding which is called the *natural embedding*.

1.3. Let (R, ϱ) be a space, $\mathscr{R}$ be a partition of R and φ be the canonical mapping of R onto $\mathscr{R}$. For $\mathscr{X}, \mathscr{Y} \subseteq \mathscr{R}$ we put $\mathscr{X}\bar{\varrho}\mathscr{Y}$ iff there exist $X, Y \subseteq R$ such that $X\varrho Y$ and $\varphi(X) = \mathscr{X}$, $\varphi(Y) = \mathscr{Y}$. Then $(\mathscr{R}, \bar{\varrho})$ is a space of parallelism which will be called a *quotient space* of the space (R, ϱ). The mapping φ is a homomorphism called the *canonical homomorphism* of (R, ϱ) on $(\mathscr{R}, \bar{\varrho})$. Obviously, $\bar{\varrho}$ is the smallest relation in $2^{\mathscr{R}}$ such that φ is a homomorphism.

1.4. Let (R, ϱ) be a space. An element $u \in R$ is called a *universal element* of (R, ϱ) if for every $X, Y \subseteq R$ such that $u \in X$ it holds both $X\varrho Y$ and $Y\varrho X$.

Let us define a space (R_U, ϱ_U) in the following way: if there is a universal element of (R, ϱ) we put $R_U = R$ and $\varrho_U = \varrho$. If there is no universal element of (R, ϱ) we put $R_U = R \cup \{w\}$ where $w \notin R$ and for $X, Y \subseteq R_U$ such that $X, Y \subseteq R$ we put $X\varrho_U Y$ iff $X\varrho Y$, otherwise we put $X\varrho_U Y$ and $Y\varrho_U X$. Obviously, (R_U, ϱ_U) is an extension of (R, ϱ) which has a universal element.

1.5. Let $I \neq \emptyset$ and for every $i \in I$ let (R_i, ϱ_i) be a space. Let $R = \prod R_i$ $(i \in I)$ denote the cartesian product and p_i the projection of R on R_i. For $X, Y \subseteq R$ we put $X\varrho Y$ iff $p_i(X)\,\varrho_i p_i(Y)$ for every $i \in I$. Then (R, ϱ) is a space of parallelism which is called a *product of spaces* (R_i, ϱ_i), $i \in I$. Projections p_i $(i \in I)$ are homomorphisms and ϱ is obviously the largest relation in 2^R such that ϱ_i is homomorphism for every $i \in I$.

Let S denote disjoint union of sets R_i, i.e. $S = \bigcup R'_i$ $(i \in I)$ where $R'_i = \{(r, i) \mid r \in R_i\}$. For $X, Y \subseteq S$ we put $X\sigma Y$ iff there exists $j \in I$ such that $X, Y \subseteq R'_j$ and $\{r \in R_j \mid (r, j) \in X\}\ \varrho_j \{r \in R_j \mid (r, j) \in Y\}$. Then (S, σ) is a space of parallelism which is called a *coproduct of spaces* (R_i, ϱ_i), $i \in I$. If we put $s_i(r) = (r, i)$ for every $r \in R_i$ we get a mapping of R_i into S, called injection. Clearly $s_i\colon (R_i, \varrho_i) \to (S, \sigma)$ and σ is the smallest relation in 2^S such that s_i is a homomorphism for every $i \in I$.

2. THE CATEGORY $\mathfrak{P}$ OF SPACES OF PARALLELISM

2.1. The spaces of parallelism with homomorphisms form a category where the composition of morphisms is the usual composition of mappings. This category will be called a *category of spaces of parallelism* and denoted by $\mathfrak{P}$.

It is obvious that null objects are one-element total spaces, conull object is the empty discrete space and hence there are no zero objects in $\mathfrak{P}$. Also quite natural are constructions of equalizers, coequalizers, pullbacks and pushouts in $\mathfrak{P}$.

2.2.1. *Let $f: (R, \varrho) \to (P, \pi)$, $g: (R, \varrho) \to (P', \pi')$. Then $f \uparrow g$ in $\mathfrak{P}$ iff it holds: $r, s \in R, f(r) = f(s) \Rightarrow g(r) = g(s)$.*

2. *Let $f: (R, \varrho) \to (P, \pi)$, $g: (R', \varrho') \to (P, \pi)$. Then $f \downarrow g$ in $\mathfrak{P}$ iff $f(R) \supseteq g(R')$.*

Proof. Sufficiency of both conditions is obvious. To prove necessity we take in 2.2.1. a discrete space $(\{x\}, \varrho)$ and put $u(x) = r$, $v(x) = s$. In 2.2.2. we take a total space $(\{x, y\}, \varrho)$ and put $u(p) = x$ for every $p \in P$ and further we put $v(p) = x$ for every $p \in f(R)$, and $v(p) = y$ for every $p \in P - f(R)$.

2.3. *Let $f: (R, \varrho) \to (P, \pi)$. Then*
1. *f is a monomorphism in $\mathfrak{P}$ iff f is a one-to-one mapping*
2. *f is an epimorphism in $\mathfrak{P}$ iff f is an onto mapping.*

Both assertions follow from 2.2. by putting $g = 1_R$ or $g = 1_P$, respectively. Moreover, from 2.3. it follows that $\mathfrak{P}$ *is not a balanced category.*

2.4. *Let $f: (R, \varrho) \to (P, \pi)$. Then f is a strict monomorphism in $\mathfrak{P}$ iff f is an embedding of (R, ϱ) into (P, π). Consequently, f is an isomorphism in the category $\mathfrak{P}$ iff f is an isomorphism.*

Proof. If f is a strict monomorphism it is a one-to-one mapping, according to 2.3.1. Let $X, Y \subseteq R$ such that $f(X) \pi f(Y)$. Let $S = f(R)$ and σ be the restriction of π on 2^S. By 2.2.2., $f \downarrow j$ where $j: (S, \sigma) \to (P, \pi)$ is the natural embedding and hence there exists h: $(S, \sigma) \to (R, \varrho)$ such that $f \circ h = j$. Obviously, it holds $h(f(X)) \varrho h(f(Y))$. Since h is a bijection, we have $h(f(X)) = X$, $h(f(Y)) = Y$ and consequently $X \varrho Y$. This shows that f is an embedding.

Conversely, if f is an embedding of (R, ϱ) into (P, π) then f is a monomorphism and if $f \downarrow g$ where $g: (R', \varrho') \to (P, \pi)$ then by 2.2.2. we have $f(R) \supseteq g(R')$. Hence for every $r' \in R'$ there exists exactly one $r \in R$ such that $f(r) = g(r')$. If we put $h(r') = r$ we get $h: (R', \varrho') \to (R, \varrho)$ and $f \circ h = g$.

The second assertion follows immediately from the first one and from 2.2.2.

2.5. *Let $f: (R, \varrho) \to (P, \pi)$. Then f is a strict epimorphism in $\mathfrak{P}$ iff there exists a quotient space $(\mathcal{R}, \bar{\varrho})$ of the space (R, ϱ) and an isomorphism $g: (\mathcal{R}, \bar{\varrho}) \to (P, \pi)$ such that $g \circ \varphi = f$ where φ denotes the canonical homomorphism of (R, ϱ) on $(\mathcal{R}, \bar{\varrho})$.*

Proof. Let f be a strict epimorphism, let $\mathcal{R}$ be the partition of R induced by the mapping f and $(\mathcal{R}, \bar{\varrho})$ be the corresponding quotient space. For an arbitrary $\mathfrak{A} \in \mathcal{R}$

we put $g(\mathfrak{A}) = f(a)$ where $a \in R$ such that $\varphi(a) = \mathfrak{A}$. Clearly, $g: (\mathscr{R}, \bar{\varrho}) \to (P, \pi)$, g is an epimorphism and $g \circ \varphi = f$. We have $f \uparrow \varphi$ and hence there exists $h: (P, \pi) \to \to (\mathscr{R}, \bar{\varrho})$ such that $h \circ f = \varphi$. Since φ is an epimorphism we get $h \circ g = 1_R$ and thus g is an isomorphism.

Conversely, let $(\mathscr{R}, \bar{\varrho})$ be a quotient space of (R, ϱ) and $g: (\mathscr{R}, \bar{\varrho}) \to (P, \pi)$ be an isomorphism such that $g \circ \varphi = f$. Then f is an epimorphism. Let $k: (R, \varrho) \to (P', \pi')$ and $f \uparrow k$. If for every $\mathfrak{A} \in \mathscr{R}$ we put $h(\mathfrak{A}) = k(x)$ where $x \in R$ such that $\varphi(x) = \mathfrak{A}$, then it is easy to show that $h: (\mathscr{R}, \bar{\varrho}) \to (P', \pi')$, $h \circ \varphi = k$ and hence $k = (h \circ g^{-1}) \circ f$. This shows that f is a strict epimorphism.

2.6. *Let $f: (R, \varrho) \to (P, \pi)$. Then f is an essential monomorphism in $\mathfrak{P}$ iff either f is an isomorphism or f is an embedding and $P - f(R) = \{u\}$ where u is the only universal element of (P, π).*

Proof. If f is an essential monomorphism then it is also an embedding. Let $S = f(R)$, σ be the restriction of π on 2^S and

$$g(t) = \begin{matrix} t & \text{for } t \in S \\ w & \text{for } t \in P - S \end{matrix}$$

where w denotes a fixed universal element of (S_U, σ_U). Then $g: (P, \pi) \to (S_U, \sigma_U)$, $g \circ f$ is a strict monomorphism and hence g is a strict monomorphism, i.e. an embedding. Thus either $S = P$ and f is an isomorphism or $P - S = \{u\}$ where u is the only universal element of (P, π).
To prove the converse is straightforward.

2.7. Definition. An extension (P, π) of the space (R, ϱ) is called an *essential extension* iff the natural embedding is an essential monomorphism. Further, (R, ϱ) is called a *retract of an extension* (P, π) iff the natural embedding is a coretraction.

2.8. *Let (R, ϱ) be a nonempty space of parallelism. Then the following statements are equivalent.*

(1) *(R, ϱ) has a universal element*
(2) *(R, ϱ) is strictly injective in $\mathfrak{P}$*
(3) *(R, ϱ) is a retract of every extension*
(4) *(R, ϱ) has no proper essential extensions*

Proof. (1) ⇒ (2). Let u be a universal element of (R, ϱ). Let $f: (P, \pi) \to (R, \varrho)$, $g: (P, \pi) \to (S, \sigma)$, g be a strict monomorphism. For an arbitrary $s \in S$ we put

$$h(s) = \begin{matrix} f(x) & \text{for } s \in g(P) \text{ where } x \in P, g(x) = s \\ u & \text{for } s \in S - g(P). \end{matrix}$$

Then $h: (S, \sigma) \to (R, \varrho)$ and $h \circ g = f$.

$(2) \Rightarrow (3)$. Let (R, ϱ) be strictly injective and (P, π) be an arbitrary extension of (R, ϱ). The natural embedding j: $R \to P$ is a strict monomorphism and hence there exists h: $(P, \pi) \to (R, \varrho)$ such that $h \circ j = 1_R$.

$(3) \Rightarrow (4)$. If (P, π) is a proper essential extension of (R, ϱ) then by 2.6. we have $P - R = \{u\}$ where u is the only universal element of (P, π). According to (3) there exists f: $(P, \pi) \to (R, \varrho)$ such that $f \circ j = 1_R$ where j denotes the natural embedding. Then $f(u)$ is a universal element of (P, π) and $f(u) \neq u$ which is a contradiction.

$(4) \Rightarrow (1)$. (R_U, ϱ_U) is clearly an essential extension of (R, ϱ). By (4) it holds $R_U = R$ and hence (R, ϱ) has a universal element.

2.9. *Let (R, ϱ) be a space of parallelism. Then*

1. *(R, ϱ) is injective in $\mathfrak{P}$ iff it is a total space*
2. *(R, ϱ) is projective in $\mathfrak{P}$ iff it is a discrete space.*

Proof. 1. Let (R, ϱ) be a total space, let f: $(P, \pi) \to (R, \varrho)$, g: $(P, \pi) \to (S, \sigma)$, g be a monomorphism. For $s \in g(P)$ there is $x \in P$ such that $g(x) = s$. We then put $h(s) = f(x)$. For $s \in S - g(P)$ we put $h(s)$ being an arbitrary element of R. Then h: $(S, \sigma) \to (R, \varrho)$ and $h \circ g = f$. Conversely, let (R, ϱ) be injective in $\mathfrak{P}$ and not a total space, i.e. there exist $X, Y \subseteq R$, X *non* ϱY. If we put $\tau = 2^R \times 2^R$ then from injectivity of (R, ϱ) we get 1_R: $(R, \tau) \to (R, \varrho)$ which implies $X \varrho Y$, a contradiction.

2. Let (R, ϱ) be projective in $\mathfrak{P}$ and not a discrete space. Then there exist $X, Y \subseteq R$, $X \varrho Y$. If we put $\omega = \emptyset$ then 1_R: $(R, \varrho) \to (R, \varrho)$, 1_R: $(R, \omega) \to (R, \varrho)$, 1_R is an epimorphism and there is obviously no homomorphism of (R, ϱ) into (R, ω). The converse is clear.

2.10. Let $I \neq \emptyset$ and for every $i \in I$ let (R_i, ϱ_i) be a space of parallelism. If (R, ϱ) denotes the product of spaces (R_i, ϱ_i), and p_i the projections then it is easy to show that $\{p_i: (R, \varrho) \to (R_i, \varrho_i)\}_{i \in I}$ is the product of the family $\{(R_i, \varrho_i)\}_{i \in I}$ in the category $\mathfrak{P}$.

Similarly, if (S, σ) denotes the coproduct of spaces (R_i, ϱ_i), and s_i the injections then $\{s_i: (R_i, \varrho_i) \to (S, \sigma)\}_{i \in I}$ is the coproduct of the family $\{(R_i, \varrho_i)\}_{i \in I}$ in $\mathfrak{P}$.

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P. Horák
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia