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ON ZEROS OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y'' + f(t, y)\, g(y') = 0$

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1. Consider the differential equation

$$y'' + f(t, y)\, g(y') = 0 \tag{1}$$

where $f \in C^1(D)$, $D = \{(t, y) : t \in [a, \infty), y \in R\}$, $f(t, y) = -f(t, -y)$ in D, $f(t, y)\, y > 0$ for $y \neq 0$, $g \in C_0(-\infty, \infty)$, $g(v) > 0$ for $v \in R$.

A non-trivial solution y of (1) is called oscillatory if there exists a sequence of numbers $\{t_k\}_1^\infty$ such that $a \leqq t_k < t_{k+1}$, $y(t_k) = 0$, $y(t) \neq 0$ on (t_k, t_{k+1}), $k = 1, 2, 3, \ldots$, $\lim\limits_{k \to \infty} t_k = \infty$ holds.

In all the paper we shall omit the trivial solution $y \equiv 0$ from our considerations.

Let y be an oscillatory solution of (1) and $\{t_k\}_1^\infty$ the sequence of all its zeros. Then there exists exactly one sequence of numbers $\{\tau_k\}_1^\infty$ called the sequence of extremants of y, such that $t_k < \tau_k < t_{k+1}$, $y'(\tau_k) = 0$ holds and

$$\begin{aligned} f(t, y(t))\, y'(t) &> 0 \qquad \text{on } (t_k, \tau_k), \\ f(t, y(t))\, y'(t) &< 0 \qquad \text{on } (\tau_k, t_{k+1}) \end{aligned} \tag{2}$$

(see [1], [2]).

Put $\Delta_i = t_{i+1} - t_i$, $\delta_i = \tau_i - t_i$, $\gamma_i = t_{i+1} - \tau_i$, $i = 1, 2, 3, \ldots$, $D_1 = \{(t, y) : (t, y) \in D, y \geqq 0\}$, $D_2 = \{(t, y) : (t, y) \in D, y > 0\}$. Thus $\Delta_i = \delta_i + \gamma_i$. Our aim lies in finding conditions under which the sequence $\{\Delta_i\}_1^\infty$ is monotone. This problem was studied e.g. in [3], [4]. The necessary results of [3] are stated in the following

Theorem 1. *Let y be an oscillatory solution of* (1) *and let* $\dfrac{\partial}{\partial t} | f(t, y) | \leqq 0$ $\left(\dfrac{\partial}{\partial t} | f(t, y) | \geqq 0\right)$.

(i) *If* $g(v) = g(-v)$ *for* $v \in R$, *then* $\delta_k \leqq \gamma_k$ $(\delta_k \geqq \gamma_k)$ $k = 1, 2, 3, \ldots$ *and*

$$|y(^1t)| \leqq |y(^2t)| \qquad (|y(^1t)| \geqq |y(^2t)|),$$

where $^1t \in [\tau_k, t_{k+1}]$, $^2t \in [t_{k+1}, \tau_{k+1}]$, $|y'(^1t)| = |y'(^2t)|$.

(ii) *If* $\frac{\partial}{\partial y} f(t, y) \geqq 0$ *in* D, $\frac{\partial}{\partial y} f(t, y)$ *is non-increasing with respect to* y *in* D_1, $\frac{\partial}{\partial y} f(t, y)$ *is non-increasing (non-decreasing) with respect to* t *in* D_1, *then*

$$\gamma_k \leqq \delta_{k+1} \qquad (\gamma_k \geqq \delta_{k+1}), \qquad k = 1, 2, 3, \ldots$$

If, in addition, $g(v) = g(-v)$, $v \in R$, *then*

$$\Delta_k \leqq \Delta_{k+1} \qquad (\Delta_k \geqq \Delta_{k+1}), \qquad k = 1, 2, 3, \ldots$$

Bihari [4] deals with the differential equation

$$y'' + h(t) f(y) g(y') = 0 \tag{3}$$

where $h \in C^1[a, \infty)$, $f \in C^1(R)$, $g \in C_0(R)$, $f(y)\, y > 0$ for $y \neq 0$, $h > 0$, $g > 0$, $f(y) = -f(-y)$. He proved that $\{\Delta_k\}_1^\infty$ is non-increasing under the more restrictive assumptions (as in Theorem 1) on the functions g and h and under the different assumptions on the function f.

2. Now, we prove the monotonicity of $\{\Delta_k\}_1^\infty$ under less restrictive assumptions on $\frac{\partial f}{\partial y}$ considered as the function of y.

Theorem 2. *Let* y *be an oscillatory solution of* (1) *and suppose that* $\frac{\partial f}{\partial t} \leqq 0$ *in* D_1,

$$\frac{1}{f} \frac{\partial f}{\partial y} \text{ is non-increasing with respect to } y \text{ in } D_2$$

and

$$\frac{1}{f} \frac{\partial f}{\partial y} \text{ is non-increasing with respect to } t \text{ in } D_2.$$

Then $\gamma_k \leqq \delta_{k+1}$, $k = 1, 2, 3, \ldots$
If, in addition, $g(v) = g(-v)$, $v \in R$, *then* $\Delta_k \leqq \Delta_{k+1}$, $k = 1, 2, 3, \ldots$

Proof. Denote by $^1t(y')$ $(^2t(y'))$ the inverse function to $y'(t)$, $t \in [\tau_k, t_{k+1}]$ $(t \in [t_{k+1}, \tau_{k+1}])$. These functions exist because $y''(t) = 0 \Leftrightarrow y(t) = 0 \Leftrightarrow t = t_{k+1}$ on $[\tau_k, \tau_{k+1}]$. Suppose that $y'(t) > 0$ holds on (τ_k, τ_{k+1}). If $y'(t) < 0$, the statement can be proved similarly. Then $y(t) < 0$, $f < 0$, $y''(t) > 0$ on $[\tau_k, t_{k+1})$, $y(t) > 0$, $f > 0$, $y''(t) < 0$ on $(t_{k+1}, \tau_{k+1}]$ (use (2)). By use of f being odd with respect to y the following estimation holds for $y' \in J = [0, y'(t_{k+1})]$

$$\frac{d}{dy'}\left\{\frac{|y''(^1t)|}{g(y')}-\frac{|y''(^2t)|}{g(y')}\right\}=$$

$$=\frac{d}{dy'}\{|f(^1t, y(^1t))|-f(^2t, y(^2t))\}=$$

$$=\frac{\partial}{\partial t}|f(^1t, y(^1t))|\cdot\frac{1}{y''(^1t)}+\frac{y'}{y''(^1t)}\frac{\partial}{\partial y}|f(^1t, y(^1t))|-$$

$$-\frac{1}{y''(^2t)}\frac{\partial}{\partial t}|f(^2t, y(^2t))|-\frac{y'}{y''(^2t)}\frac{\partial}{\partial y}f(^2t, y(^2t))\leqq$$

$$\leqq y'\left\{-\frac{\frac{\partial}{\partial y}f(^1t, |y(^1t)|)}{y''(^1t)}-\frac{\frac{\partial}{\partial y}f(^2t, y(^2t))}{y''(^2t)}\right\}=$$

$$=\frac{y'}{g(y')}\left\{-\frac{\frac{\partial}{\partial y}f(^1t, |y(^1t)|)}{f(^1t, |y(^1t)|)}+\frac{\frac{\partial}{\partial y}f(^2t, y(^2t))}{f(^2t, y'(^2t))}\right\}.$$

As $|y(^1t)|\leqq y(^2t)$ holds according to Theorem 1, we can see that $\frac{\partial}{\partial y'}G(y')\leqq 0$ on J and $G(y'(t_{k+1}))=0$ where $G(y')=\frac{|y''(^1t(y'))|}{g(y')}-\frac{|y''(^2t(y'))|}{g(y')}$. From this $G(y')\geqq 0$, $y'\in J$ and

$$y''(^1t)\geqq|y''(^2t)|,\qquad y'\in J. \tag{4}$$

Consider two functions $z_1(y')=t_{k+1}-{}^1t(y')$, $z_2(y')={}^2t(y')-t_{k+1}$, $y'\in J$. According to (4)

$$\frac{d}{dy'}[z_1-z_2]=-\frac{1}{y''(^1t)}-\frac{1}{y''(^2t)}\geqq 0,\qquad y'\in[0, y'(t_{k+1})).$$

Thus z_1-z_2 is non-decreasing and with respect to $z_1(y')=z_2(y')=0$ for $y'=y'(t_{k+1})$ we can conclude that $z_1\leqq z_2$ and the first part of the statement $\gamma_k\leqq\delta_{k+1}$ is proved. The rest follows from this and from Theorem 1

$$\Delta_k=\gamma_k+\delta_k\leqq\delta_{k+1}+\gamma_k\leqq\gamma_{k+1}+\delta_{k+1}=\Delta_{k+1}.$$

The theorem is proved.

The following theorem can be proved similarly to Theorem 2.

Theorem 3. *Let y be an oscillatory solution of* (1) *and suppose that* $\frac{\partial f}{\partial t}\geqq 0$ *in* D_1, $\frac{1}{f}\frac{\partial f}{\partial y}$ *is non-increasing with respect to* y *in* D_2

and

$$\frac{1}{f}\frac{\partial f}{\partial y} \text{ is non-decreasing with respect to } t \text{ in } D_2 \cdot$$

Then $\gamma_k \geqq \delta_{k+1}$, $k = 1, 2, \ldots$
If, in addition, $g(v) = g(-v)$, $v \in R$, *then* $\Delta_k \geqq \Delta_{k+1}$, $k = 1, 2, 3, \ldots$

Corollary. *Let* y *be an oscillatory solution of* (3) *and let* $h'(t) \leqq 0$ $(h'(t) \geqq 0)$ *for* $t \in [a, \infty)$.

(i) *If* $g(v) = g(-v)$, $v \in R$, *then* $\delta_k \leqq \gamma_k$ $(\delta_k \geqq \gamma_k)$, $k = 1, 2, \ldots$

(ii) *If* $f'(y) > 0$, $y \in R$ *and* $\dfrac{f'(y)}{f(y)}$ *is non-increasing for* $y > 0$, *then* $\gamma_k \leqq \delta_{k+1}$ $(\gamma_k \geqq \delta_{k+1})$, $k = 1, 2, 3, \ldots$ *If, in addition,* $g(v) = g(-v)$, $v \in R$, *then* $\Delta_k \leqq \Delta_{k+1}$ $(\Delta_k \geqq \Delta_{k+1})$, $k = 1, 2, 3, \ldots$

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