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# ON AN OSCILLATION CRITERION OF HARTMAN, WINTNER AND POTTER

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## SUMMARY

In §§ 1 and 2 we generalize an oscillation criterion of Hartman, Wintner and Potter for the linear second order ordinary differential equation and give some related results. In § 3 we compare it with a general form of the Leighton – Wintner criterion. We assume that the coefficients of the differential equation are integrable resp. absolutely continuous, since the proofs do not require continuity or differentiability as is usually assumed in theorems of this kind. Some of the expressions built from the coefficients are known in affine differential geometry and thus are invariant against change of the independent variable. Therefore the validity of Potter's criterion for finite intervals follows immediately. In § 4 we give a survey of the relevant facts of geometry. There we also explain the duality which is met for instance in Potter's paper. Many of the references can be found in Swanson's book [18].

## § 1. GENERALIZATION OF A CRITERION OF HARTMAN, WINTNER AND POTTER

We consider ordinary linear differential equations of the 2nd order. Such an equation is said to be oscillatory in a interval $[\alpha, \beta)$, $\beta < \infty$ or $\beta = \infty$, if some and hence every solution oscillates, i.e. has a sequence of zeros $t_n \to \beta$. By multiplication with a suitable factor the equation receives the self-adjoint form

$$(p(t)\,\dot{x})^{\cdot} + q(t)\,x = 0. \tag{1.1}$$

We assume that $p > 0$, $q > 0$ are absolutely continuous in $[\alpha, \beta)$. Setting $y = p\dot{x}$ the equation can be transformed into the system

$$\dot{x} = p^{-1}y$$
$$\dot{y} = -qx.$$

Obviously $x$ oscillates iff $y$ oscillates. The system has continuous solutions which are obviously $C^1$-solutions. Therefore (1.1) as well as

$$(1.2) \qquad (q^{-1}\dot{y})^{\cdot} + p^{-1}y = 0$$

are fulfilled everywhere. There are two distinguished transformations: firstly,

$$(1.3) \qquad s(t) = \int_\alpha^t p(\tau)^{-1}\,\mathrm{d}\tau, \qquad x'(s) = \mathrm{d}x(t(s))/\mathrm{d}s.$$

This gives

$$x' = y$$
$$y' = -pqx$$

and

$$(1.4) \qquad x'' + pqx = 0, \qquad ((pq)^{-1}y')' + y = 0.$$

Secondly,

$$(1.5) \qquad s(t) = \int_\alpha^t q(\tau)\,\mathrm{d}\tau, \qquad x'(s) = \mathrm{d}x(t(s))/\mathrm{d}s,$$

which gives

$$x' = (pq)^{-1}y$$
$$y' = -x$$

and

$$(1.6) \qquad (pqx')' + x = 0, \qquad y'' + (pq)^{-1}y = 0.$$

Here $p(t(s))\,q(t(s))$ is absolutely continuous for both transformations, since $t(s)$ is absolutely continuous and monotone (cf. e.g. Caratheodory [2], § 495).

The following theorem generalizes a criterion which was given in various forms by Hartman–Wintner [7], Potter [16], and Breuer–Gottlieb [1]. We define

$$\overline{\lim_{t\to\beta}} f(t) = \lim_{\tau\to\beta} \operatorname{ess\,sup}_{t\geq\tau} f(t)$$

and likewise $\underline{\lim}$. The following integrals are to be understood as improper integrals with respect to the upper limit.

**Theorem 1:** *Let $p, q > 0$ be absolutely continuous in $[\alpha, \beta)$, and set*

$$L = \overline{\lim_{t\to\beta}}\, p((pq)^{-1/2})^{\cdot}, \qquad l = \underline{\lim_{t\to\beta}}\, p((pq)^{-1/2})^{\cdot}.$$

(1.7) *If* $\int_\gamma^\beta p(t)^{-1}\,\mathrm{d}t = \infty$ *and* $L < 2$, *then* (1.1) *oscillates on* $[\alpha, \beta)$.

(1.8) *If* $l > 2$, *then* (1.1) *is nonoscillatory in* $[\alpha, \beta)$.

(1.9) *If* $\int_\gamma^\beta q(t)\,\mathrm{d}t = \infty$ *and* $l > -2$, *then* (1.1) *oscillates on* $[\alpha, \beta)$.

(1.10) *If* $L < -2$, *then* (1.1) *is nonoscillatory in* $[\alpha, \beta)$.

**Remark 1:** The duality between (1.7, 8) and (1.9, 10) is explained in § 4.

**Remark 2:** If $\int_\alpha^\beta p(t)^{-1}\,\mathrm{d}t = \infty$ then $L \geqq 0$, and if $\int_\alpha^\beta q(t)\,\mathrm{d}t = \infty$ then $l \leqq 0$. These assumptions can therefore be dropped in corollaries 3 and 4 of [1], compare [16], p. 474. A proof is given after the proof of theorem 1. In order to compare these corollaries with theorem 1 note

$$p((pq)^{-1/2})^{\cdot} = -q^{-1}((pq)^{1/2})^{\cdot} \tag{1.11}$$

which follows from $(pq)^{1/2}(pq)^{-1/2} = 1$ by differentiation.

Proof of theorem 1: We first prove (1.7, 8) for $p \equiv 1$: Let

$$x'' + cx = 0, \tag{1.12}$$

$c > 0$ absolutely continuous, be given. We have to show

(1.13) If $\overline{\lim_{s\to\infty}}\,(c^{-1/2})' < 2$, then (1.12) oscillates on $[\alpha, \infty)$.

(1.14) If $\underline{\lim_{s\to\beta}}\,(c^{-1/2})' > 2$, then (1.12) is non-oscillatory on $[\alpha, \beta)$.

To show (1.13) we choose $\Lambda < 2$ such that $(c^{-1/2})' < \Lambda$ for $s > s_0(\Lambda)$ a.e. (almost everywhere). Integration gives

$$0 < c(s)^{-1/2} < \Lambda(s - s_0) + c(s_0)^{-1/2},$$
$$c(s) > \Lambda^{-2}(s + k)^{-2}.$$

Similarly, to show (1.14) we choose $\lambda > 2$ such that $(c^{-1/2})' > \lambda$ for $s > s_1(\lambda)$ a.e. Integration gives

$$c(s) < \lambda^{-2}(s - s_1)^{-2}.$$

If we now compare (1.12) with the Euler equation

$$x'' + \gamma(s - \delta)^{-2}x = 0,$$

we get (1.13, 14). From this we get (1.7, 8) by the transformation (1.3). Notice that if

$$\int_\alpha^\beta p(t)^{-1}\,\mathrm{d}t = \infty$$

then $t \to \beta$ corresponds to $s \to \infty$. By transformation (1.5) we get (1.6) for $y = p\dot{x}$. Since $x$ oscillates iff $y$ oscillates we get (1.9, 10) from (1.13, 14) and (1.11). It remains to show the assertion of remark 2. If $c(s)\,(> 0)$ is absolutely continuous and $K =$

$= \overline{\lim\limits_{s\to\infty}} (c^{-1/2})'$, then

$$(c^{-1/2})' < K + \varepsilon \qquad \text{for } s > s_0(\varepsilon) \text{ a.e.}$$

By integration

$$0 < c^{-1/2} < (K + \varepsilon)(s - s_0) + c(s_0)^{-1/2},$$

and therefore $K \geqq 0$. Transformations (1.3, 5) give $L \geqq 0$ resp. $l \leqq 0$.

## § 2. SOME RELATED RESULTS

We consider (1.1) under the same assumptions as in § 1.

**Corollary 1:** Assume $L = l = \lim\limits_{t\to\beta} p((pq)^{-1/2})^{\cdot}$.

(2.1) If $|L| < 2$ and $\int\limits_\alpha^\beta (q/p)^{1/2}\,dt = \infty$, then (1.1) oscillates in $[\alpha, \beta)$.

(2.2) If $|L| > 2$ then (1.1) is non-oscillatory in $[\alpha, \beta)$.

(For an extension to the most general linear second order equation see the end of this paper.)

Proof: This follows immediately from theorem 1 and

$$\left(\int\limits_\alpha^\beta (q/p)^{1/2}\,dt\right)^2 \leqq \int\limits_\alpha^\beta p^{-1}\,dt \int\limits_\alpha^\beta q\,dt.$$

**Remark:** The following lemma shows that corollary 1 is not more special than theorem 1 provided $L$ exists. We shall use lemma 1 also for the proof of theorem 2.

**Lemma 1:** *Let $p, q > 0$ be absolutely continuous in $[\alpha, \beta)$. If $\int\limits_\alpha^\beta p^{-1}\,dt = \infty$ and $p((pq)^{-1/2})^{\cdot}$ is essentially bounded from above, or if $\int\limits_\alpha^\beta q\,dt = \infty$ and $p((pq)^{-1/2})^{\cdot}$ is essentially bounded from below, then*

$$\int\limits_\alpha^\beta (q/p)^{1/2}\,dt = \infty.$$

Proof: By transformation (1.3) we have a.e. in $[\alpha, \beta)$ resp. $[0, \infty)$

$$p((pq)^{-1/2})^{\cdot} = ((pq)^{-1/2})' \leqq M,$$
$$0 < (pq)^{-1/2} \leqq Ms + N,$$
$$\int\limits_\alpha^\beta (q/p)^{1/2}\,dt = \int\limits_0^\infty (pq)^{1/2}\,ds \geqq \int\limits_0^\infty (Ms + N)^{-1}\,ds = \infty.$$

Similarly, by transformation (1.5) we have a.e. in $[\alpha, \beta)$ resp. $[0, \infty)$

$$-p((pq)^{-1/2})^{\cdot} = q^{-1}((pq)^{1/2})^{\cdot} = ((pq)^{1/2})' \leqq M,$$

where we have made use of (1.11). The rest follows as above.

**Remark:** With lemma 1 we see that the integral condition of Potter's theorem 1.3 can be omitted, [16] or [18], p. 79. Appearently Ráb [17] noticed this already.

**Corollary** 1 has a simple form for equation

(2.3) $$\ddot{x} + b\dot{x} + x = 0, \qquad b \text{ locally integrable.}$$

**Corollary 2:** Suppose lim $b$ exists ($\underline{\lim}\, b = \overline{\lim}\, b$).
If $|\lim_{t \to \infty} b| < 2$ then (2.3) oscillates in $[\alpha, \infty)$.
If $|\lim_{t \to \beta} b| > 2$ then (2.3) is non-oscillatory in $[\alpha, \beta)$.

Proof: We multiply (2.3) by a suitable factor such that we receive the self-adjoint form

$$(a\dot{x})^{\cdot} + ax = 0.$$

Obviously $\dot{a} = ab$ and $a$ is absolutely continuous. The integral condition in (2.1) is fulfilled for an infinite interval. Corollary 1 gives the two assertions.

As is well known $\int_\alpha^\beta (q/p)^{1/2}\, dt = \infty$ is not sufficient for oscillation; consider Euler's equation

$$\ddot{x} + (2t)^{-2}x = 0.$$

It is also not necessary. For there is an unbounded function $q$ with

$$\int_1^\infty q\, dt = \infty, \qquad \int_1^\infty q^{1/2}\, dt < \infty.$$

The equation $\ddot{x} + qx = 0$ is then oscillating in $[1, \infty)$ according to the criterion of Leighton and Wintner (see § 3).

A necessary condition, i.e. a non-oscillation criterion can be derived from a theorem of Moore [12], [18] p. 73, which reads under our assumptions $p, q > 0$ as follows: If $\int_\alpha^\beta p^{-1}\, dt < \infty$ and $\int_\alpha^\beta q\, dt < \infty$, then (1.1) is non-oscillatory in $[\alpha, \beta)$. Moore establishes this for $[\alpha, \infty)$. But the validity for $[\alpha, \beta)$ follows from the invariance of the conditions under changes of the independent variable. See the explanations in § 4.

**Theorem 2:** *Let $p, q > 0$ be absolutely continuous in $[\alpha, \beta)$ and* $\int_\alpha^\beta (q/p)^{1/2}\, dt < \infty$.

(a) *If* $p((pq)^{-1/2})^{\cdot} \geqq 0$ *or* $p((pq)^{-1/2})^{\cdot} \leqq 0$ *a.e. or* (b) *if* $|p((pq)^{-1/2})^{\cdot}|$ *is essentially bounded, then* (1.1) *is non-oscillatory in* $[\alpha, \beta)$.

**Proof:** (a) The hypothesis implies that $pq$ is a monotone function. A theorem of Leighton [11], [18] p. 71, gives nonoscillation on $[\alpha, \infty)$, and since $p((pq)^{-1/2})^{\cdot}$ is invariant under changes of the independent variable (see § 4), this is true also on finite intervals.

(b) From Lemma 1 we have

$$\int_{\alpha}^{\beta} p^{-1}\, dt < \infty \qquad \text{and} \qquad \int_{\alpha}^{\beta} q\, dt < \infty.$$

Moore's theorem cited above gives the assertion.

## § 3. COMPARISON WITH OTHER CRITERIA

If $b$ is absolutely continuous, corollary 2 can be derived from a generalization of the well-known criterion of Leighton and Wintner.

**Criterion of Leighton [10], [11] and Wintner [20]:** Let $p$ be continuous and $> 0$, and $q$ be locally integrable. If (as improper integrals)

$$\int_{\alpha}^{\beta} p^{-1}\, dt = \infty \qquad \text{and} \qquad \int_{\alpha}^{\beta} q\, dt = \infty,$$

then (1.1) is oscillatory in $[\alpha, \beta)$.

Wintner considered the special case $p = 1$ and $\beta = \infty$.

His proof works under our assumptions and transformation (1.3) gives the criterion as stated. For $p = 1$ and $q \geqq 0$ the criterion was given by Fite [4] and later also by Gagliardo [5]. Pfaff [15] gives a similar criterion for $p \equiv 1$ and a distribution $q$ which is the derivative of a locally square integrable function. For literature on extension to systems see Etgen and Pawlowski [3].

Kreith [8] gives the corresponding criterion for the general second order equation. The following corollary is a corrected version of his theorem 2.3, p. 14.

**Corollary 3:** Assume that $p_1$ is continuous and $> 0$, $q_0$ is absolutely continuous and $p_0$ is integrable. Then

$$-(p_1 \dot{x})^{\cdot} + q_0 \dot{x} + p_0 x = 0$$

is oscillatory on $[\alpha, \beta)$ if

$$\int_{\alpha}^{\beta} p_1^{-1}\, dt = \infty$$

and

$$\lim_{t \to \beta} \left\{ \frac{q_0(t)}{2} - \int_{\alpha}^{t} \left( p_0 + \frac{q_0^2}{4p_1} \right) d\tau \right\} = \infty.$$

Kreith proves this (for $\beta = \infty$) using a Riccati equation. We derive it here from the Leighton–Wintner criterion: By the usual transformation

$$x(t) = u(t) \exp\left( -\int^t \frac{q_0}{2p_1} \, d\tau \right)$$

we get

$$-(p_1 \dot{u})^{\cdot} + \left( p_0 - \frac{\dot{q}_0}{2} + \frac{q_0^2}{4p_1} \right) u = 0.$$

(One has to be careful not to use more than is required of the coefficient functions.) The assertion then follows from the Leighton–Wintner criterion.

The following nice formulation is a trivial consequence.

**Corollary 4:** Let $A$, $B$ be absolutely continuous, $A > 0$, and $C$ be integrable. If $B/A$ is bounded from above and

$$\int_\alpha^\infty (AC - B^2)/A^2 \, dt = \infty,$$

then

$$A\ddot{x} + 2B\dot{x} + Cx = 0$$

is oscillatory on $[\alpha, \infty)$.

The following simple example shows that corollary 4, though it is only a special case, may have an advantage.

Consider

$$(t^2 \dot{x})^{\cdot} + t^2 x = 0 \qquad \text{in } [1, \infty).$$

Since $\int_1^\infty t^{-2} \, dt < \infty$, Leighton's criterion gives no decision, and the same is true for Opial's extension [14] of Leighton's criterion.

On the other hand, $B/A = t/t^2 = 1/t$ is bounded on $[1, \infty)$, and

$$\int_1^\infty (AC - B^2)/A^2 \, dt = \int_1^\infty (t^4 - t^2)/t^4 \, dt = \int_1^\infty (1 - 1/t^2) \, dt = \infty$$

so that we have oscillation. The same result is given by theorem 1 since

$$\int_1^\infty q \, dt = \int_1^\infty t^2 \, dt = \infty \qquad \text{and} \qquad p((pq)^{-1/2})^{\cdot} = t^2 (t^{-2})^{\cdot} = -2/t \to 0.$$

**Corollary 5:** Let $b$ be absolutely continuous and bounded from above on $[\alpha, \infty)$. If

$$\int_\alpha^\infty (1 - b^2/4) \, dt = \infty,$$

then

(3.1) $$\ddot{x} + b\dot{x} + x = 0$$

is oscillatory on $[\alpha, \infty)$.

This is merely the special case $A = C = 1$ of corollary 4. We state it explicitely because of its relationship to § 1. The question arises whether here or generally in corollary 4 the boundedness condition can be omitted.

If $\lim_{t\to\infty} b$ exists and $|\lim_{t\to\infty} b| < 2$, then

$$\int_{\alpha}^{\infty} (1 - b^2/4)\,ds = \infty$$

so that (3.1) oscillates on $[0, \infty)$. This gives the oscillation part of corollary 2. On the other hand, if $b = 2(1 - 1/t)^{1/2}$, then $\lim_{t\to\infty} b = 2$ and corollary 2 does not give a decision. But

$$\int_{1}^{\infty} (1 - b^2/4)\,dt = \int_{1}^{\infty} \frac{dt}{t} = \infty,$$

and corollary 5 shows that

$$\ddot{x} + 2(1 - 1/t)^{1/2}\,\dot{x} + x = 0$$

oscillates on $[1, \infty)$.

Corollary 5 even implies the oscillation part of theorem 1, provided that $L = l$ and $p$ and $q$ are, for instance, $C^2$-functions. For the transformation

(3.2) $$s(t) = \int_{\alpha}^{t} (q(\tau)/p(\tau))^{1/2}\,d\tau$$

gives (1.1) the form

(3.3) $$x'' + b(s)\,x' + x = 0 \qquad (' = d/ds),$$

where

$$b(s(t)) = -p((pq)^{-1/2})^{\cdot}$$

is $C^1$. Thus, under these assumptions the oscillation part of corollary 2 is equivalent to that of corollary 1 and, in view of Lemma 1, to that of theorem 1.

## § 4. RELATIONS TO GEOMETRY

In the preceeding paragraphs we made occasionally use of the following reasoning: If the differential equation and the conditions occurring in a criterion are invariant against transformation of the independent variable, then the criterion is not only valid for an infinite interval but also for a finite interval. Appearently the validity of Potter's criterion for finite intervals has escaped notice, maybe because the invariance of $p((pq)^{-1/2})^{\cdot}$ is not evident whereas the invariance of the conditions in Leighton's criterion is pretty obvious. Wintner's criterion and its generalization (corollary 4) are not valid for finite intervals because the form of the differential equation respectively the integral condition is not invariant. We now give some known facts from affine differential geometry which interpret differential equations and conditions and explain the duality met in theorem 1 and elsewhere in the literature. Confer also [6], [9], and [13], e.g.

We consider equations (1.1) and (1.2). Two independent solutions of a differential equation are considered as coordinates of a plane curve. The introduction of a parameter invariant under an affine group amounts to a specific change of the independent variable.

First we choose $SL_2$, the group of unimodular linear mappings. Transformation

(1.3) $$s(t) = \int_{\alpha}^{t} p(\tau)^{-1}\,d\tau$$

yields a parameter which is invariant with respect to $SL_2$ and can be interpreted as twice the area between curve and origin. The differential equation with respect to this parameter is

(1.4) $$x'' + pqx = 0.$$

Since $y = p\dot{x} = x'$ we can find a geometric description of the relation between $x$ and $y$: A polarity with respect to the unit circle followed by a rotation of a right angle transforms $x$ into $y$. Thus, the duality can be interpreted as that of projective geometry. Transformation

(1.5) $$s^*(t) = \int_{\alpha}^{t} q(\tau)\,d\tau$$

yields also a parameter invariant with respect to $SL_2$. The differential equation now becomes

(1.6) $$(pqx')' + x = 0, \qquad y'' + (pq)^{-1} y = 0 \ (' = d/ds^*).$$

The last equation shows that $s^*$ has the same geometric meaning for $y$ as $s$ has for $x$. Since the equations are determined up to a constant factor, the same is true for $s$, $s^*$. Geometrically this corresponds to the choice of unit area. But conditions like $\int q\,dt = \infty$ are not affected by this ambiguity.

Now we choose $GL_2$, the group of non-degenerate linear mappings. The invariant parameter is given by (3.2) and the differential equation takes the form

(3.3) $$x'' + bx' + x = 0 \qquad \text{or} \qquad (ax')' + ax = 0,$$

where $a = \sqrt{pq}$, $b = a'/a = -p((pq)^{-1/2})^{\cdot}$. This parameter is also invariant parameter of $y$ and one has

$$y'' - by' + y = 0 \qquad \text{or} \qquad (a^{-1}y')' + a^{-1}y = 0.$$

This illuminates theorem 1.

If the differential equation is given in its most general form

$$A\ddot{x} + B\dot{x} + Cx = 0, \tag{4.1}$$

$b$ and the equation for $y$ can be written in a form which is simpler than is to be expected:

$$s = \int^{t} (C/A)^{1/2}\, d\tau, \qquad b = -A((AC)^{-1/2})^{\cdot} + (B - A^{\cdot})/(AC)^{1/2}, \tag{4.2}$$

$$(C^{-1}\dot{y})^{\cdot} + ((A^{\cdot} - B)/AC)\,\dot{y} + A^{-1}y = 0,$$

or

$$A\ddot{y} + (A^{\cdot} - B - AC^{\cdot}/C)\,\dot{y} + Cy = 0.$$

(4.2) may be used to extend corollary 1 to equation (4.1).

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