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A NOTE ON THE CONVERGENCE OF A PAIR OF SEQUENCES OF MAPPINGS

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The aim of this note is to investigate conditions under which the convergence of a pair of sequences of mappings to two mappings S and T of a metric space into itself implies the convergence of the ircommon fixed points to the common fixed point of S and T.

In his recent paper, G. Jungck [4] introduced the relation

$$d(Sx, Sy) \leqq k\, d(Tx, Ty), \quad k \in (0, 1)$$

for a pair of mappings (S, T) from a metric space (X, d) into itself and for every $x, y \in X$. Mappings satisfying such a relation will be called 'Jungck mappings' and k as 'Jungck constant'. If $S(X) = T(X)$ then commuting continuous Jungck mappings (S, T) have a unique common fixed point [4].

Theorem 1. Let S_n and T_n be Jungck mappings of a metric space (X, d) into itself with Jungck constant k and with at least one common fixed point u_n for each $n = 1, 2, \ldots$. If the sequences $\{S_n\}$ and $\{T_n\}$ converge respectively pointwise to $S, T : X \to X$ with common fixed point u, then u is the unique common fixed point of S and T, and the sequence $\{u_n\}$ converges to u.

We remark that the restriction that every pair of Jungck mappings (S_n, T_n) has the same Jungck constant k is strong. We relax the restriction in the following.

Theorem 2. Let (X, d) be a metric space, and let S_n and T_n be Jungck mappings of X into itself with Jungck constant k_n and with at least one common fixed point u_n for each $n = 1, 2, \ldots$. Furthermore, if $k_{n+1} \leqq k_n$ for $n = 1, 2, \ldots$, and the sequences $\{S_n\}$ and $\{T_n\}$ converge respectively pointwise to $S, T : X \to X$ with common fixed point u, then u is the unique common fixed point of S and T and the sequence $\{u_n\}$ converges to u.

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Remark 1. If the Jungck constants are such that $k_{n+1} \geqq k_n$ for each n, the theorem is, in general, false. The following example illustrates this remark.

Example 1. Let S_n, $T_n : E^1 \to E^1$ be defined as

$$S_n x = \frac{3n}{n+1} p + \frac{n^2 - n + 1}{n^2 + 2n + 1} x \quad \text{and} \quad T_n x = p + \frac{n}{n+1} x$$

for all $x \in E^1 = (-\infty, +\infty)$, $n = 1, 2, \ldots$, and $p > 0$.

We see that (S_n, T_n) are Jungck mappings with Jungck constants $k_n = (n^2 - n + 1)/(n^2 + 2n + 1)$ and with common fixed points $u_n = (n + 1) p$. Also $k_{n+1} \geqq k_n$, $Sx = \lim_{n\to\infty} S_n x = 3p + x$ and $Tx = \lim_{n\to\infty} T_n x = p + x$ for every $x \in E^1$. Since S and T are translation maps, neither possesses a fixed point. Moreover,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (n + 1) p = \infty \notin E^1.$$

Theorem 3. Let (X, d) be a metric space, and let S_n and T_n be mappings of X into itself with at least one common fixed point u_n for each $n = 1, 2, \ldots$. Suppose there are nonnegative real numbers a, b, c, e and f $(c + e + f \neq 1)$ such that

$$\text{(3.1)} \qquad \mathrm{d}(S_n x, S_n y) \leqq a\, \mathrm{d}(S_n x, T_n x) + b\, \mathrm{d}(S_n y, T_n y) + c\, \mathrm{d}(S_n x, T_n y) + e\, \mathrm{d}(S_n y, T_n x) + f\, \mathrm{d}(T_n x, T_n y)$$

for all $x, y \in X$ and $n = 1, 2, \ldots$ If the sequences $\{S_n\}$ and $\{T_n\}$ converge respectively pointwise to $S, T : X \to X$ with common fixed point u, then u is the unique common fixed point of S and T, and the sequence $\{u_n\}$ converges to u.

We remark that if S_n and T_n are commuting continuous mappings and satisfy (3.1) with

$$\text{(3.2)} \qquad 0 < a + b + c + e + f < 1$$

then they have a unique common fixed point in X (see [6]). But in the above theorem, continuity, commutativity for S_n, T_n, S and T and the condition (3.2) are not essential. It is simply required that (S_n, T_n) should have a common fixed point. It may be mentioned that the limiting mappings S and T may commute even if S_n and T_n are not commutative (see Example 2 below).

Proof of Theorem 1 follows from Theorem 3 by setting $a = b = c = e = 0$ and $f = k$ in (3.1). Theorem 2 follows from Theorem 1 by noticing that the Jungck constants $k_{n+1} \leqq k_n$ $n = 1, 2, \ldots$, and $k_1 = k$ will serve the purpose of Jungck constant for every pair of Jungck mappings (S_n, T_n).

Proof of Theorem 3. Sequences $\{S_n\}$ and $\{T_n\}$ converge respectively pointwise to S and T. Therefore for $\varepsilon > 0$ and $u \in X$, there is a positive integer N such that $n \geqq N$ implies

$$\text{(3.3)} \quad \mathrm{d}(S_n u, Su) < \frac{1 - c - e - f}{2(1 + b + e)} \varepsilon \quad \text{and} \quad \mathrm{d}(T_n u, Tu) < \frac{1 - c - e - f}{2(b + c + f)} \varepsilon.$$

Now for all $n \geqq N$,

$$\begin{aligned} d(u_n, u) &= \mathrm{d}(S_n u_n, Su) \leqq \\ &\leqq \mathrm{d}(S_n u_n, S_n u) + \mathrm{d}(S_n u, Su) \leqq \\ &\leqq a\,\mathrm{d}(S_n u_n, T_n u_n) + b\,\mathrm{d}(S_n u, T_n u) + c\,\mathrm{d}(S_n u_n, T_n u) + \\ &+ e\,\mathrm{d}(S_n u, T_n u_n) + f\,\mathrm{d}(T_n u_n, T_n u) + \mathrm{d}(S_n u, Su) \leqq \\ &\leqq b\big(\mathrm{d}(S_n u, Su) + \mathrm{d}(Tu, T_n u)\big) + c\big(\mathrm{d}(u_n, u) + \mathrm{d}(Tu, T_n u)\big) + \\ &\quad + e\big(\mathrm{d}(S_n u, Su) + \mathrm{d}(u, u_n)\big) + f\big(\mathrm{d}(u_n, u) + \mathrm{d}(Tu, T_n u)\big) + \\ &\quad + \mathrm{d}(S_n u, Su), \text{ (since } S_n u_n = u_n = T_n u_n \text{ and } Su = u = Tu) \\ &= (1 + b + e)\,\mathrm{d}(S_n u, Su) + (b + c + f)\,\mathrm{d}(T_n u, Tu) + \\ &\quad + (c + e + f)\,\mathrm{d}(u_n, u) \end{aligned}$$

which gives

$$\mathrm{d}(u_n, u) \leqq \frac{1 + b + e}{1 - c - e - f}\,\mathrm{d}(S_n u, Su) + \frac{b + c + f}{1 - c - e - f}\,\mathrm{d}(T_n u, Tu).$$

Therefore, in view of (3.3), for $n \geqq N$,

$$d(u_n, u) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $\{u_n\}$ converges to u.

To show the uniqueness of u, let v be another common fixed point of S and T. Then in a way similar to the above, $\{u_n\}$ converges to v which implies $u = v$.

Remark 2. Let T_n be identity mappings. Then:

(i) Theorem 6.11 of Singh [7] is obtained.

(ii) If $a = b$ and $c = e = f = 0$, we obtain a result due to Dube and Singh [2].

(iii) If $a = b = f = 0$, we get a result due to Collins [1].

(iv) If $c = e = 0$, we get a result due to Reich [5].

Results (ii) – (iv) have been quoted from Singh [7].

Example 2. Let $S_n, T_n : [0, 2] \to [0, 2]$ with usual metric be defined as

$$S_n x = 1 + \frac{x}{2(n + 1)} \qquad \text{and} \qquad T_n x = \frac{n}{n + 1}x + \frac{2}{2n + 1}$$

for every $x \in [0, 2]$ and $n = 1, 2, \ldots$

The common fixed point u_n of S_n and T_n is given by

$$u_n = (2n + 2)/(2n + 1) \qquad \text{for each } n = 1, 2, \ldots$$

Also $Sx = \lim_{n\to\infty} S_n x = 1$ and $Tx = \lim_{n\to\infty} T_n x = x$ for all $x \in [0, 2]$, and thus $u = \lim_{n\to\infty} u_n = 1$ is the unique common fixed point of S and T.

It is easily seen that S_n and T_n satisfy the condition (3.1) with the proper choice of constants, inparticular with $a = b = c = e = 0$ and $f = 1/2$ for all points in [0, 2]. This shows that Theorem 1 is applicable with-Jungck constant $k = 1/2$. We note that Theorem 2 may be applied by taking $k_n = 1/2n$.

Theorem 4. Let S_n and T_n be mappings from a metric space (X, d) into itself with at least one common fixed point u_n for each $n = 1, 2, \ldots$ Let $S, T : X \to X$ be mappings with common fixed point u such-that

$$(4.1) \qquad \mathrm{d}(Sx, Sy) \leqq a\,\mathrm{d}(Sx, Tx) + b\,\mathrm{d}(Sy, Ty) + \\ + c\,\mathrm{d}(Sx, Ty) + e\,\mathrm{d}(Sy, Tx) + f\,\mathrm{d}(Tx, Ty) \qquad \text{for all } x, y \in X,$$

where a, b, c, e and f are nonnegative real numbers such that $c + e + f \neq 1$. If the sequences $\{S_n\}$ and $\{T_n\}$ converge uniformly to S and T respectively, then the sequence $\{u_n\}$ converges to u uniquely.

Proof. Since $\{S_n\}$ and $\{T_n\}$ converge uniformly to S and T respectively, given $\varepsilon > 0$ there is a positive integer N such that $n \geqq N$ implies

(4.2)

$$\mathrm{d}(S_n u_n, Su_n) < \frac{1 - c - e - f}{2(1 + a + c)}\,\varepsilon \qquad \text{and} \qquad \mathrm{d}(T_n u_n, Tu_n) < \frac{1 - c - e - f}{2(a + e + f)}\,\varepsilon.$$

We have for any n,

$$\begin{aligned} \mathrm{d}(u_n, u) &= \mathrm{d}(S_n u_n, Su) \leqq \\ &\leqq \mathrm{d}(S_n u_n, Su_n) + \mathrm{d}(Su_n, Su) \leqq \\ &\leqq \mathrm{d}(S_n u_n, Su_n) + a\,\mathrm{d}(Su_n, Tu_n) + b\,\mathrm{d}(Su, Tu) + c\,\mathrm{d}(Su_n, Tu) + \\ &\quad + e\,\mathrm{d}(Su, Tu_n) + f\,\mathrm{d}(Tu_n, Tu) \leqq \\ &\leqq \mathrm{d}(S_n u_n, Su_n) + a(\mathrm{d}(Su_n, S_n u_n) + \mathrm{d}(T_n u_n, Tu_n)) + \\ &\quad + c(\mathrm{d}(Su_n, S_n u) + \mathrm{d}(u_n, u)) + e(\mathrm{d}(u, u_n) + \mathrm{d}(T_n u_n, Tu_n)) + \\ &\quad + f(\mathrm{d}(Tu_n, T_n u_n) + \mathrm{d}(u_n, u)) \\ &\quad (\text{since } Su = u = Tu \text{ and } S_n u_n = u_n = T_n u_n). \end{aligned}$$

This gives

$$\mathrm{d}(u_n, u) \leqq \frac{1 + a + c}{1 - c - e - f}\,\mathrm{d}(S_n u_n, Su_n) + \frac{a + e + f}{1 - c - e - f}\,\mathrm{d}(T_n u_n, Tu_n).$$

Therefore, in view of (4.2), for $n \geqq N$,

$$d(u_n, u) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $\{u_n\}$ converges to u. Proof of uniqueness of u follows easily.

Remark 3. Let T_n and T be identity mappings. Then:

(i) Theorem 6.12 of Singh [7] is obtained.

(ii) If $a = b$, $c = e$ and (3.2) holds, we obtain Theorem 2 of Iséki [3].

(iii) If $a = b$ and $c = e = f = 0$, we obtain a theorem due to Dube and Singh [2] (quoted from Singh [7]).

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