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Archivum Mathematicum, Vol. 14 (1978), No. 4, 207--214

Persistent URL: http://dml.cz/dmlcz/107012

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIV: 207—214, 1978

ELEMENTARY THEORY OF DIFFERENTIAL INVARIANTS

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(Received September 12, 1977)

In this note, the objects such as covariants, invariant tensors, generally invariant functions, concominants, or invariants of fibre bundles and natural maps, etc., are considered. We generally call them the differential invariants. These are equivariant maps of left L_n^r-spaces, where L_n^r is the group of r-jets of local diffeomorphisms of the real, n-dimensional euclidean space with source and target at the origin.

Many examples of the objects of this kind can be given. A well-known one is provided by the map of the second jet prolongation of the bundle of the second order, symmetric, covariant tensors on a manifold to the bundle of the fourth order, covariant tensors on the same manifold same manifold, defined by the Levi–Civita curvature tensor considered as a function of the components of a metric tensor and their first and second derivatives.

We present a straightforward geometric approach to the theory of differential invariants which makes use of the categories of fibre bundles and the Lie derivatives of maps.

Several discussions of the differential invariants from this point of view have been given [2–8]. We continue these discussions to clarify the relation between the differential invariants and some classes of the morphisms of fibre bundles.

We state a one-to-one correspondence between the set of the differential invariants and the set of the natural transformations of certain lifting functors in a category of fibre bundles. In particular, this allows to alternatively define the differential invariants as the maps transforming geometric objects to geometric objects; in this sense the differential invariants are geometric objects themselves. As an immediate consequence we obtain a description of the differential invariants by their behaviour under the local one-parameter transformation groups of the underlying base manifolds.

1. FUNDAMENTAL CATEGORIES

All manifolds, considered in this paper, are real, finite-dimensional, Hausdorff, and second countable. Our considerations belong to the category $\mathscr{C}^\infty$. The real, n-dimensional euclidean space is denoted by R^n, and we write $R^1 = R$. If $\mathscr{C}$ is a category we denote by Ob $\mathscr{C}$ the set of objects, and by Mor $\mathscr{C}$ the set of morphisms of $\mathscr{C}$.

$\mathscr{D}_n$ denotes the category formed by n-dimensional manifolds and their injective immersions. If G is a Lie group, then $\mathscr{PB}_n(G)$ denotes the category, consisting of all principal G-bundles over n-dimensional manifolds, and their G-homomorphisms over the morphisms of $\mathscr{D}_n$. It is the purpose of this section to describe a category of fibre bundles $\mathscr{FB}_n(G)$, which will be needed later.

Let $\pi \in \mathrm{Ob}\,\mathscr{PB}_n(G)$ be a principal G-bundle, $\pi : Y \to X$, let P be a left G-space. The fibre bundle with fibre P, associated to the principal G-bundle π, will be denoted by π_P or $\pi_P : Y_P \to X$. Recall that the points of the manifold Y_P are equivalence classes of pairs $(y, p) \in Y \times P$, typically denoted by $[y, p]$, with respect to the equivalence relation defined by the right action $G \times (Y \times P) \ni (g, (y, p)) \to (y \cdot g, g^{-1} \cdot y) \in Y \times P$ of G on $Y \times P$. The projection π_p is defined by

$$\pi_P([y, p]) = \pi(y).$$

The *objects* of the category $\mathscr{FB}_n(G)$ are all fibre bundles associated to the principal G-bundles belonging to the set Ob $\mathscr{PB}_n(G)$.

To define the morphisms of the category $\mathscr{FB}_n(G)$, consider a principal G-bundle $\pi_i : Y_i \to X_i$, $\pi_i \in \mathrm{Ob}\,\mathscr{PB}_n(G)$, and a left G-space P_i, $i = 1, 2$. Denote by $\pi_{iP_i} : Y_{iP_i} \to X_i$ the fibre bundle with fibre P_i associated to π_i. Now a pair (φ, φ_0) of maps $\varphi : Y_{1P_1} \to Y_{2P_2}$, $\varphi_0 : X_1 \to X_2$ is a *morphism* of the category $\mathscr{FB}_n(G)$ if there exist a morphism $\Phi \in \mathrm{Ob}\,\mathscr{PB}_n(G)$, $\Phi : Y_1 \to Y_2$, and a G-equivariant map $F : P_1 \to P_2$ such that φ_0 is the projection of Φ,

$$\pi_2 \circ \Phi = \varphi_0 \circ \pi_1,$$

and for every $z \in Y_{1P_1}$, $z = [y, p]$,

$$\varphi(z) = [\Phi(y), F(p)].$$

These objects and morphisms obviously form a category with respect to the composition of maps which we have denoted by $\mathscr{FB}_n(G)$.

2. DIFFERENTIAL INVARIANTS

Let L_n^r be the Lie group of r-jets of local diffeomorphisms of R^n with source and target $0 \in R^n$ [1]. Our aim is to study the subject, introduced in the following.

Definition. A *differential invariant* $F : P \to Q$ is an L^r_n-equivariant map of a left L^r_n-space P to a left L^r_n-space Q.

To fix the notation, let us briefly recall some definitions. Let $X \in Ob\, \mathscr{D}_n$, $x \in X$. An *r-frame* at the point x of X is an invertible r-jet with source $0 \in R^n$ and target x. The set $\mathscr{F}^r X$ of all r-frames at the points of X together with the natural projection map of $\mathscr{F}^r X$ onto X, denoted by $\pi_{X,r}$, carries a natural structure of a principal L^r_n-bundle, and is called the *bundle of r-frames* over X. Denote by $*$ the composition of jets. Then the right action of L^r_n on $\mathscr{F}^r X$ is defined by the map $\mathscr{F}^r X \times L^r_n \ni (y, g) \to \to y * g \in \mathscr{F}^r X$.

Let $\alpha \in \text{Mor}\, \mathscr{D}_n$, $\alpha : X_1 \to X_2$, $x \in X_1$, and let $j^r_x \alpha$ denote the r-jet of α at the point x. α gives rise to the map

$$\mathscr{F}^r X_1 \ni y \to \mathscr{F}^r \alpha(y) = j^r_{\pi_{X_1,r}(y)} \alpha * y \in \mathscr{F}^r X_2$$

whose projection is equal to α, i.e.,

$$\pi_{X_2,r} \circ \mathscr{F}^r \alpha = \alpha \circ \pi_{X_1,r}.$$

This map is obviously an element of the set Mor $\mathscr{P}\mathscr{B}_n(L^r_n)$. The correspondence $X \to \to \mathscr{F}^r X$, $\alpha \to \mathscr{F}^r \alpha$, where $X \in \text{Ob}\, \mathscr{D}_n$ and $\alpha \in \text{Mor}\, \mathscr{D}_n$, defines a *lifting of order r* [4] — a covariant functor from the category $\mathscr{D}_n$ to the category $\mathscr{P}\mathscr{B}_n(L^r_n)$. This lifting is denoted by $\mathscr{F}^r$.

If P is a left L^r_n-space, then the fibre bundle with fibre P, associated to the principal L^r_n-bundle $\pi_{X,r} : \mathscr{F}^r X \to X$, $X \in \text{Ob}\, \mathscr{D}_n$, is denoted by $\pi_{X,r,P} : \mathscr{F}^r_P X \to X$.

Let $\alpha \in \text{Mor}\, \mathscr{D}_n$, $\alpha : X_1 \to X_2$. Then $\mathscr{F}^r \alpha \in \text{Mor}\, \mathscr{P}\mathscr{B}_n(L^r_n)$, $\mathscr{F}^r \alpha : \mathscr{F}^r X_1 \to \mathscr{F}^r X_2$. There arises a map

(1) $$\mathscr{F}^r_P X_1 \ni z \to \mathscr{F}^r_P \alpha(z) = [\mathscr{F}^r \alpha(y), p] \in \mathscr{F}^r_P X_2,$$

where $z = [y, p]$. The pair $(\mathscr{F}^r_P \alpha, \alpha)$ belongs to the set Mor $\mathscr{F}\mathscr{B}_n(L^r_n)$ and is said to be *induced* by α.

The correspondence $X \to \mathscr{F}^r_P X$, $\alpha \to (\mathscr{F}^r_P \alpha, \alpha)$, where $X \in Ob\, \mathscr{D}_n$ and $\alpha \in \text{Mor}\, \mathscr{D}_n$, defines a covariant functor from $\mathscr{D}_n$ to $\mathscr{F}\mathscr{B}_n(L^r_n)$ which is called the *P-lifting* associated to the lifting $\mathscr{F}^r$. This P-lifting is denoted by $\mathscr{F}^r_P$.

Let P and Q be two left L^r_n-spaces and $F : P \to Q$ a differential invariant. For every $X \in \text{Ob}\, \mathscr{D}_n$, the formula

(2) $$F_X(z) = [y, F(p)],$$

where $z = [y, p]$, defines a morphism $(F_X, id_X) \in \text{Mor}\, \mathscr{F}\mathscr{B}_n(L^r_n)$ such that F_X maps $\mathscr{F}^r_P X$ to $\mathscr{F}^r_Q X$. We call the pair (F_X, id_X) the *realization* of the differential invariant F on the manifold X.

If $X \in \text{Ob}\, \mathscr{D}_n$ and if U is an open subset of X, then $\mathscr{F}^r_P U$ is an open subset of $\mathscr{F}^r_P X$. Obviously,

$$F_X |_{\mathscr{F}^r_P U} = F_U.$$

If $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\alpha : U_1 \to U_2$, where U_i is an open subset of $X_i \in \operatorname{Ob} \mathscr{D}_n$, then $F_{U_2} \circ \mathscr{F}^r_P\alpha = \mathscr{F}^r_Q\alpha \circ F_{U_1}$ which we also write as

(3) $$F_{X_2} \circ \mathscr{F}^r_P\alpha = \mathscr{F}^r_Q\alpha \circ F_{X_1}.$$

Our aim is to characterize the differential invariants as the natural transformations of the liftings, associated to $\mathscr{F}^r$. Recall that a *natural transformation* t of $\mathscr{F}^r_P$ to $\mathscr{F}^r_Q$ consists of a collection of morphisms $(t_X, \mathrm{id}_X) \in \operatorname{Mor} \mathscr{F}\mathscr{B}_n(L^r_n)$, where $X \in \operatorname{Ob} \mathscr{D}_n$, such that for every $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\alpha : X_1 \to X_2$,

(4) $$t_{X_2} \circ \mathscr{F}^r_P\alpha = \mathscr{F}^r_Q\alpha \circ t_{X_1}.$$

Theorem 1. *Let P and Q be two left L^r_n-spaces, $F : P \to Q$ a differential invariant. Then*:

I. *The correspondence $t_F : X \to F_X$, where $X \in \operatorname{Ob} \mathscr{D}_n$, is a natural transformation of the P-lifting $\mathscr{F}^r_P$ to the Q-lifting $\mathscr{F}^r_Q$.*

II. *The correspondence $F \to t_F$ is a bijection between the set of differential invariants from P to Q and the set of natural transformations of $\mathscr{F}^r_P$ to $\mathscr{F}^r_Q$.*

Proof. Let P, Q, and F be as above. For every $X \in \operatorname{Ob} \mathscr{D}_n$, F_X is obviously an element of $\operatorname{Mor} \mathscr{F}\mathscr{B}_n(L^r_n)$. To prove the first assertion it thus suffices to show that for every $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\alpha : X_1 \to X_2$, the relation (4) holds. This is, however, a direct consequence of our definitions (1), (2).

Let us prove the second statement. Firstly, let us show that the correspondence $F \to t_F$ is injective. Assuming that for some differential invariants F_1, F_2, the equality $t_{F_1} = t_{F_2}$ holds, we obtain $F_{1X} = F_{2X}$ for all $X \in \operatorname{Ob} \mathscr{D}_n$. For some $X \in \operatorname{Ob} \mathscr{D}_n$ and $z \in \mathscr{F}^r_P X$, $z = [y, p]$, we obtain $F_{1X}(z) = [y, F_1(p)] = F_{2X}(z) = [y, F_2(p)]$ proving that the correspondence $F \to t_F$ is injective. Secondly, let us show that the correspondence $F \to t_F$ is surjective. Let us assume that we are given a natural transformation t of $\mathscr{F}^r_P$ to $\mathscr{F}^r_Q$. Choose $X \in \operatorname{Ob} \mathscr{D}_n$ and $y \in \mathscr{F}^r X$. This choice gives rise to a map $T_y : P \to Q$ defined by the relation

(5) $$t_X(z) = [y, T_y(p)],$$

where $z = [y, p]$. Note that for every open subset U of X such that $y \in \mathscr{F}^r U$,

(6) $$t_U(z) = [y, T_y(p)].$$

Let $X_i \in \operatorname{Ob} \mathscr{D}_n$, $y_i \in \mathscr{F}^r X_i$, $i = 1, 2$. There always exist an open subset U_i of X_i containing $\pi_{X_i,r}(y_i)$, and $\alpha \in \operatorname{Mor} \mathscr{D}_n$, $\alpha : U_1 \to U_2$, such that

(7) $$\mathscr{F}^r\alpha(y_1) = y_2.$$

For such an α and every $p \in P$ we obtain, using (5), (6), (4), (1), and (7)

$$t_{X_2}([y_2, p]) = [y_2, T_{Y_2}(p)] = t_{U_2} \circ \mathscr{F}^r_P\alpha([y_1, p]) = \mathscr{F}^r_Q\alpha \circ t_{U_1}([y_1, p]) =$$
$$= \mathscr{F}^r_Q\alpha([y_1, T_{Y_1}(p)]) = [\mathscr{F}^r\alpha(y_1), T_{y_1}(p)] = [y_2, T_{y_1}(p)]$$

which implies that $T_{y_1} = T_{y_2}$. There must exist a map $T : P \to Q$ such that for every $X \in \mathrm{Ob}\, \mathscr{D}_n$ and $y \in \mathscr{F}^r X$,

$$T = T_y. \tag{8}$$

Let us study the behaviour of this map under the action of L_n^r on P. Choose $X \in \mathrm{Ob}\, \mathscr{D}_n$, $z \in \mathscr{F}_P^r X$, $z = [y, p]$, and $g \in L_n^r$, Then, by definition,

$$t_X([y, g \cdot p]) = [y, T(g \cdot p)] = t_X([y \cdot g, p]) = \\ = [y \cdot g, T(p)] = [y, g \cdot T(p)],$$

and

$$T(g \cdot p) = g \cdot T(p).$$

This shows that T is a differential invariant. On comparison of (8), (5), and (2) we obtain that the realization of the differential invariant T on X, T_X, is equal to $t_X \in \mathrm{Mor}\, \mathscr{F}\mathscr{B}_n(L_n^r)$. This shows that the correspondence $F \to t_F$ is surjective which finishes the proof of Theorem 1.

Let P and Q be two left L_n^r-spaces, let $f : P \to Q$ be any map. Denote by l_n^r the Lie algebra of L_n^r. Let $\xi \in l_n^r$ and let g_t be the one-parameter group generated by ξ. Then the formula

$$\partial_\xi f(f(p)) = \left\{ \frac{\mathrm{d}}{\mathrm{d}t} g_{-t} \cdot f(g_t \cdot p) \right\}_0$$

defines a vector field along f which is called the *Lie derivative* of f with respect to ξ [9].

Similarly let $X \in \mathrm{Ob}\, \mathscr{D}_n$, let ξ be a vector field on X, denote by α_t the local one-parameter group generated by ξ, and consider a morphism $(G, \mathrm{id}_X) \in \mathrm{Mor}\, \mathscr{F}\mathscr{B}_n(L_n^r)$, where $G : \mathscr{F}_P^r X \to \mathscr{F}_Q^r X$. For every $z \in \mathscr{F}_P^r X$, $t \to (\mathscr{F}_Q^r \alpha_{-t} \circ G \circ \mathscr{F}_P^r \alpha_t)(z)$ is a curve in $\mathscr{F}_Q^r X$ passing through the point $G(z)$. The arising vector field along G,

$$\partial_\xi G(G(z)) = \left\{ \frac{\mathrm{d}}{\mathrm{d}t} (\mathscr{F}_Q^r \alpha_{-t} \circ G \circ \mathscr{F}_P^r \alpha_t)(z) \right\}_0$$

is called the *Lie derivative* of G with respect to ξ.

Recall that the group L_n^r consists of two components. The first one, $L_n^{r(+)}$, the maximal connected subgroup of L_n^r, is formed by the r-jets of local diffeomorphisms of R^n whose Jacobian is positive. The second component, $L_n^{r(-)}$, is the complement of $L_n^{r(+)}$ in L_n^r. Every element $g_0 \in L_n^{r(-)}$ gives rise to the diffeomorphism $L_n^{r(-)} \ni g \to \to g_0 * g \in L_n^{r(+)}$.

Let P and Q be two left L_n^r-spaces, let $f : P \to Q$ be a map. It is immediately proved that the following two conditions are equivalent:

I. f is a differential invariant.

II. For every $\xi \in l_n^r$,

$$\partial_\xi f = 0,$$

and there exists $g_0 \in L_n^{r(-)}$ such that

$$f(g_0 \cdot p) = g_0 \cdot f(p)$$

for all $p \in P$.

We shall state a similar result for the morphisms of fibre bundles, associated to the bundles of r-frames.

Theorem 2. *Let $X \in \operatorname{Ob} \mathscr{D}_n$ be connected, let $(G, \mathrm{id}_X) \in \operatorname{Mor} \mathscr{F}\mathscr{B}_n(L_n^r)$, $G : \mathscr{F}^r_P X \to \mathscr{F}^r_Q X$. The following three conditions are equivalent:*

I. *For every local diffeomorphism α of X,*

$$\mathscr{F}^r_Q \alpha \circ G = G \circ \mathscr{F}^r_P \alpha. \tag{9}$$

II. *For every vector field ξ defined on an open subset of X,*

$$\partial_\xi G = 0,$$

and there exist a point $x_0 \in X$, a point $y_0 \in \pi^{-1}_{X,r}(x_0)$, and a local diffeomorphism α_0 of X defined on an open neighbourhood of x_0 such that $\alpha_0(x_0) = x_0$, the element $g_0 \in L^r_n$ defined by the relation $\mathscr{F}^r \alpha_0(y_0) = y_0 * g_0$ belongs to $L_n^{r(-)}$, and

$$\mathscr{F}^r_Q \alpha_0 \circ G = G \circ \mathscr{F}^r_P \alpha_0. \tag{10}$$

III. *There exists a differential invariant $F : P \to Q$ whose realization F_X on X is equal to G,*

$$F_X = G.$$

Proof. Firstly, it is obvious that II follows from I.

Secondly, let us assume that the second condition is satisfied. Let ξ be a vector field defined on an open subset of X, α_t its one-parameter group. Then our assumption leads to the relation

$$\mathscr{F}^r_Q \alpha_t \circ G = G \circ \mathscr{F}^r_P \alpha_t \tag{11}$$

taking place for all t.

Let $x \in X$ be any point. Every $y \in \pi^{-1}_{X,r}(x)$ defines a map $G_y : P \to Q$ by the relation

$$G(z) = [y, G_y(p)], \tag{12}$$

where $z = [y, p]$. Our aim is to study the map $y \to G_y$.

Let x_0 and y_0 be as in the second condition of Theorem 2. Let α_t be any local one-parameter group of transformations of X, defined on a neighbourhood of x_0, assume that for all t, $\alpha_t(x_0) = x_0$. Then (11), (12), and (1) give

$$G_{y_0} = G_{\mathscr{F}^r \alpha_t(y_0)}$$

which shows that the function $y \to G_y$ is constant along the curve $t \to \mathscr{F}^r \alpha_t(y_0)$. Let us clarify which points of the fibre $\pi^{-1}_{X,r}(x_0)$ can be joined with y_0 by such curves.

Note that every such a local one-parameter group α_t defines a local one-parameter subgroup g_t in $L_n^{r(+)}$ by the relation

$$y_0 * g_t^{-1} = \mathscr{F}^r\alpha_t(y_0).$$

Conversely, let g_t be a local one-parameter subgroup of L_n^r; obviously, $g_t \in L_n^{r(+)}$ for all t. Then there always exists a local one-parameter transformation group χ_t in R^n such that $g_t = j_0^r\chi_t$. χ_t may be chosen in the form of appropriate polynomials with coefficients depending on t. Let φ_0 be a local diffeomorphism from R^n to X, defined on a neighbourhood of $0 \in R^n$ and such that $y_0 = j_0^r\varphi_0$. Then the one-parameter system of maps

$$\alpha_t = \varphi_0\chi_{-t}\varphi_0^{-1}$$

is a local one-parameter transformation group, defined on a neighbourhood of $x_0 \in X$. Evidently, $\alpha_t(x_0) = x_0$ and

$$y * g_t^{-1} = j_0^r(\varphi_0\chi_t^{-1}) = j_{x_0}^r(\varphi_0\chi_t^{-1}\varphi_0^{-1}) * j_0^r\varphi_0 = \mathscr{F}^r\alpha_t(y_0)$$

Since local one-parameter subgroups of a Lie group fill a neighbourhood of the identity of the group, this relation shows that

$$G_{y_0} = G_{y_0 * g} \tag{13}$$

for every $g \in L_n^{r(+)}$.

Let α_0 be a local diffeomorphism of X, satisfying the second part of the condition II. Then for every $z \in \pi_{X,r,P}^{-1}(x_0)$, $z = [y_0, p]$,

$$\mathscr{F}_Q^r\alpha_0 \circ G(z) = [\mathscr{F}^r\alpha_0(y_0), G_{y_0}(p)] = G \circ \mathscr{F}_P^r\alpha_0(z) =$$
$$= [\mathscr{F}^r\alpha_0(y_0), G_{\mathscr{F}^r\alpha_0(y_0)}(p)] = [\mathscr{F}^r\alpha_0(y_0), G_{y_0 * g_0}(p)]$$

which shows that

$$G_{y_0} = G_{y_0 * g_0}. \tag{14}$$

(13) and (14) together show that the function $y \to G_y$ is invariant under the action of L_n^r on $\pi_{X,r}^{-1}(x_0)$, or, in other words, that the equality

$$G_{y_1} = G_{y_2}$$

holds for all $y_1, y_2 \in \pi_{X,r}^{-1}(x_0)$. Consequently, the map G_{y_0} depends on $x_0 = \pi_{X,r}(y_0)$ only, and we may denote

$$G_{y_0} = G_{x_0}.$$

Let us now verify that G_x can be defined in the same way at every point $x \in X$. Let $x_0, y_0 = j_0^r\varphi_0$, $g_0 = j_0^r\chi_0$, and α_0 be as before. Then $j_0^r\chi_0 = j_0^r(\varphi_0^{-1}\alpha_0\varphi_0)$, by (10). Let $x \in X$. Since X is connected we can join x_0 with x by a curve. We can construct the tangent vector field to this curve, and prolong it to a vector field ξ defined on an open subset of X. Let β_t be the local one-parameter group generated by ξ. There is t_0 such that $\beta_{t_0}(x_0) = x$. Consider the point $y = j_0^r(\beta_{t_0}\varphi_0) \in \pi_{X,r}^{-1}(x)$, and the map $\alpha = \beta_{t_0}\alpha_0\beta_{t_0}^{-1}$, defined on a neighbourhood of x. We obtain

$$\mathscr{F}^r\alpha(y) = j_x^r(\beta_{t_0}\alpha_0\beta_{t_0}^{-1}) * j_0^r(\beta_{t_0}\varphi_0) = j_0^r(\beta_{t_0}\varphi_0) * j_0^r(\varphi_0^{-1}\alpha_0\beta_{t_0}^{-1}\beta_{t_0}\varphi_0) = y * g_0 .$$

Moreover, it follows from (9) and (10) that

$$\mathscr{F}^r_Q\alpha \circ G = G \circ \mathscr{F}^r_P\alpha$$

which shows that the second part of the condition II is satisfied at the point $x \in X$.

Consider now the maps G_{x_0}, G_x. With the help of the local one-parameter group β_t it is directly obtained that $G_{x_0} = G_x$. We set

$$F = G_x. \tag{15}$$

It follows from (15) and (12) that F is a differential invariant, and (2) shows that $F_X = G$. We have thus seen that III is a consequence of II.

Thirdly, if III holds then II must also hold, by (3).

This completes the proof of Theorem 2.

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