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REPRESENTATION OF ORDERED CLASSES BY CLASSES OF CONNECTED UNARY ALGEBRAS

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1. Notation. We denote by Ord the class of all ordinals. The natural order of ordinals is denoted by $\leqq$. If $\alpha \in \mathrm{Ord}$ then we put $W(\alpha) = \{\beta \in \mathrm{Ord};\ \beta < \alpha\}$. Further, we put $N = W(\omega_0)$. If A is a set we denote by $|A|$ the cardinal of A.

Let $\mathscr{A}$ be a category. Then we denote by $\mathscr{A}$ also the class of objects of $\mathscr{A}$ and, for arbitrary $P, Q \in \mathscr{A}$, by $[P, Q]_{\mathscr{A}}$ the set of all morphisms from P into Q. The sign $\cong$ means an isomorphism of categories and $\subseteq$ a full inclusion functor. If $\mathscr{A}$ is a category such that for each $P, Q \in \mathscr{A}$, there holds $|[P, Q]_{\mathscr{A}}| \leqq 1$, then $\mathscr{A}$ is called a *thin* category or a *quasi-ordered class*. If $\mathscr{A}$ is a thin category such that, for each $P, Q \in \mathscr{A}$, there holds $[P, Q]_{\mathscr{A}} \neq \emptyset$, $[P, Q]_{\mathscr{A}} \neq \emptyset$ implies $P = Q$, then $\mathscr{A}$ is called an *ordered class*. If $\mathscr{A}$ is a thin category (an ordered class resp.), then for each $P, Q \in \mathscr{A}$ we put $P\pi_{\mathscr{A}}Q$ ($P \leqq_{\mathscr{A}} Q$ resp.) if $[P, Q]_{\mathscr{A}} \neq \emptyset$. An ordered class $\mathscr{A}$ is called a *chain* if $P \leqq_{\mathscr{A}} Q$ or $Q \leqq_{\mathscr{A}} P$ for each $P, Q \in \mathscr{A}$.

If $\mathscr{A}$ is a category, then a thin category $\mathscr{A}(b)$ such that the class of objects of $\mathscr{A}(b)$ is equal to that of objects of $\mathscr{A}$ and $P\pi_{\mathscr{A}(b)}Q$ iff $[P, Q]_{\mathscr{A}} \neq \emptyset$ fo reach $P, Q \in \mathscr{A}$ is called a *basic* category for $\mathscr{A}$. (Therefore, a basic category $\mathscr{A}(b)$ for $\mathscr{A}$ is a thin category with the same objects and the same existence of morphisms.)

Let A be a quasi-ordered set (with the quasi-order π_A). If $a, b \in A$ are arbitrary, then we put $a\varrho_A b$ iff $a\pi_A b$ and $b\pi_A a$. Then ϱ_A is an equivalence on A. Further, if $T, T' \in A/\varrho_A$ are arbitrary, we put $T\pi_{A/\varrho_A}T'$ iff $a\pi_A b$ for each $a \in T$ and each $b \in T'$. Then π_{A/ϱ_A} is an order on A/ϱ_A. (See, for example, [1], I., § 4.) We say that the *order* π_{A/ϱ_A} *is defined by the quasi-order* π_A.

If A, B are ordered sets, then the cardinal power of A and B is denoted by A^B.

The *lexicographic sum* $\sum^{l}_{G \in \mathscr{G}} \mathscr{A}_G$ of a system $\{\mathscr{A}_G;\ G \in \mathscr{G}\}$ of mutually disjoint thin categories where $\mathscr{G}$ is an ordered class is the class $\mathscr{A} = \bigcup_{G \in \mathscr{G}} \mathscr{A}_G$ of objects where, for each $G_1, G_2 \in \mathscr{G}$, $P \in \mathscr{A}_{G_1}$, $Q \in \mathscr{A}_{G_2}$, there holds $P\pi_{\mathscr{A}}Q$ iff (1) $G_1 \leqq_{\mathscr{G}} G_2$ and (2) $G_1 = G_2$ implies $P\pi_{\mathscr{A}_{G_1}}Q$. Further, if $\mathscr{G} = \{1, \ldots, n\}$ is a chain with the natural order then we put $\sum^{l}_{G \in \mathscr{G}} \mathscr{A}_G = \mathscr{A}_1 \oplus \ldots \oplus \mathscr{A}_n$. If $\{\mathscr{A}, \mathscr{B}\}$ is a non-indexed system of two disjoint thin categories then we suppose $\{\mathscr{A}, \mathscr{B}\}$ to be a chain with $\mathscr{A} <_{(\mathscr{A},\mathscr{B})} \mathscr{B}$ and we can define $\mathscr{A} \oplus \mathscr{B}$. (See [6].)

Let A be a set, f a partial map from A into A. Then the ordered pair $A = (A, f)$ is called a *partial unary algebra*. We put $DA = A - \operatorname{dom} f$. If $DA = \emptyset$ then A is called a *complete* unary algebra. If $A = (A, f)$, $B = (B, g)$ are partial unary algebras and $F : A \to B$ a map ($\operatorname{dom} F = A$), then F is called a *homomorphism* of A into B if $x \in \operatorname{dom} f$ implies $Fx \in \operatorname{dom} g$ and $Ffx = gFx$.

Let $A = (A, f)$ be a partial unary algebra. We put $f^0 = \mathrm{id}_A$. Suppose that we have defined a partial map f^{n-1} from A into A for $n \in N - \{0\}$. We denote by f^n the following partial map from A into A: if $x \in \operatorname{dom} f^{n-1}$ and $f^{n-1}x \in \operatorname{dom} f$, then we put $f^n x = ff^{n-1}x$. A is called a *connected* partial unary algebra if, for each $x, y \in A$, there are $m, n \in N$ such that $x \in \operatorname{dom} f^m$, $y \in \operatorname{dom} f^n$ and $f^m x = f^n y$. The category of all connected partial unary algebras is denoted by $\mathscr{U}^c$ and the category of all connected complete unary algebras is denoted by $\mathscr{V}^c$ where morphisms are homomorphisms.

2. Problems. *Let M be an ordered set.*

(a) (P. Goralčík, see [2], § 3, remark 2.) *Find necessary and sufficient conditions for the existence of $\mathscr{M} \subseteq \mathscr{V}^c$ such that $M \cong \mathscr{M}(b)$.*

(b) *Find necessary and sufficient conditions for the existence of $\mathscr{M} \subseteq \mathscr{U}^c$ such that $M \cong \mathscr{M}(b)$.*

By our considerations, we can apply results of [7], [9], [10] and [11].

Let $\infty_1, \infty_2 \in \mathrm{Ord}$ and let us suppose that $\alpha < \alpha_1 < \infty_2$ for each $\alpha \in \mathrm{Ord}$. Let $A = (A, f) \in \mathscr{U}^c$. Then we define the sets $ZA = \{x \in A;$ there is $n \in N - \{0\}$ such that $f^n x = x\}$, $KA = \{x \in A - ZA;$ there is a sequence $(x_i)_{i \in N}$ such that $x_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $fx_{i+1} = x_i$ for each $n \in N\}$ and $A^0 = \{x \in A;$ $f^{-1}x = \emptyset\}$; if $\alpha \in \mathrm{Ord} - \{0\}$ is arbitrary and if the sets A^λ have been defined for all $\lambda \in W(\alpha)$ then we put $A^\alpha = \{x \in A - \bigcup_{\lambda \in W(\alpha)} A^\lambda; f^{-1}x \subseteq \bigcup_{\lambda \in W(\alpha)} A^\lambda\}$. Further, we put $RA = |ZA|$, $\vartheta A = \min\{\lambda \in \mathrm{Ord}; A^\lambda = \emptyset\}$ and if we put $A^{\infty_1} = KA$, $A^{\infty_2} = ZA$, then we define the map $SA : A \to \mathrm{Ord} \cup \{\infty_1, \infty_2\}$ by the condition $SAx = \lambda$ for each $x \in A^\lambda$, $\lambda \in W(\vartheta A) \cup \{\infty_1, \infty_2\}$.

(i) *Let $A \in \mathscr{U}^c$. Then the following assertions hold.*

(a) $|DA| \leqq 1$; *we denote by dA the only point with the property $\{dA\} = DA$.*

(b) *If $RA = 0$, $KA = \emptyset$ then $DA \neq \emptyset$ iff ϑA is isolated.*

(c) *If $RA = 0$, $KA = \emptyset$, $DA \neq \emptyset$ then $SAdA = \vartheta A - 1$.*

(See [7], 2.1, 2.26 (c) and [9] 1.12.)

(ii) *If $\alpha \in \mathrm{Ord}$ is isolated then there is $A \in \mathscr{U}^c$ such that $RA = 0$, $KA = \emptyset$ and $\vartheta A =$ $= \alpha$.* (See [11], 4.8.)

If $(\alpha_n)_{n \in N}$ is a sequence of ordinals then we write (α_n) instead of $(\alpha_n)_{n \in N}$. We say that a sequence of ordinals (α_n) is an *end* of a sequence of ordinals (β_n) if there is $m \in N$ such that $\alpha_n = \beta_{m+n}$ for each $n \in N$.

3. Definition. Let $\alpha \in \mathrm{Ord}$ be limit and cofinal with ω_0 and let $\leqq_0$ be the order on the cardinal power $\mathrm{W}(\alpha)^N$. We put $\mathcal{O}_0(\alpha) = \{(\alpha_n) \in \mathrm{W}(\alpha)^N;\ \lim_{n \in N} \alpha_n = \alpha\}$. We define an order $\leqq_1$ on $\mathcal{O}_0(\alpha)$ such that, for each (α_n), $(\beta_n) \in \mathcal{O}_0(\alpha)$, we put $(\alpha_n) \leqq_1 (\beta_n)$ iff (α_n) is an end of (β_n).

4. Definition. We put $\mathcal{O}^i = \{\alpha \in \mathrm{Ord} - \{0\};\ \alpha \text{ isolated}\}$, $\mathcal{O}^l = \{\alpha \in \mathrm{Ord};\ \alpha \text{ limit and cofinal with } \omega_0\}$. Let d, $\bar{d} \notin \mathrm{Ord}$ and we suppose that $\alpha < d < \bar{d}$ for each $\alpha \in \mathrm{Ord}$. Let $\mathcal{N}$ be an arbitrary set disjoint with $\mathrm{Ord} \cup \{d, \bar{d}\}$ and equivalent with $N - \{0\}$ and $'\colon N - \{0\} \to \mathcal{N}$ a bijection. Let $\mathcal{O}(\alpha) = \{\alpha\}$ for each $\alpha \in \mathcal{O}^i$ and let $\mathcal{O}(\alpha)$ for each $\alpha \in \mathcal{O}^l$ be the ordered set with the order $\leqq_{\mathcal{O}(\alpha)}$ which is defined by the quasi-order $\leqq_0 \circ \leqq_1$ (the composition) on $\mathcal{O}_0(\alpha)$ (see 1). We define the ordered class

(a) $\mathcal{O} = \mathcal{O}^i \cup \mathcal{O}^l \cup \{d, \bar{d}\}$ such that, for each $\alpha, \beta \in \mathcal{O}$, $\alpha \leqq_{\mathcal{O}} \beta$ iff (1) $\alpha \leqq \beta$ and (2) $\alpha \in \mathcal{O}^l \cup \{\bar{d}\}$ implies $\beta \in \mathcal{O}^l \cup \{\bar{d}\}$,

(b) $\mathcal{N}$ such that, for each m', $n' \in \mathcal{N}$ where $m, n \in N - \{0\}$, $m' \leqq_{\mathcal{N}} n'$ iff $n \mid m$,*)

(c) $\mathcal{C} = (\sum_{\alpha \in \mathcal{O}}^{l} \mathcal{O}(\alpha)) \oplus \mathcal{N}$. (Clearly, $\mathcal{C}$ is an ordered class.)

Further, we define the subcategory $\mathcal{C}^* = \bigcup_{\alpha \in \mathcal{O}^l} \mathcal{O}(\alpha) \cup \{\bar{d}\} \cup \mathcal{N}$ of $\mathcal{C}$.

(iii) *Let $A = (A, f) \in \mathcal{U}^c$ be such that $RA = 0$, $KA = \emptyset$ and $\vartheta A \in \mathcal{O}^l$. Then there is $\mu \in \mathcal{O}(\vartheta A)$ such that $(SAf^n x) \in \mu$ for each $x \in A$.* (See [10], 2.16(b).)

5. Definition. (a) We define the object function $\chi\colon \mathcal{U}^c \to \mathcal{C}$ in this way: if $A = (A, f) \in \mathcal{U}^c$, we put

$$\chi A = \begin{cases} RA & \text{if } RA \neq 0. \\ \bar{d} & \text{if } RA = 0,\ KA \neq \emptyset,\ DA = \emptyset, \\ d & \text{if } RA = 0,\ KA \neq \emptyset,\ DA \neq \emptyset, \\ \vartheta A & \text{if } RA = 0,\ KA = \emptyset,\ \vartheta A \in \mathcal{O}^i, \\ \mu \in \mathcal{O}(\vartheta A) & \text{if } RA = 0,\ KA = \emptyset,\ \vartheta A \in \mathcal{O}^l,\ x \in A,\ (SAf^n x) \in \mu. \end{cases}$$

(b) If $a \in \mathcal{C}$ is arbitrary then we put $a - \mathcal{U}^c = \{A \in \mathcal{U}^c;\ \chi A = a\}$.

Let $\mathcal{A}$ be a category. Then it is called a category *with non-empty homs* if, for each P, $Q \in \mathcal{A}$, there holds $[P, Q]_{\mathcal{A}} \neq \emptyset$. The following assertion holds for basic categories of subcategories of $\mathcal{U}^c$.

(iv) *$\mathcal{U}^c(b) = \sum_{a \in \mathcal{C}}^{l} [a - \mathcal{U}^c(b)]$ and $\mathcal{V}^c(b) = \sum_{a \in \mathcal{C}^*}^{l} [a - \mathcal{U}^c(b)]$, where $a - \mathcal{U}^c(b)$ is a category with non-empty homs for each $a \in \mathcal{C}$.* (See [10], 2.24 and 2.25.)

6. Lemma. (a) *$a - \mathcal{U}^c \neq \emptyset$ for each $a \in \mathcal{C}$.*

(b) *$\chi \mathcal{U}^c = \mathcal{C}$, $\chi \mathcal{V}^c = \mathcal{C}^*$.*

Proof of (a). The assertion is evident for each $a \in \{d, \bar{d}\} \cup \mathcal{N}$. If $a \in \mathcal{O}^i$, then the assertion follows from 5 and (ii).

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*) $n \mid m$ means that n is a divisor of m.

Therefore, let $a = \mu \in \mathcal{O}(\alpha)$ where $\alpha \in \mathcal{O}^l$. Then μ is a set of increasing sequences of ordinals (α_n) with $\lim_{n\in N} \alpha_n = \alpha$ by 3 and 4. Let $(\alpha_n) \in \mu$ be arbitrary. Then, for each $n \in N$, there is $\boldsymbol{A}_n = (A_n, f_n) \in \mathcal{U}^c$ such that $R\boldsymbol{A}_n = 0$, $K\boldsymbol{A}_n = \emptyset$ and $\vartheta\, \boldsymbol{A}_n = \alpha_n + 1$ by (ii). We can suppose that A_n are mutually disjoint. Then $D\boldsymbol{A}_n \neq \emptyset$ and $S\boldsymbol{A}_n d\boldsymbol{A}_n = \alpha_n$ for each $n \in N$ by (i). We define $\boldsymbol{B} = (B, g) \in \mathcal{U}^c$ such that $B = \bigcup_{n\in N} A_n$ and, for each $x \in B$,

$$gx = \begin{cases} f_n x & \text{if } x \in A_n - d\boldsymbol{A}_n \text{ where } n \in N, \\ d\boldsymbol{A}_{n+1} & \text{if } x = d\boldsymbol{A}_n \text{ where } n \in N. \end{cases}$$

Now, since $g^{-1}d\boldsymbol{A}_0 = f_0^{-1}d\boldsymbol{A}_0$ we have $S\boldsymbol{B}d\boldsymbol{A}_0 = S\boldsymbol{A}_0 d\boldsymbol{A}_0 = \alpha_0$.

Let $n \in N - \{0\}$ be arbitrary. Then $g^{-1}d\boldsymbol{A}_n = \{d\boldsymbol{A}_{n-1}\} \cup f_n^{-1}d\boldsymbol{A}_n$, i.e. $S\boldsymbol{B}g^{-1}d\boldsymbol{A}_n = S\boldsymbol{B}d\boldsymbol{A}_{n-1} \cup S\boldsymbol{B}f_n^{-1}d\boldsymbol{A}_n = \{\alpha_{n-1}\} \cup S\boldsymbol{A}_n f_n^{-1} d\boldsymbol{A}_n$ and since $\alpha_{n-1} < \alpha_n$ we have $S\boldsymbol{B}d\boldsymbol{A}_n = S\boldsymbol{A}_n\, d\boldsymbol{A}_n = \alpha_n$.

Therefore, for each $n \in N$, there holds $S\boldsymbol{B}g^n d\boldsymbol{A}_0 = S\boldsymbol{B}d\boldsymbol{A}_n = \alpha_n$. Hence $(S\boldsymbol{B}g^n d\boldsymbol{A}_0) \in \mu \in \mathcal{O}(\alpha)$ and since $R\boldsymbol{B} = 0$, $K\boldsymbol{B} = \emptyset$ and $\vartheta\boldsymbol{B} \in \mathcal{O}^l$ we have $\chi\boldsymbol{B} = \mu$ by 5. From this follows $\boldsymbol{B} \in a - \mathcal{U}^c$.

(b) follows directly from (a) and (iv).

7. Lemma. *Let $\mathcal{M} \subseteq \mathcal{U}^c(b)$ be arbitrary. Then the following assertions are equivalent:*

(A) *$\mathcal{M}$ is an ordered class.*

(B) *$\chi \mid \mathcal{M}$ is injective.*

(C) *$\chi \mid \mathcal{M}$ is an isomorphisms of the quasi-ordered classes $\mathcal{M}$ and $\chi\mathcal{M}$.*

Proof. (A) implies (B). Indeed, if we had $\chi\boldsymbol{A} = \chi\boldsymbol{B}$ for some $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}$, $\boldsymbol{A} \neq \boldsymbol{B}$ then we should have $\boldsymbol{A}, \boldsymbol{B} \in \chi\boldsymbol{A} - \mathcal{U}^c(b)$ and thus, $\boldsymbol{A} \leqq_{\mathcal{U}^c(b)} \boldsymbol{B}$, $\boldsymbol{B} \leqq_{\mathcal{U}^c(b)} \boldsymbol{A}$ by (iv), which is a contradiction.

(B) implies (C). Indeed, if $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}$ are arbitrary then $\boldsymbol{A} \leqq_{\mathcal{U}^c(b)} \boldsymbol{B}$ iff $\chi\boldsymbol{A} \leqq_{\mathcal{C}} \chi\boldsymbol{B}$ by (iv) which implies that $\chi \mid \mathcal{M}: \mathcal{M} \to \chi\mathcal{M}$ is an isomorphism.

(C) implies (A). Since $\chi\mathcal{M} \subseteq \mathcal{C}$ is an ordered class hence $\mathcal{M} \subseteq \mathcal{U}^c(b)$ is an ordered class.

8. Lemma. (a) *Let $\mathcal{D} \subseteq \mathcal{C}$ be arbitrary. Then there is $\mathcal{M} \subseteq \bigcup_{a\in\mathcal{D}} [a - \mathcal{U}^c(b)]$ such that $\mathcal{M} \cong \mathcal{D}$.*

(b) *Let $\mathcal{C}' \subseteq \mathcal{C}$ be arbitrary and let M be an ordered class. Then there is $\mathcal{M} \subseteq \bigcup_{a\in\mathcal{C}'} [a - \mathcal{U}^c(b)]$ such that $M \cong \mathcal{M}$ if and only if there is $\mathcal{D} \subseteq \mathcal{C}'$ such that $M \cong \mathcal{D}$.*

Proof of (a). $a - \mathcal{U}^c(b)$ is non-empty for each $a \in \mathcal{D}$ by 6. We take $\boldsymbol{A}_a \in a - \mathcal{U}^c(b)$ arbitrary and put $\mathcal{M} = \{\boldsymbol{A}_a;\ a \in \mathcal{D}\}$. Then $\mathcal{M} \subseteq \bigcup_{a\in\mathcal{D}} [a - \mathcal{U}^c(b)]$ and since $\chi \mid \mathcal{M}$ is injective there holds $\mathcal{M} \cong \chi\mathcal{M} = \mathcal{D}$ by 7.

Proof of (b). If $M \cong \mathcal{M}$ for some $\mathcal{M} \subseteq \bigcup_{a\in\mathcal{C}'} [a - \mathcal{U}^c(b)]$ then $\chi \mid \mathcal{M}$ is an isomorphism by 7 and we have $M \cong \mathcal{M} \cong \chi\mathcal{M} \subseteq \mathcal{C}'$. Let, on the other hand, $M \cong \mathcal{D}$

for some $\mathscr{D} \subseteq \mathscr{C}'$. Then, by (a), there is $\mathscr{M} \subseteq \bigcup_{a \in \mathscr{D}} [a - \mathscr{U}^c(b)] \subseteq \bigcup_{a \in \mathscr{C}'} [a - \mathscr{U}^c(b)]$ such that $\mathscr{M} \cong \mathscr{D}$ and we have $M \cong \mathscr{M}$.

The following assertions expressing a representation of ordered classes by classes of connected partial and complete unary algebras give an answer to the problems 2.

Theorem. *Let M be an ordered class. Then there exists* $\mathscr{M} \subseteq \mathscr{V}^c(b)$ ($\mathscr{M} \subseteq \mathscr{U}^c(b)$ resp.) *such that $M \cong \mathscr{M}$ if and only if M can be embedded into the ordered class* $\mathscr{C}^*$ ($\mathscr{C}$ resp.).

These assertions follow directly from 8(b).

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