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CHARACTERIZATIONS OF CERTAIN MONOUNARY ALGEBRAS
(Part II)

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This is a continuation of the paper [5] where definitions of used notions and other necessary details can be found.

3. REDUCED MONOUNARY c-ALGEBRAS

We shall introduce first a certain modification of the construction described in [11] p. 228 (Def. 2.7) which we use for the definition of a reduced monounary c-algebra.

Let (A, f) be a connected monounary algebra such that $R(A, f) = 1$, and (B, g) a connected monounary algebra with $A \cap B = \emptyset$. Let $c \in B_g^0$. Then $(A, f) \oplus_c (B, g)$ denotes a monounary algebra (C, h) defined in this way : $C = B \cup (A - A_f^{\infty 2})$ and for every $x \in C$ it holds

$$h(x) = \begin{cases} f(x) & \text{for } x \in A - (A_f^{\infty 2} \cup f^{-1}(A_f^{\infty 2})), \\ c & \text{for } x \in f^{-1}(A_f^{\infty 2}) - A_f^{\infty 2}, \\ g(x) & \text{for } x \in B. \end{cases}$$

3.1. Definition: A connected monounary algebra (A, f) is said to be reduced if it has exactly one of the following forms:

i) $f^2 = f$ (i.e. (A, f) is idempotent),

(ii) Either $A = A_f^{\infty 1}$ or $A = A_f^{\infty 1} \cup A_f^0$, where $(A_f^{\infty 1}, \leqq_f)$ is a chain of the type $\omega_0^* \oplus \omega_0$ and $A_f^0 \neq \emptyset$.

(iii) $(A, f) = (A_1, f_1) \oplus_c (A_2, f_2)$, where f_1 is a constant mapping and $(A_2, \leqq_{f_2})$ is a chain of the type ω_0 with the first element c.

The below stated first characterization of a reduced c-algebra (Theorem 3.6) is given by the use of the endomorphism semigroup. We shall prove three lemmas first. We say that a transformation semigroup $S(A) \subseteq T(A)$ acts transitively on the set A if for every pair of elements $a, b \in A$ there exists $f \in S(A)$ such that $f(a) = b$. An ideal I of a semigroup S is said to be half-prime if rad $I = I$. For $f \in T(A)$ we put $\langle f \rangle' = \langle f \rangle - \{\mathrm{id}_A\}$ and $S(f) = \langle f \rangle . \langle \mathrm{Id}\, C(f) \rangle$. Let S be a subsemigroup of $T(A)$. In accordance with [6] we denote it by S^1 if S is a monoid (i.e. if it contains an identity), and in the opposite case S means $S \cup \{\mathrm{id}_A\}$. Thus $\langle f \rangle'^1 = \langle f \rangle^1 = \langle f \rangle$. A principal ideal of S generated by $f \in S$ is denoted by $I_S(f)$, if it is danger of confusion. Evidently, for a principal ideal there holds $I_S(f) = S^1 . f . S^1$ (see [6] p. 21).

3.2. Lemma. *Let (A,f) be a monounary c-algebra, $A \neq A_f^{\infty_2}$. Then $A = A_f^{\infty_1}$ iff the monoid $C(f)$ acts transitively on the set A.*

Proof. Let $A = A_f^{\infty_1}$, $a, b \in A$. For every $n \in \mathbf{N}_0$ it holds $S_f(f^n(a)) = S_f(f^n(b)) = \infty_1$ thus by Proposition 1.4 [5] there exists an endomorphism g of the algebra (A,f) such that $g(a) = b$, i.e. the monoid $C(f)$ acts transitively on the set A.

Assume the last condition is satisfied. Since for each endomorphism g of (A,f) and $x \in A_f^{\infty_2}$ there holds $g(x) \in A_f^{\infty_2}$, $(A_f^{\infty_1} = \emptyset)$, we have $R(A,f) = 0$. Further, by Lemma 2.8 [13] $x \in A$, $g \in C(f)$ implies $S_f(x) \leqq S_f(g(x))$, thus $A_f^0 = \emptyset$, hence $S_f(x) = \infty_1$ for each $x \in A$, i.e. $A = A_f^{\infty_1}$.

It is easy to see that Lemma 3.2 is contained in Theorem 1 [18], part (a), but the proof is based on some other considerations.

3.3. Lemma. *Let (A,f) be a c-algebra with $R(A,f) \leqq 1$ and such that $\langle f \rangle'$ is an ideal of $C(f)$. Then $x, y \in A$, $\delta(x, y) = 0$* is followed by $f(x) = f(y)$.

Proof. Suppose on the contrary, there exists a pair of elements $x, y \in A$ with $\delta(x, y) = 0$ and $f(x) = f(y)$. If $A_f^{\infty_1} \neq \emptyset$, then we denote by a such an element of $A_f^{\infty_1}$ that $\delta(a, x) = 0$ and by b an element of the set $\{x, y\}$ with $f(a) = f(b)$. Since $S_f(f^n(a)) = \infty_1 \geqq S_f(f^n(b))$ for each $n \in \mathbf{N}_0$, by Proposition 1.4. [5], there exists an endomorphism g of the algebra (A,f) with the property $g(b) = a$. Then $f . g(b) = f^k(b)$ for any $k \in \mathbf{N}_0$, thus $f . g \notin \langle f \rangle'$, which contradicts the inclusion $\langle f \rangle' . C(f) \subseteq \langle f \rangle'$.

Let $A_f^{\infty_1} = \emptyset$. Denote by a, b elements of A with properties $f(a) \neq f(b), f^2(a) = f^2(b)$ and $\delta(a, b) = 0$. It is evident that such a pair exists. By the definition of a degree (1.16. [11]) there exist elements $x_0, x_1 \in (a]_f$ with $S_f(x_i) = i$ for $i = 0, 1$ and $f(x_0) = x_1$. Since $f^k(x_0) <_f f^k(b)$ whenever $k \geqq 2$, it holds $S_f(f^n(x_0)) \leqq S_f(f^n(b))$ for each $n \in \mathbf{N}_0$. By Proposition 1.4 [5] there exists a mapping $h \in C(f$ with the property $h(x_0) = b$. Then $f . h(x_0) = f(b) \neq f^k(x_0)$ for any $k \in \mathbf{N}_0$ thus $f . h \notin \langle f \rangle'$ which contradicts the supposition that $\langle f \rangle'$ is an ideal of $C(f)$ again. Consequently, $\delta(x, y) = 0$ is followed by $f(x) = f(y)$, q.e.d.

Notice that the converse of the above assertion is not true. The implication converse to that stated above (in Lemma 3.3) is true only under some additional conditions, e.g. $R(A,f) = 1$ or $A_f^{\infty_1} = \emptyset$.

3.4. Lemma. *Let (A,f) be a c-algebra with* card $A \geqq 2$ *and* $R(A, f) \leqq 1$. *The following conditions are equivalent:* 1° (A,f) *is either an idempotent c-algebra or* $(A,f) = (A_1, f_1) \oplus_c (A_2, f_2)$, *where* (A_1, f_1) *is an idempotent c-algebra and* $(A_2, \leqq_{f_2})$ *is a chain of the type* ω_0. 2° $\langle f \rangle'$ *is a half-prime ideal in* $C(f)$ *and* $f^2 \neq f$ *implies* card $\langle f \rangle' = \aleph_0$.

Proof. Assume condition 1° is satisfied. If $g \in C(f)$ then for arbitrary $a \in A$ either $g(a) = f^n(a)$ with a suitable $n \in \mathrm{N}_0$ or $\delta(a, b) = \delta(g(a), b)$ for each $b \in A$. Thus for every positive integer n we have $f^n . g = g . f^n \in \langle f \rangle'$, hence $\langle f \rangle' . C(f) = C(f) . \langle f \rangle' = \langle f \rangle'$, i.e. $\langle f \rangle'$ is a proper ideal of the monoid $C(f)$ and at the same time $\mathrm{rad}_{C(f)} \langle f \rangle' = \{g \in C(f) : g^n \in \langle f \rangle' \text{ for some integer } n\} = \langle f \rangle'$, i.e. $\langle f \rangle'$ is a half-prime ideal of $C(f)$. If f is not idempotent then in our case $f^k = f^{k+1}$ for each $k \in \mathrm{N}_0$ and we have card $\langle f \rangle' = \aleph_0$. Therefore condition 2° is satisfied.

Suppose assertion 2° holds. Since $\langle f \rangle'$ is an ideal of $C(f)$ it holds by **Lemma 3.3** that $x, y \in A$, $\delta(x, y) = 0$ is followed by $f(x) = f(y)$. Admit that simultaneously $A_f^{\infty_1} \neq \emptyset$, $A_f^{\infty_2} = \emptyset$. The constant mapping h of A onto the cyclic element of (A,f) belongs to $C(f)$ and for every pair of positive integers n, m it holds $g^n . f^m = g \notin \langle f \rangle'$. This is a contradiction, thus either $A_f^{\infty_1} = \emptyset$ or $A_f^{\infty_2} = \emptyset$. Admit that $A_f^{\infty_1} = \emptyset$. Let $a, b \in A_f^{\infty_1}$ be a pair of elements with $f(b) = a$. Since $A_f^{\infty_2} = \emptyset$, thus $S_f(x) \in \mathrm{Ord}$ for each $x \in [b)_f$, by Proposition 1.4 [5] there exists $g \in C(f)$ with $g(a) = b$. Then $f . g \notin \langle f \rangle'$, which is a contradiction. Hence $A_f^{\infty_1} = \emptyset$. Now, admit that there exists an element $q \in A_f^0$ with $S_f(f(a)) \geqq 2$. With respect to Lemma 3.3 and the assumption we have $R(A,f) = 1$ iff $f^2 = f$. Hence $f^2 \neq f$ is followed by $S_f(x) \in \mathrm{Ord}$ for each $x \in A$. Let $b \in A_f^0 \cap (f(a)]_f$ be an element with $S_f(f(b)) = 1$ and $f(b) = f(a)$. Such an element b exists with respect to the definition of a degree S_f and $S_f(f(a)) \geqq 2$. Then $S_f(f^n(b)) \leqq S_f(f^n(a))$ for every $n \in \mathrm{N}_0$ and again by Proposition 1.4 [5] there exists $h \in C(f)$ with $h(b) = a$ and $h(x) \in [x)_f$ for each $x \neq b$. Then $h \notin \langle f \rangle'$ but for an integer k such that $f^k(b) = f(a)$ it holds $h^2 = f^k$ thus $h \in \mathrm{rad}_{C(f)} \langle f \rangle'$ which contradicts the assumption. Consequently the algebra (A,f) has one of the forms described in 1°.

Remark. If (A,f) is a c-algebra such that $x, y \in A$, $\delta(x, y) = 0$ is followed by $f(x) = f(y)$ then the monogenuous semigroup $\langle f \rangle'$ is a proper ideal of the semigroup $S(f)$. Indeed, $\langle f \rangle'$ is a subsemigroup of $S(f)$ and $g \in \langle \mathrm{Id}\, C(f) \rangle$, $k \in \mathrm{N}$ implies $f^k . g = g . f^k = f^k$. Then it holds $\langle f \rangle' . \langle \mathrm{Id}\, C(f) \rangle = \langle \mathrm{Id}\, C(f) \rangle . \langle f \rangle' = \langle f \rangle'$ and we have $\langle f \rangle' . S(f) = \langle f \rangle' = S(f) . \langle f \rangle'$.

3.5. Lemma. *Let (A,f) be a c-algebra with $A_f^{\infty_1} = \emptyset$, $g \in C(f)$. For every element $x \in A$ it holds $\delta(x, g(x)) \leqq 0$.*

Proof. If $R(A,f) > 0$ then $x \in A_f^{\infty_2}$ implies $g(x) \in A_f^{\infty_2}$ by Lemma 2.8 [13]. Then $\delta(x, g(x)) = 0$ for every $x \in A_f^{\infty_2}$. If we admit that there exists an element $a \in A - A_f^{\infty_2}$ with $0 < \delta(a, g(a)) = \deg(g(a)) - \deg(a)$ (see [5] § 1), we get that for the integer $n = \deg(a)$ there holds $g(f^n(a)) \notin A_f^{\infty 2}$ while $f^n(a) \in A_f^{\infty 2}$. Thus $\delta(x, g(x)) \leqq 0$ for each $x \in A$ in this case. Let $R(A,f) = 0$. Admit there exists $a \in A$ with $\delta(a, g(a)) > 0$. If $g(a) <_f a$ then for some n there holds $f^n(g(a)) = a$ and by Lemma 1.19 (a) [11], $S_f(a) \geqq S_f(g(a)) + n > S_f(g(a))$ but with respect to Lemma 2.8 [13] it is $S_f(a) \leqq S_f(g(a))$, which is a contradiction. If $g(a) \parallel_f a$ then we denote by n_0, m_0 the least integers having the property $f^{n_0}(a) = f^{m_0}(g(a))$ and we put $b = f^{n_0}(a)$. Clearly, $n_0 < m_0$. Then we have $g(b) = g(f^{n_0}(a)) = f^{n_0}(g(a)) <_f <_f f^{m_0}(g(a)) = b$ and we get a contradiction in the same way as above. Hence $x \in A$, $g \in C(f)$ is followed by $\delta(x, g(x)) \leqq 0$.

3.6. Lemma. *Let (A,f) be a c-algebra with $R(A,f) \leqq 1$. Then $A = A_f^{\infty 1} \cup A_f^0$, where $(A_f^{\infty 1}, \leqq_f)$ is a chain and $A_f^0 \neq \emptyset$ iff $\langle f \rangle'$ is an infinite proper ideal of $S(f)$, the monoid $\langle \mathrm{Id}\, C(f) \rangle$ is non-trivial and to each $g \in \langle f \rangle'$ there exists $h \in C(f)$ with $g \,.\, h \in \mathrm{Id}\, C(f)$.*

Proof. Let $A = A_f^{\infty 1} \cup A_f^0$, $(A_f^{\infty 1}, \leqq_f)$ be a chain and $A_f^0 \neq \emptyset$. Every element $a \in A_f^{\infty 1}$ is a fixed point of each $g \in C(f)$, thus $\langle \mathrm{Id}\, C(f) \rangle \subseteq C(f)$ and further $\langle \mathrm{Id}\, C(f) \rangle \,.\, \langle f \rangle' = \langle f \rangle' \,.\, \langle \mathrm{Id}\, C(f) \rangle = \langle f \rangle'$ consequently $\langle f \rangle' \,.\, S(f) = \langle f \rangle' = S(f) \,.\, \langle f \rangle'$. Since $A_f^0 \neq \emptyset$, there exists $g \in \langle \mathrm{Id}\, C(f) \rangle$ which is different from id_A. (E.g. $g(x) = x$ for $x \in A_f^{\infty 1}$, $g(x) = y \in A_f^{\infty 1}$ for $x \in A_f^0$ and for y such that $\delta(x, y) = 0$). Let $g \in \langle f \rangle'$ be arbitrary, $n \in \mathrm{N}$ such that $g = f^n$. Consider an arbitrary element $a \in A$ and put $a_1 = a$ if $a \in A_f^{\infty 1}$ and if $a \notin A_f^{\infty 1}$ then denote by a_1 an element of $A_f^{\infty 1}$ satisfying the condition $\delta(a, a_1) = 0$. Further, denote by b an element of $A_f^{\infty 1}$ with $f^n(b) = a_1$. Since $S_f(f^k(b)) = \infty_1$ for each $k \in \mathrm{N}_0$, by Proposition 1.4 [5] that there exists an endomorphism h of (A, f) with $h(a) = b$. Then $g(h(a)) = g(b) = f^n(a) = a_1$. With respect to the construction obtained in Definition 9 [13], for each $x \in A$ there holds $\delta(x, g \,.\, h(x)) = 0$. Since $g \,.\, h \in C(f)$ and $g(h(x)) \in A_f^{\infty 1}$ we have $g \,.\, h \in \mathrm{Id}\, C(f)$.

Now, we shall prove the converse implication. Suppose first $R(A,f) = 1$, $A_f^{\infty 2} = \{z_f\}$. Admit $A_f^{\infty 1} \neq \emptyset$. Then $f \,.\, h \in \mathrm{Id}\, C(f)$ iff h is a constant transformation with the value z_f, thus $h \neq f^n$ for each $n \in \mathrm{N}_0$ which contradicts the condition $\langle f \rangle' \,.\, S(f) = \langle f \rangle' \,.\, \langle f \rangle \,.\, \langle \mathrm{Id}\, C(f) \rangle \subseteq \langle f \rangle'$. Thus $A_f^{\infty 1} = \emptyset$. Suppose the set $\{n \in \mathrm{N}\colon n = \deg(x),\ x \in A_f^0\}$ is unbounded. Then by Lemma 3.5 we have $f \,.\, h \in \mathrm{Id}\, C(f)$ iff $h(x) = z_f$ for each $x \in A$, a contradiction again. Assume on the contrary there exists $a \in A_f^0$ with the property $\deg(x) \leqq \deg(a)$ for every $x \in A_f^0$. Putting $n = \deg(a)$ we get $f^{n+k} = f^n$ for each $k \in \mathrm{N}_0$, hence the semigroup $\langle f \rangle'$ is finite. This contradicts the supposition, hence $R(A,f) = 0$. Admit $A_f^{\infty 1} = \emptyset$. Then clearly for each $g \in \mathrm{Id}\, C(f)$ and every $x \in A$ there holds $\delta(x, g(x)) = 0$. Thus according to Lemma 3.5 we get $f \,.\, h \notin \mathrm{Id}\, C(f)$ for every $h \in C(f)$, hence $A_f^{\infty 1} \neq \emptyset$. Assume there exists an element $a \in A$ such that for a suitable $b \in A_f^{\infty 1}$ with $\delta(a, b) = 0$ the equality $f^k(a) = f^k(b)$ implies $k \geqq 2$. Denote by g an endomorphism of (A, f) satisfying the condition $g(a) = b$. Since $\langle f \rangle' \,.\, S(f) \subseteq \langle f \rangle'$ there exists a positive integer n

with the property $f\,.\,g(a) = f^n(a)$. But $f\,.\,g(a) = f(b) \neq f^k(a)$ for each $k \in \mathbf{N}_0$. This contradiction shows that $(A_f^{\infty 1}, \leqq_f)$ is a chain of the type $\omega_0^* \oplus \omega_0$ and $A = A_f^{\infty 1} \cup A_f^0$. Since $\mathrm{Id}\, C(f)$ is non-trivial, the set A_f^0 is non-empty.

3.7. Theorem. *Let (A, f) be a monounary c-algebra having at least two elements and such that $R(A, f) \leqq 1$. Put $S(f) = \langle f \rangle\,.\,\langle \mathrm{Id}\, C(f) \rangle$. The algebra (A, f) is reduced iff exactly one of the following conditions is satisfied:*

1° *The monoid $C(f)$ acts transitively on the set A.*

2° *$\langle f \rangle'$ is an infinite proper ideal of $S(f)$ and either it is a half-prime ideal of $C(f)$, where $f^2 \neq f$ implies $\operatorname{card} \langle f \rangle' = \aleph_0$, or the monoid $\langle \mathrm{Id}\, C(f) \rangle$ is non-trivial and to each $g \in \langle f \rangle'$ there exists $h \in C(f)$ with $g\,.\,h \in \mathrm{Id}\, C(f)$.*

Proof follows from Lemmas 3.2, 3.4 and 3.6.

Notice that if (A, f) is a reduced c-algebra with $A_f^0 \neq \emptyset$, i.e. the so called ordinal part is non-void, then the semigroup $\langle f \rangle'$ is a principal ideal generated by f in the monoid $S(f)$. Indeed, by Lemma 3.6 and the above remark we have $\langle f \rangle'\,.\,\langle \mathrm{Id}\, C(f) \rangle = \langle \mathrm{Id}\, C(f) \rangle\,.\,\langle f \rangle' = \langle f \rangle'$. Then $I_{S(f)}(f) = S^1(f)\,.\,f\,.\,S^1(f) = S(f)\,.\,f\,.\,S(f) = \langle f \rangle\,.\,\langle \mathrm{Id}\, C(f) \rangle\,.\,f\,.\,\langle f \rangle\,.\,\langle \mathrm{Id}\, C(f) \rangle = \langle f \rangle\,.\,\langle \mathrm{Id}\, C(f) \rangle\,.\,\langle f \rangle' \times \langle \mathrm{Id}\, C(f) \rangle = \langle f \rangle\,.\,\langle f \rangle' = \langle f \rangle'$.

The following theorem contains a characterization of a reduced c-algebra expressed in terms of groupoid using the binary operation ∇_f.

3.8. Theorem. *Let (A, f) be a monounary c-algebra such that $R(A, f) \leqq 1$,* $\operatorname{card} A \geqq 2$. *The algebra (A, f) is reduced iff exactly one of the following conditions is satisfied:*

1° *(A, ∇_f) is an ideal-simple groupoid without idempotents.*

2° *(A, ∇_f) is a commutative groupoid containing the least proper ideal I such that $(A/I, \nabla_I)$ is a BD-groupoid and if $I = I(a)$, $a \in A$ then $A = I \cup \sqrt{a}$ and $\mathrm{Id}\,(A, \nabla_f) \neq \emptyset$ is followed by $\mathrm{Id}\,(A, \nabla_f) = I$.*

Proof. Let (A, f) be a reduced c-algebra, $\operatorname{card} A \geqq 2$. Suppose first that (A, f) has the form (i) from Def. 3.1, $A_f^{\infty 2} = \{z_f\}$. Since $x \nabla_f z_f = f(x) = z_f = z_f \nabla_f x$ for every element $x \in A$, the singleton $\{z_f\}$ is the least proper ideal of the groupoid (A, ∇_f) and the factor-groupoid $(A/\{z_f\}, \nabla_f)$ is isomorphic to (A, ∇_f). Putting $I = \{z_f\}$, we get by Lemma 1.3. [5] that $(A/I, \nabla_I)$ is a BD-groupoid. Since $x \in A - I$ implies $x \nabla_f x = f(x) = z_f$ it holds $A = \sqrt{z_f} = I \cup \sqrt{z_f}$. The commutativity of the operation ∇_f is evident in this case. Thus (A, ∇_f) satisfies the condition 2°.

Suppose that the algebra (A, f) satisfies condition (ii) from Definition 3.1. If $A = A_f^{\infty 1}$ then for every element $x \in A$ it holds $x \nabla_f x = f(x) = x$. Admit that (A, ∇_f) contains a proper ideal I. For arbitrary $a \in A - I$ there exists $b \in A$, $b \neq a$ with $f(b) = a$. Since $x \in I$ implies $f(x) = x \nabla_f x \in I$, i.e. I is a subalgebra of (A, f), and since (A, f) is connected, there exists $k \in \mathbf{N}_0$ with $f^k(b) \in I$. From the definition of an ideal it follows $a = f(b) = b \nabla_f f^k(b) \in I$, which is a contradiction. Thus the groupoid (A, ∇_f) is ideal-simple.

Assume $A = A_f^{\infty 1} \cup A_f^0$, where $(A_f^{\infty 1}, \leqq_f)$ is a chain (of the type $\omega_0^* \oplus \omega_0$) and $A_f^0 \neq \emptyset$. Since $a \in A$, $b \in A$, $\delta(a, b) = 0$ is followed by the alternative $f(a) = f(b)$ or $a = b$, the groupoid (A, ∇_f) is commutative. For each element $x \in A$ there is $f(x) \in A_f^{\infty 1}$ thus $a \nabla_f x \in A_f^{\infty 1}$ for every pair of elements $a \in A_f^{\infty 1}$, $x \in A$ hence $A_f^{\infty 1}$ is an ideal of (A, ∇_f). Admit that there exists an ideal I of (A, ∇_f) with $I \subsetneqq A_f^{\infty 1}$. Let $a \in A_f^{\infty 1} - I$. If it were $f^n(a) \notin I$ for each $n \in \mathrm{N}$ then there would exist a natural number k and an element $b_k \in I$ such that $f^k(b_k) = a$. Let k be the least integer with this property. Then $b_k \in I$, $f(b_k) \notin I$ and thus $b_k \nabla_f a \notin I$, which is a contradiction.

Assume there is an integer $m_0 \geqq 1$ with $f^{m_0}(a) \in I$. Let $b \in A$, $f(b) = a$. Then $b \nabla_f f^{m_0}(a) = f(b) = a \notin I$, which is a contradiction again. Therefore $A_f^{\infty 1}$ is the least ideal of the groupoid (A, ∇_f). Clearly, $A_f^{\infty 1}$ contains more than only one generator. Denote by $(A/A_f^{\infty 1}, \nabla)$ the corresponding factorgroupoid of the groupoid (A, ∇_f). Then for a suitable idempotent c-algebra (B, g) we have $(A/A_f^{\infty 1}, \nabla) \cong$ $\cong (B, \nabla_g)$ thus $(A/A_f^{\infty 1}, \nabla)$ is a BD-groupoid by Lemma 1.3 [5].

Suppose that (A, f) satisfies condition (iii) in Definition 3.1. Without loss of generality we can suppose that $A_1 \neq \emptyset$. It is easy to see that A_2 is a principle ideal of (A, ∇_f) generated by the element c. Since $A_2 - \{c\}$ is not an ideal of (A, ∇_f) (if $a \in A - A_2$, $b \in A_2$ then $a \nabla_f b = f(a) = c$) and $A_2 - X$, where $X \subset A_2, c \notin X$, is not any carrier set of a subgroupoid we have that A_2 is the least ideal of (A, ∇_f). Further $(A/A_2, \nabla) \cong (A_1, \nabla_{f_1})$, where (A_1, f_1) is a c-algebra from (iii) def. 3.1, thus by Lemma 1.3 [5] $(A/A_2, \nabla)$ is a BD-groupoid. Let $b \in A - A_2 = A_f^0$. Then $b \nabla_f b = f(b) = c$, i.e. $A = I \cup \sqrt{c}$ where $I = A_2 = I(c)$ – the principal ideal generated by the element c. Therefore the condition 2° is satisfied again. If $\mathrm{Id}\,(A, \nabla_f) \neq \emptyset$ then $\mathrm{Id}\,(A, \nabla_f) = \{z_f\}$, where z_f is the only cyclic element of the c-algebra (A, f). Since (A, f) is reduced, it holds $f^2 = f$, hence $I = \{z_f\}$.

Now suppose that (A, f) is a c-algebra such that $R(A, f) \leqq 1$, card $A \geqq 2$ and (A, ∇_f) is an ideal-simple groupoid without idempotents (i.e. 1° holds). Then clearly $R(A, f) = 0$. Admit $A_f^0 = \emptyset$. Let $a \in A_f^0$. Put $B = A - \{a\}$. If $x \in A$, $y \in B$ are arbitrary elements then $x \nabla_f y \in B$, $y \nabla_f x \in B$ for $f(A) \subseteq B$, thus B is a proper ideal of (A, ∇_f) which contradicts the assumption. Hence $A = A_f^{\infty 1}$.

Suppose the groupoid (A, ∇_f) satisfies condition 2° where I is a principal ideal generated by $a \in A$. If $R(A, f) = 1$ then denoting by z_f the cyclic element of (A, f) and with respect to the minimality of I, we get $I = \{z_f\}$ and for each $x \in A$ it holds $f(x) = z_f \nabla_f x = z_f$, thus $f^2 = f$. Hence condition (i) from Definition 3.1 is satisfied.

Let $R(A, f) = 0$. Then $\mathrm{Id}\,(A, \nabla_f) = \emptyset$.

Suppose $A_f^{\infty 1} = \emptyset$. From the commutativity of the groupoid (A, ∇_f) it follows that for each $x \in A$ the set $f^{-1}(x) - A_f^0$ contains at most one element. Indeed, $x, y \in f^{-1}(a) - A_f^0$, $x \neq y$ implies the existence of a pair of different elements $x_1 \in$

$\in f^{-1}(x)$, $y_1 \in f^{-1}(y)$ such that $x_1 \nabla_f y_1 = f(y_1) = y \neq x = f(x_1) = y_1 \nabla_f x_1$. Then for each element $x \in A$ by the definition of S_f it holds $S_f(x) < \omega_0$, thus with respect to the connectedness of (A,f) there is $a \in A$ with $\emptyset \neq f^{-1}(a) \subseteq A_f^0$. Consider the set $I = \{f^k(a): k = 0, 1, 2, \ldots\}$. Since $f(x) \in I$ for every $x \in A$, I is an ideal of the groupoid (A, ∇_f). It can be easily shown, similarly as in the first part of this proof, that I is the least ideal of (A, ∇_f) and the factor-groupoid $(A/I, \nabla_I)$ is a BD-groupoid. The ideal I is a principal ideal generated by the element a, thus for each $x \in A$ with $x \neq f^n(a)$, $n \in \mathrm{N}_0$ from $A = I \cup \sqrt{a}$ it follows $f(x) = x \nabla_f x = a$. Therefore the algebra (A,f) is of the form (iii) from Definition 3.1.

Let $A \neq A_f^{\infty 1} \neq \emptyset$. Admit $I = I(a)$, where $a \in A$. If $b \in A$ is an element with the property $\delta(a, b) > 0$ then for each $x \in I$ it holds $\delta(x, b) > 0$ because $I(a) = \{f^k(a): k = 0, 1, 2, \ldots\}$, thus $x \nabla_f b = f(b) \neq f^n(a)$ for each $n \in \mathrm{N}_0$, i.e. $x \nabla_f b \notin I$, which is a contradiction. Consequently the ideal I is not principal. Admit there exists an element $x \in A_f^0$ with $f(x) \notin A_f^{\infty 1}$. Then there exists $y \in A_f^{\infty 1}$ with $\delta(x, y) = 0$, $f(x) \neq f(y)$ consequently $x \nabla_f y = f(y) \neq f(x) = y \nabla_f x$, which contradicts the commutativity. Hence $f(A_f^0) \subset A_f^{\infty 1}$. It follows also from the commutativity of the operation ∇_f that if $x, y \in A_f^{\infty 1}$, $\delta(x, y) = 0$, then $x = y$. Thus $A = A_f^{\infty 1} \cup A_f^0$, where $(A_f^{\infty 1}, \leqq_f)$ is a chain, i.e. the algebra (A,f) is reduced. The proof is complete.

We shall formulate another characterization (similar to Theorem 2.5 [5]) of a reduced c-algebra using notion of a weak radical in a groupoid (defined in § 1 [5]). The following theorem is a certain modification of the preceding one.

3.9. Theorem. *Let (A,f) be a monounary algebra minimal c-algebras of that are singletons and* card $A \geqq 2$. *Then (A,f) is a reduced c-algebra iff the grupoid (A, ∇_f) is either left ideal-simple without idempotents or it contains a proper minimal ideal I such that*

a) $\mathrm{rad}_w I = A$,

b) each element of I which is not the only generator of I possesses the unique square root in (I, ∇_f),

c) if I is a principal ideal generated by $a \in A$ then $x \in I$, $x \neq a$ is followed by $\sqrt{x} \subset I$ in (A, ∇_f).

Proof. Suppose (A,f) is a monounary algebra such that the groupoid (A, ∇_f) is left ideal-simple and does not contain idempotents. Since for every two components (A_1, f_1), (A_2, f_2) of a monounary algebra (A,f) and for $a \in A_1$, $b \in A_2$ there holds $a \nabla_f b = f(b)$, $b \nabla_f a = f(a)$ (by the assumption $R(A_i, f_i) \leqq 1$, $i = 1, 2$), the algebra (A,f) is connected. Hence condition 1° in Theorem 3.8 is satisfied. Suppose that (A, ∇_f) contains a minimal proper left ideal I with $\mathrm{rad}_w I = A$ and I is not principal. Since each component of (A,f) is a left ideal of (A, ∇_f) and the set $[a^n]$ is contained in the component containing a for each $n \in \mathrm{N}$, we get again that (A,f) is connected. It holds $f(a) = a \nabla_f [a^{n-1}] \in [a^n]$ for every integer $n \geqq 2$. Then $[a^n] \subset I$ for some

$n \geqq 2$ is followed by $f(a) \in I$, consequently $A - I = A_f^0$ with respect to the minimality of the ideal I. Since each $x \in I$ has the property card $(\sqrt{x} \cap I) = 1$, by Theorem 2.5 [5] (I, f_I) is a nested subalgebra of (A, f); it is a two-way infinite chain. Then $A = A_f^0 \cup A_f^{\infty 1}$, where $A_f^{\infty 1} = I$, thus (A, f) is a reduced c-algebra. If moreover $I = I(a)$ then evidently (I, f_I) is a one-way infinite chain and $A - I = \sqrt{a}$. Then (A, f) is of the form (iii) from Def. 3.1 thus (A, f) is reduced, too. From $I \neq \neq \mathrm{Id}\,(A, \nabla_f) \neq \emptyset$ it follows $R(A, f) = 1$ and for the cyclic element z_f of (A, f) it holds card $\sqrt{z_f} = 2$, which is a contradiction. Condition 2° from Theorem 3.8 is satisfied, therefore (A, f) is a reduced c-algebra.

Now suppose that (A, f) is a reduced c-algebra. If $A = A_f^{\infty 1}$ then the groupoid (A, ∇_f) is ideal-simple by Theorem 3.8 and since $x, y \in A$, $x \leqq {}_f y$ implies $x \nabla_f y = = y \nabla_f x$ we get easily that (A, ∇_f) is left ideal-simple. Further $\mathrm{Id}\,(A, \nabla_f) = \emptyset$. Assume $A \neq A_f^{\infty 1}$. Then condition 2° from Theorem 3.8 is satisfied. Let I be a proper ideal considered in 2° Theorem 3.8. Suppose I is not principal and $a \in \in A - I$. Since $x \in I$, $x \leqq {}_f y$ is followed by $y \in I$, there exists $b \in I$ such that $a < {}_f b$. Then $\delta(a, b) < 0$, $a \nabla_f a = f(a) = a \nabla_f b \in I$ and $[a^n] \subset I$ for each integer $n \geqq 2$. Then $a \in \mathrm{rad}_w I$, i.e. $\mathrm{rad}_w I = A$. Let $a \in I$. Since (A, ∇_f) is commutative, we have that $x, y \in A$, $\delta(x, y) = 0$ implies $f(x) = f(y)$. From the minimality of I it follows that (I, f_I) is a nested c-algebra (it is a two-way infinite chain). According to Theorem 2.5 [5] with respect to the fact that $\mathrm{Id}\,(A, \nabla_f) \neq \emptyset$ implies $I = \mathrm{Id}\,(A, \nabla_f)$, we get that each element of I possesses the unique square root in (I, ∇_f). Let $I = = I(\dot{a})$, $a \in A$. Similarly as above we get that $\mathrm{rad}_w I = A$ and $x \in I$ implies card $(\sqrt{x} \cap I) = 1$. Moreover, from the equality $A = I \cup \sqrt{a}$ it follows that $x \in I$, $x \neq a$ implies $\sqrt{x} \subset I$, q.e.d.

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