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ON FOCAL POINTS AND LIMIT BEHAVIOR OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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1. FOCAL POINTS OF HIGHER ORDER LINEAR EQUATIONS

The notion of focal point of a differential equation

(1) $$y^{(n)} + p_1(t)\, y^{(n-1)} + \dots + p_n(t)\, y = 0$$

generalizes the notion of dispersion of second or third kind studied by O. Borůvka [1] for second order equations. The notion has recently been used in several studies, see e.g., [2].

Let $x = x(t)$ be a vector of n linearly independent solutions of (1). The determinant of n vectors $v_1, \dots, v_n$ is denoted by $\det(v_1, \dots, v_n)$. For an equation (1) defined on $t_0 < t < \infty$, put $D_k(t, u) = \det\big(x(t), \dots, x^{(k-1)}(t), x^{(k)}(u), \dots, x^{(n-1)}(u)\big)$. The admissible changes of $x(t)$ are by linear transformations with nondegenerate matrix. In such a change, $D_k(t, u)$ is multiplied by the (non vanishing) determinant of the matrix. The zeros of $D_k(t, u)$ therefore have intrinsic meaning for (1).

Definition: u is a *k-focal point* of t for (1) if $u > t$ and $D_k(t, u) = 0$. *The* k-focal point of t is $f_k(t) = \inf u$, $D_k(t, u) = 0$, $u > t$. If no k-focal point exists, $f_k(t) = \infty$.

Proposition 1: *If $f_k(t) < \infty$ then for every k-focal point u there exists a scalar solution of* (1) *for which*

(2) $$y(t) = \dots = y^{(k-1)}(t) = y^{(k)}(u) = \dots = y^{(n-1)}(u) = 0.$$

Conversely, the validity of (2) *for a solution of* (1) *implies that u is a k-focal point of t for* (1).

In R^n we take a coordinate system for which the $(x_1, \dots, x_{n-1})$ hyperplane is an hyperplane through 0 and the endpoints of the vectors $x(t), \dots, x^{(k-1)}(t)$,

$x^{(k)}(u), \ldots, x^{(n-1)}(u)$. The x_n-coordinate axis has an arbitrary direction linearly independent of the directions in the hyperplane. Then the coordinate function $y = x_n(t)$ defined by the vector solution $x(t)$ has the desired property. Conversely, any vector of solutions of (1) that contains the given $y(t)$ as coordinate function $x_n(t)$ can be used to obtain $D_k(t, u) = 0$.

Focal points of second order differential equations are isolated. The next propositions show that the situation is more complicated for higher order equations.

In the following, we assume that the coefficients $p_i(t)$ are continuous. Then $D_k(t, u)$ is a continuously differentiable function of t and u. If $D_k(t, u) = 0$ and $(D_k(t, u))_u \neq 0$, we obtain locally a function $u = u(t)$ which is differentiable. In that case, $u' = -(D_k)_t/(D_k)_u$. For $n = 2$, these are the differentiation formulae of Borůvka ([1], § 13 no. 3). If $(D_k)_u = 0$, it may not be possible to obtain u as continuous function of t in an interval containing the given point $u = u_0$. In fact, if the $p_i(t)$ are continuously differentiable then $D_k(t, u)$ has a second order Peano–Taylor development

$$D_k(t + h, u + k) = D_k(t, u) + h(D_k(t, u))_t + k(D_k(t, u))_u +$$
$$+ \frac{1}{2}[h^2(D_k(t + \Theta h, u + \Theta k))_{tt} + 2hk(D_k(t + \Theta h, u + \Theta k))_{tu} +$$
$$+ k^2(D_k(t + \Theta h, u + \Theta k))_{uu}]$$

with $0 \leqq \Theta \leqq 1$. If $D_k(t, u) = (D_k(t, u))_u = (D_k(t, u))_{uu} = 0$, a variation of t by h in the equation $D_k(t, u) = 0$ implies a variation of u in the order of magnitude of $-(D_k)_t/(D_k)_{tu}$ which is bounded away from zero if $\operatorname{grad} D_k(t, u) \neq 0$. All these developments are valid in particular for $f_k(t)$:

Proposition 2: *A k-focal point u of t is isolated if* $\partial D_k(t, u)/\partial u \neq 0$. *The k-focal point* f_k *is differentiable if* $\partial D_k(t, f_k)/\partial u \neq 0$. *In that case,* $f_k'(t) = -(D_k(t, f_k))_t/(D_k(t, f_k))_u$.

Since the Wronskian of a basis of (1) is different from zero, the vector $x(t)$ and its first $n - 1$ derivatives form a basis of R^n and no arbitrarily short arc of the curve of endpoints of $x(t)$ can be restricted to an hyperplane of R^n. The vector product of $n - 1$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ in R^n is defined as the unique vector $c = [\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}]$ for which

$$c \,.\, \mathbf{v} = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{v})$$

for all $\mathbf{v}$ in R^n where . denotes the euclidean scalar product. To a vector function $x(t)$ derived from an equation (1) we associate the vector functions

$$x^*(t) = [x(t), \ldots, x^{(n-2)}(t)]$$

which is a solution vector for the adjoint equation of (1) and

$$x^+(t) = [x'(t), \ldots, x^{(n-1)}(t)],$$

which is the centroaffine polar as defined in [3] and satisfies a differential equation of the type of (1) if and only if $p_n(t) \neq 0$ on the interval of definition of $\boldsymbol{x}^*(t)$ (equation (9) of [3]). Clearly, $D_1(t, u) = \boldsymbol{x}(t) \,.\, \boldsymbol{x}^*(u)$. These notions are used in the next proposition.

Proposition 3: *If $p_n(t) \neq 0$ then the 1-focal points are isolated. In that case, $f_1(t)$ is differentiable unless $\boldsymbol{x}(t) \,.\, \boldsymbol{x}^*(f_1) = 0$.*

Under our hypotheses, $\boldsymbol{x}^*(u)$ is not restricted to an hyperplane of $\boldsymbol{R}^n$ for u in any neighborhood of a 1-focal point. Therefore, $\boldsymbol{x}(t_0) \,.\, \boldsymbol{x}^*(u) = 0$ can have only isolated zeros for fixed $t = t_0$.

By a change of parameter, (1) can be transformed into an equation

$$y^{(n)} + P_2(s)\, y^{(n-2)} + \dots + P_n(s)\, y = 0,$$

where $P_n(s) = 0$ if and only if $p_n(t(s)) = 0$. In terms of this new variable, defined by $\det\left(\boldsymbol{x}(t), \dots, \boldsymbol{x}^{(n-1)}(t)\right) (\mathrm{d}t/\mathrm{d}s)^{\frac{1}{2}n(n-1)} = 1$, we have

$$\frac{\partial}{\partial u} D_1(s, u) = P_n(u) \det\left(\boldsymbol{x}(s), \boldsymbol{x}'(u), \dots, \boldsymbol{x}^{(n-2)}(u), \boldsymbol{x}(u)\right) = (-1)^n P_n(u)\, \boldsymbol{x}(s) \,.\, \boldsymbol{x}^*(u).$$

The second statement of the proposition is now implied by proposition 2.

Remark: No result parallel to proposition 3 holds for $(n - 1)$-focal points. For a counter-example, we may take any n-th order equation with a polynomial solution of degree $\leqq n - 2$ (e.g., any equation with $p_n = 1$, $p_{n-1} = t$.) Then there exists a constant vector $\boldsymbol{c}$ for which $\boldsymbol{x}^{(n-1)} \,.\, \boldsymbol{c} = 0$ since $\boldsymbol{x}^{(n-1)}$ has a coordinate function which is identically zero. If we take intitial conditions at $t = t_0$ so that $\boldsymbol{x}^*(t_0) = \boldsymbol{c}$, it follows from $D_{n-1}(t, u) = \boldsymbol{x}^*(t) \,.\, \boldsymbol{x}^{(n-1)}(u)$ that $D_{n-1}(t_0, u) = 0$.

Similar theorems can be obtained for focal points defined by exchanging the places of t and u.

2. NOT-ROTARY SOLUTIONS OF LINEAR TWO–BY–TWO SYSTEMS

Let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a 2-vector and $P(t)$ a 2 by 2 matrix function. The system

$$\boldsymbol{x}'' = -P(t)\, \boldsymbol{x} \qquad (3)$$

describes the motion of a particle of mass 1 under the influence of the force $-P(t)\, \boldsymbol{x}$. It is also possible to reduce equations (1) to the form (3) for $n = 3,4$. Some properties of systems (3) relating to conjugate and focal points have been studied by the author in a recent book ([4], Sec. 8). Here we study some properties of solutions of (3) for $t \to \infty$. We assume $P(t)$ to be defined as continuous function for all t (or all $t \geqq t_0$). We introduce the vector $\boldsymbol{v}(\alpha) = \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}$ and the two quadratic forms

$$R(t, \alpha) = \mathbf{v}(\alpha)^T P(t)\, \mathbf{v}(\alpha),$$
$$Q(t, \alpha) = \det [\mathbf{v}(\alpha), P(t)\, \mathbf{v}(\alpha)].$$

For a solution of (3) we define $r(t)$ and $\alpha(t)$ by

(4) $$\boldsymbol{x}(t) = r(t)\, \mathbf{v}[\alpha(t)].$$

For typographical convenience, α shall always stand for $\alpha(t)$. For the Wronskian determinant

$$W(t) = \det(\boldsymbol{x}, \boldsymbol{x}') = r^2\alpha',$$

we obtain

(5) $$W' = -r^2 Q(t, \alpha).$$

A solution $\boldsymbol{x}(t)$ is *oscillatory* if $\boldsymbol{x}(t) \,.\, \boldsymbol{a} = 0$ has an infinity of solutions for some fixed vector $\boldsymbol{a}$. The solution is *rotary* if it is oscillatory for all $\boldsymbol{a}$. It follows from (5) that $W(t) \neq 0$ for all t if, e.g., Q is positive (negative) definite for all t and $W(t_0) \leqq 0$ $(\geqq 0)$. A study of these conditions can be found in [6].

By differentiation of (4) we obtain

(6) $$r'' + (R(t, \alpha) - \alpha'^2)\, r = 0.$$

For the scalar case, $P(t) = -q(t)\, I$ where I is the identity matrix, this is formula (5) of [1], § 5 no. 2.

We are interested in estimating the growth of r for $t \to \infty$ as function of the oscillatory or rotary behavior of a trajectory.

Proposition 4: *A trajectory $\boldsymbol{x}(t)$ of* (3) *that passes at most a finite number of times through the origin is rotary if and only if* $\left| \int_{t_0}^{\infty} W r^{-2}\, \mathrm{d}t \right| = \infty$.

For scalar equations, this is theorem 6.3 of [5]. The integral is $\left| \int_{t_0}^{\infty} \mathrm{d}\alpha(t) \right|$. The trajectory is rotary if and only if $\lim_{t\to\infty} \alpha(t) = \pm\infty$.

If $W(t) \neq 0$ and $\boldsymbol{x}(t)$ is not rotary then α' is of constant sign but $\lim_{t\to\infty} \alpha(t) < \infty$: $\boldsymbol{x}(t)$ cannot be oscillatory.

Proposition 5: *If $\boldsymbol{x}(t)$ is not rotary then* $\int_{t_0}^{\infty} W^2 r^{-4}\, \mathrm{d}t$ *is finite or infinite together with* $\int_{t_0}^{\infty} R[t, \alpha(t)]\, \mathrm{d}t$.

If $\boldsymbol{x}(t)$ is not rotary, the trajectory passes through 0 at most a finite number of times. Hence, the solution r of (6) is nonoscillatory and by a theorem of Leighton ([7], Chap. 1, prop. 6; [4], prop. 8–6)

$$\int_{t_0}^{\infty} (R - \alpha'^2)\, \mathrm{d}t < \infty.$$

The inequality would be violated if one of the integrals of proposition 5 were infinite and the other finite.

If $x(t)$ is not rotary and $W(t)$ does not change sign then α is monotone with a finite limit, $\alpha'(t) \to 0$. But that means that for large t we have $| Wr^{-2} | < 1$, i.e. $W^2r^{-4} < | Wr^{-2} |$. By proposition 4, $\int_{t_0}^{\infty} W^2r^{-4}\,\mathrm{d}t < \infty$. It follows that the integral over R cannot be infinite. By contraposition we obtain:

Proposition 6: *If* $\int_{t_0}^{\infty} R\,\mathrm{d}t = \infty$ *along a non-rotary trajectory* $x(t)$ *then* $\det(x(t), x'(t))$ *changes sign an infinity of times on every interval* (a, ∞).

Remark: If W is of constant sign and R is negative definite for all t then it follows from (6) that r either increases exponentially or decreases exponentially, depending on the initial conditions. In the first case, x is non-oscillatory and $\int^{\infty} r^{-n}\,\mathrm{d}t$ is finite for all n; in the second case, x is rotary and $\int^{\infty} r^{-n}\,\mathrm{d}t$ is infinite for all natural n.
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