Archivum Mathematicum

Svatoslav Staněk

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*Archivum Mathematicum*, Vol. 14 (1978), No. 2, 109--122

Persistent URL: http://dml.cz/dmlcz/106997

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIV: 109—122, 1978

# PHASE AND DISPERSION THEORY OF THE DIFFERENTIAL EQUATION $y'' = q(t)y$ IN CONNECTION WITH THE GENERALIZED FLOQUET THEORY

SVATOSLAV STANĚK, Olomouc

(Received June 17, 1977)

## 1. INTRODUCTION

The classical Floquet theory of the equations

$$\text{(q)} \qquad y'' = q(t)\,y, \qquad q \in C^0_{\mathbf{R}}, \qquad \mathbf{R} = (-\infty, \infty)$$

describes the properties of solutions of (q) when the function $q$ (i.e. the carrier of (q)) is periodic, usually with period $\pi : q(t + \pi) = q(t)$ for $t \in \mathbf{R}$. Then $u(t + \pi)$ is a solution of (q) for every solution $u$ of (q), too. According to the Floquet theory a quadratic algebraic equation can be uniquely associated to (q), whose roots – the so-called characteristic multipliers of (q) – are of importance in investigating the properties of solutions of (q). Provided that (q) is both-sided oscillatory, the characteristic multipliers of (q) can be calculated by means of the (first) phase and of the basic central dispersion (of the 1st kind) of (q). Cf. [2]–[5], [8].

O. Borůvka established in [1] all functions $X$, the so-called dispersions (of the 1st kind) of (q) possessing the property, by which $\frac{u[X(t)]}{\sqrt{|X'(t)|}}$ is a solution of (q) on $\mathbf{R}$ for every solution $u$ of the both-sided oscillatcry equation (q) again.

M. Laitoch generalized in [6] on the above basis the Floquet theory even to equations (q) whose carrier $q$ is in general no periodic function. To such (q) and $X$, it is possible uniquely associate an algebraic quadratic equation whose roots determine the behaviour of solutions of (q). (See [6]).

The main point of the present article is to calculate the above roots using the phase and dispersion theory for the 2nd order linear differential equations, getting thus as special cases the results of [2]–[5] and [8]. Next we investigate qualitative properties of the roots mentioned, making use of the dispersion $X$ and of the central dispersions of (q).

## 2. BASIC CONCEPTS AND RELATIONS

In what follows we shall investigate equations (q) being both-sided oscillatory on $\mathbf{R}$ only (i.e. every nonnull solution of (q) has an infinite number of zeros on the left and on the right of every $t_0 \in \mathbf{R}$). The trivial solution will be excluded.

In keeping with [1] we say that a function $\alpha : \mathbf{R} \to \mathbf{R}$, $\alpha \in C^0_{\mathbf{R}}$ is a (first) phase of (q) if there exist independent solutions $u, v$ of (q):

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)} \qquad \text{on } \mathbf{R} - \{t \in \mathbf{R};\ v(t) = 0\}.$$

Every phase $\alpha$ of (q) satisfies:

$$\alpha \in C^3_{\mathbf{R}}, \qquad \alpha'(t) \neq 0 \qquad \text{for } t \in \mathbf{R}, \qquad \alpha(\mathbf{R}) = \mathbf{R}.$$

If $\alpha$ is a phase of (q), then $\dfrac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $\dfrac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}}$ are independent solutions of (q) and $k_1 \dfrac{\sin(\alpha(t) + k_2)}{\sqrt{|\alpha'(t)|}}$ ($k_1, k_2$ = constants) is its general solution.

The set of phases of the equation $y'' = -y$ will be denoted by $\mathfrak{E}$. It holds: $\varepsilon(t + \pi) = \varepsilon(t) + \pi \,.\, \operatorname{sign} \varepsilon'$ for every $\varepsilon \in \mathfrak{E}$.

Let $t_0 \in \mathbf{R}$ and $u$ be a solution of (q), $u(t_0) = 0$. Let us denote by $\varphi(t_0)$ the first zero of $u$ lying on the right of $t_0$. Then the function $\varphi$ is defined on $\mathbf{R}$ and is called the basic central dispersion (of the 1st kind) of (q). The basic central dispersion $\varphi$ of (q) has the following properties:

$$\varphi \in C^3_{\mathbf{R}}, \qquad \varphi(t) > t, \qquad \varphi'(t) > 0 \qquad \text{for } t \in \mathbf{R}.$$

$\varphi_n(t)$ is the composite function $\underbrace{\varphi \ldots \varphi}_{n}(t)$ and $\varphi_{-n}(t)$ stands for the inverse function to $\varphi_n(t)$; $\varphi_0(t) \equiv t$ for $t \in \mathbf{R}$. The functions $\varphi_n$ ($n = \pm 1, \pm 2, \ldots$) are called the central dispersions (of the 1st kind) of (q).

Let $\alpha$ be a phase and $\varphi$ be the basic central dispersion of (q). Then $\alpha[\varphi_n(t)] = \alpha(t) + n\pi \,.\, \operatorname{sign} \alpha'$ for $t \in \mathbf{R}$, $n$ being an integer.

The funtion $X \in C^3_{\mathbf{R}}$, $X' \neq 0$ representing a solution of the nonlinear differential equation

$$\text{(qq)} \qquad \sqrt{|X'|} \left( \frac{1}{\sqrt{|X'|}} \right)'' + X'^2 \,.\, q(X) = q(t)$$

is called the dispersion (of the 1st kind) of (q). Especially the basic central dispersion $\varphi$ of (q) and $\varphi_n$ ($n = 0, \pm 1, \pm 2, \ldots$) are dispersions of (q). If the function $t + \pi$ is a dispersion of (q), then it is solution of (qq) whence we have: $q(t + \pi) = q(t)$ and conversely also: if $q(t + \pi) = q(t)$ for $t \in \mathbf{R}$, then $t + \pi$ is a dispersion of (q).

Let $\alpha$ be a phase of (q). Then $X$ is a dispersion of (q) precisely if $X = \alpha^{-1}\varepsilon\alpha$, $\varepsilon \in \mathfrak{E}$. Consequently $\alpha^{-1}\mathfrak{E}\alpha := \{\alpha^{-1}\varepsilon\alpha;\ \varepsilon \in \mathfrak{E}\}$ are all solutions of (qq).

Every dispersion $X$ of (q) maps $\mathbf{R}$ on $\mathbf{R}$ and possesses the following important property: Let $u$ be an arbitrary solution of (q), then the function $\dfrac{\mu[X(t)]}{\sqrt{|X'(t)|}}$ is a solution of this equation again. In case $X = \varphi_n$ it is even valid the formula

$$\frac{u[\varphi_n(t)]}{\sqrt{\varphi_n'(t)}} = (-1)^n u(t), \qquad t \in \mathbf{R}. \tag{1}$$

All the above results has been proved in [1].

## 3. PREPARATORY LEMMAS

Let $X(t) \not\equiv t$ be a dispersion of (q), $\varphi$ be the basic central dispersion of (q) and $u$, $v$ its independent solutions. Then $\dfrac{u[X(t)]}{\sqrt{|X'(t)|}}$ and $\dfrac{v[X(t)]}{\sqrt{|X'(t)|}}$ are independent solutions of (q) and

$$\begin{aligned} \frac{u[X(t)]}{\sqrt{|X'(t)|}} &= a_{11}u(t) + a_{12}v(t), \\ \frac{v[X(t)]}{\sqrt{|X'(t)|}} &= a_{21}u(t) + a_{22}v(t), \qquad t \in \mathbf{R}, \end{aligned} \tag{2}$$

where $a_{ij}$ ($i, j = 1, 2$) are real numbers and $\det a_{ij} = a_{11}a_{22} - a_{12}a_{21} \neq 0$. Let $y$ be a solution of (q) with $\dfrac{y[X(t)]}{\sqrt{|X'(t)|}} = \tau \,.\, y(t)$ for $t \in \mathbf{R}$, where $\tau$ is a (generally complex) number. Then $\tau$ is a root of the equation

$$\varrho^2 - (a_{11} + a_{22})\,\varrho + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{3}$$

The coefficients of (3) do not depend on the choice of the independent solutions $u$, $v$ of (q). We call equation (3) the characteristic equation of (q) relative to the dispersion $X$ and its roots as the characteristic multipliers of (q) relative to the dispersion $X$. In Lemma 4 we shall prove: $a_{11}a_{22} - a_{12}a_{21} = \operatorname{sign} X'$.

If $\varrho_{-1}, \varrho_1$ are characteristic multipliers of (q) relative to $X$, then it follows from [6] that there exist independent solutions $u, v$ of (q) satisfying either

$$(4)\qquad \frac{u[X(t)]}{\sqrt{|X'(t)|}} = \varrho_{-1}u(t), \qquad \frac{v[X(t)]}{\sqrt{|X'(t)|}} = \varrho_1 v(t), \qquad \varrho_{-1}\cdot\varrho_1 = \pm 1,$$

or

$$(5)\qquad \frac{u[X(t)]}{\sqrt{|X'(t)|}} = \varrho_1 u(t), \qquad \frac{v[X(t)]}{\sqrt{|X'(t)|}} = u(t) + \varrho_1 v(t), \qquad \varrho_{-1} = \varrho_1 = \pm 1.$$

Let $n$ be an integer. Say that $x \in \mathbf{R}$ is a number of type $n$ of (q) relative to the dispersion $X$ if $X(x) = \varphi_n(x)$.

**Lemma 1.** *Let $\alpha$ be a phase of* (q), $X = \alpha^{-1}\varepsilon\alpha$, *where* $\varepsilon \in \mathfrak{E}$. *Then $x$ is a number of type $n$ of* (q) *relative to the dispersion $X$ exactly if* $\varepsilon(x_1) = x_1 + n\pi \,.\, \operatorname{sign}\alpha'$ *for* $x_1 := \alpha(x)$.

Proof. ($\Rightarrow$) Let $X(x) = \varphi_n(x)$. Then $\alpha^{-1}\varepsilon\alpha(x) = \varphi_n(x) = \alpha^{-1}[\alpha(x) + n\pi \,.\, \operatorname{sign}\alpha']$. Herefrom we obtain $\varepsilon(x_1) = x_1 + n\pi \,.\, \operatorname{sign}\alpha'$ for $x_1 := \alpha(x)$.

($\Leftarrow$) Let $x_1 := \alpha(x)$ and $\varepsilon(x_1) = x_1 + n\pi \,.\, \operatorname{sign}\alpha'$. Then $\varepsilon\alpha(x) = \alpha(x) + n\pi \,.\, \operatorname{sign}\alpha'$, $\alpha^{-1}\varepsilon\alpha(x) = \alpha^{-1}[\alpha(x) + n\pi \,.\, \operatorname{sign}\alpha']$. From this $X(x) = \varphi_n(x)$.

**Corollary 1.** *Let* $\operatorname{sign} X' = 1$ *and let $x$ be a number of type $n$ of* (q) *relative to the dispersion $X$. Then $\varphi_i(x)$ is also number of type $n$ of* (q) *relative to the dispersion $X$, for every integer $i$.*

Proof. Let $x$ be a determined number of type $n$ of (q) relative to dispersion $X$, and so $X(x) = \varphi_n(x)$. Let $\alpha$ be a phase of (q) and let $X = \alpha^{-1}\varepsilon\alpha$, $\varepsilon \in \mathfrak{E}$. Then $\operatorname{sign}\varepsilon' = 1$ and we have from Lemma 1 $\varepsilon\alpha(x) = \alpha(x) + n\pi \,.\, \operatorname{sign}\alpha'$. It holds for every integer $i$ that $\varepsilon\alpha\varphi_i(x) = \varepsilon[\alpha(x) + i\pi \,.\, \operatorname{sign}\alpha'] = \varepsilon\alpha(x) + i\pi \,.\, \operatorname{sign}\alpha' = \alpha(x) + i\pi \,.\, \operatorname{sign}\alpha' + n\pi \,.\, \operatorname{sign}\alpha' = \alpha\varphi_i(x) + n\pi \,.\, \operatorname{sign}\alpha'$. From Lemma 1 and from $\varepsilon\alpha\varphi_i(x) = \alpha\varphi_i(x) + n\pi \,.\, \operatorname{sign}\alpha'$ we observe that $\varphi_i(x)$ is a number of type $n$ of (q) relative to the dispersion $X$, for every integer $i$.

**Lemma 2.** *Let* $\operatorname{sign} X' = 1$. *Then all number of* (q) *relative to the dispersion $X$* (*so far such exist*) *are of the same type.*

Proof. Suppose that $x$ and $y$ are numbers of types $n$ and $m$, respectively, of (q) relative to the dispersion $X$. This implies that $X(x) = \varphi_n(x)$, $X(y) = \varphi_m(y)$. Let $X = X = \alpha^{-1}\varepsilon\alpha$, $\varepsilon \in \mathfrak{E}$. Then $\operatorname{sign}\varepsilon' = 1$ and we get from Lemma 1: $\varepsilon(x_1) = x_1 + n\pi \,.\, \operatorname{sign}\alpha'$, $\varepsilon(y_1) = y_1 + m\pi \,.\, \operatorname{sign}\alpha'$, where $x_1 := \alpha(x)$, $y_1 := \alpha(y)$. From $\varepsilon(t + \pi) = \varepsilon(t) + \pi$ and $\varepsilon(x_1) = x_1 + n\pi \,.\, \operatorname{sign}\alpha'$ we obtain $t + (n \,.\, \operatorname{sign}\alpha' - 1)\pi < \varepsilon(t) < t + (n \,.\, \operatorname{sign}\alpha' + 1)\pi$ for $t \in \mathbf{R}$. In the special case of $t = y_1$ we have: $y_1 + (n \,.\, \operatorname{sign}\alpha' - 1)\pi < \varepsilon(y_1) = y_1 + m\pi \,.\, \operatorname{sign}\alpha' < y_1 + (n \,.\, \operatorname{sign}\alpha' + 1)\pi$ and from this $n \,.\, \operatorname{sign}\alpha' - 1 < m \,.\, \operatorname{sign}\alpha' < n \,.\, \operatorname{sign}\alpha' + 1$, which occurs exactly for $n = m$.

**Corollary 2.** *Let* sign $X' = 1$ *and let* $x$ *be a number of type* $n$ *of* (q) *relative to the dispersion* $X$. *Then*

$$\varphi_{n-1}(t) < X(t) < \varphi_{n+1}(t) \qquad \text{for } t \in \mathbf{R}.$$

Proof. Let $x$ be a number of type $n$ of (q) relative to the dispersion $X$, i.e. $X(x) = \varphi_n(x)$. Suppose now the assertion is not true. Then it follows from the continuity of the function $X$ that (q) relative to the dispersion $X$ possesses also a number of type $n - 1$ or $n + 1$, contrary to Lemma 2.

**Lemma 3.** *Let* sign $X' = -1$. *Then there exists only one* $x \in \mathbf{R}$: $X(x) = x$ *(this implies that there exists only one number of type* 0 *of* (q) *relative to the dispersion* $X$).

Proof. Equation $X(t) = t$ has only one solution on $\mathbf{R}$ for $X(\mathbf{R}) = \mathbf{R}$ and sign $X' = -1$.

**Lemma 4.** *Let* $x \in \mathbf{R}$ *and* $u, v$ *be solutions of* (q) *satisfying the initial conditions:* $u(x) = 1$, $u'(x) = 0$, $v(x) = 0$, $v'(x) = 1$. *Then*

$$\varrho^2 - \left(\frac{u[X(x)]}{\sqrt{|X'(x)|}} + \operatorname{sign} X' . \sqrt{|X'(x)|}\, v'[X(x)] - \frac{1}{2}\frac{X''(x)\, v[X(x)]}{X'(x)\sqrt{|X'(x)|}}\right)\varrho + \operatorname{sign} X' = 0 \tag{6}$$

*is the characteristic equation of* (q) *relative to the dispersion* $X$.

Proof. Let $u, v$ be solutions of (q) satisfying the initial conditions of Lemma 4. Then (2) holds where $a_{ij}$ $(i, j = 1, 2)$ are real numbers, $\det a_{ij} \neq 0$. Putting $x$ in place of $t$ in (2) we get

$$a_{11} = \frac{u[X(x)]}{\sqrt{|X'(x)|}}, \qquad a_{21} = \frac{v[X(x)]}{\sqrt{|X'(x)|}},$$

$$a_{12} = \frac{X'(x)\, u'[X(x)]}{\sqrt{|X'(x)|}} - \frac{1}{2}\frac{X''(x)\, u[X(x)]}{X'(x)\sqrt{|X'(x)|}} = \operatorname{sign} X' . \sqrt{|X'(x)|}\, u'[X(x)] - \frac{1}{2}\frac{X''(x)\, u[X(x)]}{X'(x)\sqrt{|X'(x)|}},$$

$$a_{22} = \frac{X'(x)\, v'[X(x)]}{\sqrt{|X'(x)|}} - \frac{1}{2}\frac{X''(x)\, v[X(x)]}{X'(x)\sqrt{|X'(x)|}} = \operatorname{sign} X' . \sqrt{|X'(x)|}\, v'[X(x)] - \frac{1}{2}\frac{X''(x)\, v[X(x)]}{X'(x)\sqrt{|X'(x)|}}.$$

From this

$$a_{11}a_{22} - a_{12}a_{21} = \operatorname{sign} X'[u(X(x)) . v'(X(x)) - u'(X(x)) . v(X(x))] = \operatorname{sign} X',$$

for $uv' - u'v = 1$. We get equation (6) by inserting the above results instead of $a_{ij}$ $(i, j = 1, 2)$ into (3).

**Remark 1.** In special case in which $X(t) = t + \pi$ Lemma 4 is given say in [4], [5], [7].

**Corollary 3.** *Let the assumptions of Lemma* 4 *where x is a number of type n* (*so far it exists*) *of* (q) *relative to the dispersion X, be satisfied. Then*

$$\varrho^2 - \left(\frac{u[X(x)]}{\sqrt{|X'(x)|}} + \operatorname{sign} X' \,.\, \sqrt{|X'(x)|}\, v'[X(x)]\right)\varrho + \operatorname{sign} X' = 0 \tag{7}$$

*is the characteristic equation of* (q) *relative to the dispersion X.*

Proof. Let $X(x) = \varphi_n(x)$. Then $v[X(x)] = v[\varphi_n(x)] = 0$ enabling us to write equation (6) in the form of (7).

**Corollary 4.** *If the characteristic equation* (6) *of* (q) *relative to the dispersion X has complex roots, then they are equal to* $e^{\pm a\pi i}$, $0 < a < 1$ *and* $\operatorname{sign} X' = 1$.

Proof. Let the roots of (6) be complex and equal to $\alpha \pm i\beta, \beta \neq 0$. Then $(\alpha + i\beta) - (\alpha - i\beta) = \alpha^2 + \beta^2 = \operatorname{sign} X'$. From this we get $\operatorname{sign} X' = 1, \alpha^2 + \beta^2 = 1$ and consequently $e^{\pm a\pi i}$, where $0 < a < 1$, are characteristic multipliers of (q) relative to the dispersion $X$.

## 4. MAIN RESULTS

Let $X(t) \not\equiv t$ be a dispersion of (q) and $\varphi$ be the basic central dispersion of (q).

**Theorem 1.** $e^{\pm a\pi i}$, $0 < a < 1$ *are the characteristic multipliers of* (q) *relative to the dispersion X if and only if there exists a phase* $\alpha$ *of* (q) *and an integer n:*

$$\alpha[X(t)] = \alpha(t) + (a + 2n)\,\pi, \qquad t \in \mathbf{R}.$$

Proof. ($\Rightarrow$) Let $e^{\pm a\pi i}$ be the characteristic multipliers of (q) relative to the dispersion $X$. Then there exist independent solutions $u$, $v$ of (q):

$$\begin{aligned} \frac{u[X(t)]}{\sqrt{|X'(t)|}} &= \cos a\pi \,.\, u(t) + \sin a\pi \,.\, v(t), \\ \frac{v[X(t)]}{\sqrt{|X'(t)|}} &= -\sin a\pi \,.\, u(t) + \cos a\pi \,.\, v(t). \end{aligned} \tag{8}$$

Let $\alpha \in C^0_{\mathbf{R}}$, $\operatorname{tg} \alpha(t) = \dfrac{u(t)}{v(t)}$ for $t \in \mathbf{R} - \{t \in \mathbf{R};\ v(t) = 0\}$. Then $\alpha$ is a phase of (q) and we have from (8) $\operatorname{tg} \alpha[X(t)] = \operatorname{tg}[\alpha(t) + a]$ and $\alpha[X(t)] = \alpha(t) + (a + k)\,\pi$, where $k$ is an integer. We now prove that $k$ is an even integer. First of all there exists $c \in \mathbf{R}$: $u(t) = \dfrac{c}{\sqrt{|\alpha'(t)|}} \sin \alpha(t)$, $v(t) = \dfrac{c}{\sqrt{|\alpha'(t)|}} \cos \alpha(t)$. Furthermore

$$\frac{u[X(t)]}{\sqrt{|X(t)|}} = \frac{c}{\sqrt{|\alpha'[X(t)] \,.\, X'(t)|}} \sin \alpha[X(t)] =$$

$$= \frac{c}{\sqrt{|(\alpha[X(t)])'|}} \sin [\alpha(t) + (a + k)\pi] = (-1)^k \frac{c}{\sqrt{|\alpha'(t)|}} \sin [\alpha(t) + a\pi]$$

and we get from (8)

$$\frac{u[X(t)]}{\sqrt{|X'(t)|}} = \cos a\pi \,.\, \frac{c}{\sqrt{|\alpha'(t)|}} \sin \alpha(t) + \sin a\pi \,.\, \frac{c}{\sqrt{|\alpha'(t)|}} \cos \alpha(t) =$$

$$= \frac{c}{\sqrt{|\alpha'(t)|}} \sin [\alpha(t) + a\pi].$$

Thus $(-1)^k = 1$ and $k$ is an even integer ($k = 2n$).

($\Leftarrow$) Let $0 < a < 1$, $n$ an integer. Let there exist a phase $\alpha$ of (q): $\alpha[X(t)] = \alpha(t) + (a + 2n)\pi$ for $t \in \mathbf{R}$. Then $u$, $v$, $u(t)\colon = \dfrac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $v(t)\colon = \dfrac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}}$, $t \in \mathbf{R}$ are independent solutions of (q) for which the equality of (8) holds. From (2), (3) and (8) now follows that $\varrho^2 - 2\cos a\pi \,.\, \varrho + 1 = 0$ is the characteristic equation of (q) relative to the dispersion $X$. $e^{\pm a\pi i}$ are its roots and consequently also the characteristic multipliers of (q) relative to the dispersion $X$.

**Remark 2.** Theorem 1 was proved for the dispersion $X(t) = t + \pi$ in [3] and [8].

**Corollary 5.** *Equation* (q) *relative to the dispersion $X$ possesses numbers* (*of type $n$*) *precisely if the characteristic multipliers of* (q) *relative to the dispersion $X$ are real.*

Proof. Let (q) relative to the dispersion $X$ possess real characteristic multipliers and $\tau$ be one of them. Then there exists a solution $u$ of (q): $\dfrac{u[X(t)]}{\sqrt{|X'(t)|}} = \tau \,.\, u(t)$ for $t \in \mathbf{R}$. Let $u(x) = 0$. Then $u[X(x)] = 0$, hence there exists a number $n$: $X(x) = \varphi_n(x)$ and $x$ is a number of type $n$ of (q) relative to the dispersion $X$. Let the characteristic multipliers of (q) relative to the dispersion $X$ be complex and equal to $e^{\pm a\pi i}$, $0 < a < 1$. Then there exists an integer $m$ and a phase $\alpha$ of (q): $\alpha[X(t)] = \alpha(t) + (a + 2m)\pi$. Let $x$ be a number of type $n$ of (q) relative to the dispersion $X$: $X(x) = \varphi_n(x)$. Then $\alpha[X(x)] = \alpha[\varphi_n(x)] = \alpha(x) + n\pi \,.\, \operatorname{sign} \alpha'$ which contradicts $\alpha[X(x)] = \alpha(x) + (a + 2m)\pi$.

**Theorem 2.** *Let* sign $X' = -1$, $X(x) = x$. *Then*

$$\varrho_{-1} = \frac{1}{\sqrt{-X'(x)}}, \qquad \varrho_1 = -\sqrt{-X'(x)}$$

*holds for the characteristic multipliers* $\varrho_{-1}$, $\varrho_1$ of (q) *relative to the dispersion* $X$.

Proof. Let sign $X' = -1$, $X(x) = x$ and $u$, $v$, be solutions of (q), $u(x) = 1$, $u'(x) = 0$, $v(x) = 0$, $v'(x) = 1$. Then $v[X(x)] = v(x) = 0$, $v'[X(x)] = v'(x) = 1$, $u[X(x)] = u(x) = 1$ and according to Lemma 4

$$\varrho^2 - \left(\frac{1}{\sqrt{|X'(x)|}} - \sqrt{|X'(x)|}\right)\varrho - 1 = 0 \tag{9}$$

is the characteristic equation of (q) relative to the dispersion $X$. $\dfrac{1}{\sqrt{-X'(x)}}$ and $-\sqrt{-X'(x)}$ are the roots of equation (9).

**Theorem 3.** *Let* sign $X' = 1$ *and* $x$ *be a number of type* $n$ *od* (q) *relative to the dispersion* $X$. *Then*

$$\varrho_{-1} = (-1)^n\sqrt{\frac{\varphi_n'(x)}{X'(x)}}, \qquad \varrho_1 = (-1)^n\sqrt{\frac{X'(x)}{\varphi_n'(x)}}$$

*holds for the (real) characteristic multipliers* $\varrho_{-1}$, $\varrho_1$ *of* (q) *relative to the dispersion* $X$.

Proof. Let $x$ be a number of type $n$ of (q) relative to dispersion $X$: $X(x) = \varphi_n(x)$ and $u$, $v$ be solutions of (q), $u(x) = 1$, $u'(x) = 0$, $v(x) = 0$, $v'(x) = 1$. By differentiating the latter equality

$$\frac{u[\varphi_n(t)]}{\sqrt{\varphi_n'(t)}} = (-1)^n u(t), \qquad \frac{v[\varphi_n(t)]}{\sqrt{\varphi_n'(t)}} = (-1)^n v(t), \qquad t \in \mathbf{R}, \tag{10}$$

we get

$$v'[\varphi_n(t)]\sqrt{\varphi_n'(t)} - v[\varphi_n(t)]\left(\frac{1}{\sqrt{\varphi_n'(t)}}\right)' = (-1)^n v'(t), \qquad t \in \mathbf{R}. \tag{11}$$

From (10) and (11) we have for $t = x$

$$u[X(x)] = (-1)^n\sqrt{\varphi^n(x)},$$

$$v'[X(x)] = (-1)^n\frac{1}{\sqrt{\varphi_n'(x)}}$$

for $v[\varphi_n(x)] = 0$. Therefore according to Corollary 3

$$\varrho^2 - (-1)^n\left(\sqrt{\frac{\varphi_n'(x)}{X'(x)}} + \sqrt{\frac{X'(x)}{\varphi_n'(x)}}\right)\varrho + 1 = 0$$

is the characteristic equation of (q) relative to the dispersion $X$. $(-1)^n\sqrt{\dfrac{\varphi_n'(x)}{X'(x)}}$ and

$(-1)^n \sqrt{\dfrac{X'(x)}{\varphi'(x)}}$ are its roots and thus also the characteristic multipliers of (q) relative to the dispersion $X$.

**Remark 3.** Theorem 3 generalizes the results of [2]–[5] proved for $X(t) = t + \pi$.

**Corollary 6.** *Let $\alpha$ be a phase of* (q) *and $x$ be a number of type* 0 *of* (q) *relative to the dispersion $X$, $X = \alpha^{-1}\varepsilon\alpha$, $\varepsilon \in \mathfrak{E}$. Then*

$$\varrho_{-1} = \frac{1}{\sqrt{|\varepsilon'(x_0)|}}, \qquad \varrho_1 = \operatorname{sign} \varepsilon' . \sqrt{|\varepsilon'(x_0)|} \qquad (x_0 = \alpha(x))$$

*are the characteristic multipliers of* (q) *relative to the dispersion $X$.*

Proof. Let $X = \alpha^{-1}\varepsilon\alpha$, $\varepsilon \in \mathfrak{E}$ and let $x$ be a number of type 0 of (q) relative to the dispersion $X$: $X(x) = x$. Then sign $X' =$ sign $\varepsilon'$. From Theorems 2 and 3 then follows that

$$\varrho_{-1} = \frac{1}{\sqrt{|X'(x)|}}, \qquad \varrho_1 = \operatorname{sign} \varepsilon' . \sqrt{|X'(x)|}$$

are the characteristic multipliers of (q) relative to the dispersion $X$. From $X' = \alpha^{-1\prime}\varepsilon\alpha . \varepsilon'\alpha . \alpha'$ and $\varepsilon\alpha(x) = \alpha(x)$ follows: $X'(x) = \varepsilon'\alpha(x) = \varepsilon'(x_0)$, where $x_0 := \alpha(x)$.

**Lemma 5.** *Equation* (q) *has two equal* (*real*) *characteristic multipliers relative to the dispersion $X$ and there exist independent solutions $u$, $v$ of* (q) *for which* (4) *holds iff there exists an integer $n$ such that $X(t) = \varphi_n(t)$ for $t \in \mathbf{R}$.*

Proof. ($\Rightarrow$) If this holds for an integer $n$ $X(t) = \varphi_n(t)$ for $t \in \mathbf{R}$, then it follows from Theorem 3 and (1) that (q) possesses independent solutions $u$, $v$ for which (4) holds.

($\Leftarrow$) If equation (q) relative to the dispersion $X$ has two equal characteristic multipliers, then there exists (according to Corollary 5) a number $x$ and an integer $n$: $X(x) = \varphi_n(x)$ and by Theorem 3 $(-1)^n$ is a double characteristic multiplier of (q) relative to the dispersion $X$. By our assumption, there exist independent solutions $u$, $v$ of (q) for which (4) holds. From this we find that for every solution $y$ of (q) $\dfrac{y[X(t)]}{\sqrt{X'(t)}} = (-1)^n y(t)$, respecting (1), we get $\dfrac{y[X(t)]}{\sqrt{X'(t)}} = \dfrac{y[\varphi_n(t)]}{\sqrt{\varphi_n'(t)}}$. Let $\alpha$ be a phase of (q), sign $\alpha' = 1$. Then for every $k$, $k \in \mathbf{R}$:

$$\frac{\sin(\alpha[X(t)] + k)}{\sqrt{X'(t) . \alpha'[X(t)]}} = \frac{\sin(\alpha[\varphi_n(t)] + k)}{\sqrt{\varphi_n'(t) . \alpha'[\varphi_n(t)]}}, \qquad t \in \mathbf{R}.$$

Consequently $X'(t) . \alpha'[X(t)] = \varphi_n'(t) . \alpha'[\varphi_n(t)]$, $\alpha[X(t)] = \alpha[\varphi_n(t)] + s\pi$, where $s$ is an integer. Since $X(x) = \varphi_n(x)$, it holds $s = 0$ and $X(t) = \varphi_n(t)$ for $t \in \mathbf{R}$.

**Lemma 6.** *Let* $\varrho_{-1} = \varrho_1 (= \varrho = \pm 1)$ *hold for the characteristic multipliers* $\varrho_{-1}$, $\varrho_1$ of (q) *relative to the dispersion* $X$ *and let* $u_1, v_1$ or $u_2, v_2$ *be pairs of independent solutions of* (q) *for which*

$$\frac{u_i[X(t)]}{\sqrt{X'(t)}} = \varrho \,.\, u_i(t),$$
(12)
$$\frac{v_i[X(t)]}{\sqrt{X'(t)}} = u_i(t) + \varrho \,.\, v_i(t), \qquad i = 1, 2.$$

*Then* sign $(u_1 v_1' - u_1' v_1) =$ sign $(u_2 v_2' - u_2' v_2)$.

Proof. Let $\varrho_{-1} = \varrho_1 (= \varrho = \pm 1)$ hold for the characteristic multipliers $\varrho_{-1}$, $\varrho_1$ of (q) relative to the dispersion $X$. Then it follows from Lemma 4: sign $X' = 1$. Let (12) hold for the pairs $u_1, v_1$ or $u_2, v_2$ of independent solutions of (q). Let $b_{ij}$ $(i, j = 1, 2)$, det $b_{ij} \neq 0$ be such numbers that

$$u_2(t) = b_{11} u_1(t) + b_{12} v_1(t),$$
$$v_2(t) = b_{21} u_1(t) + b_{22} v_1(t).$$

Then

$$\frac{u_2[X(t)]}{\sqrt{X'(t)}} = b_{11} \frac{u_1[X(t)]}{\sqrt{X'(t)}} + b_{12} \frac{v_1[X(t)]}{\sqrt{X'(t)}} = b_{11} \varrho u_1(t) + b_{12} (u_1(t) + \varrho v_1(t)) =$$
$$= (b_{12} + \varrho b_{11})\, u_1(t) + \varrho b_{12} v_1(t),$$

$$\frac{v_2[X(t)]}{\sqrt{X'(t)}} = b_{21} \frac{u_1[X(t)]}{\sqrt{X'(t)}} + b_{22} \frac{v_1[X(t)]}{\sqrt{X'(t)}} = b_{21} \varrho u_1(t) + b_{22} (u_1(t) + \varrho v_1(t)) =$$
$$= (b_{22} + \varrho b_{21})\, u_1(t) + \varrho b_{22} v_1(t)$$

and

$$\varrho b_{11} u_1(t) + \varrho b_{12} v_1(t) = (b_{12} + \varrho b_{11})\, u_1(t) + \varrho b_{12} v_1(t),$$
$$b_{11} u_1(t) + b_{12} v_1(t) + \varrho (b_{21} u_1(t) + b_{22} v_1(t)) = (b_{22} + \varrho b_{21})\, u_1(t) + \varrho b_{22} v_1(t).$$

Therefore $b_{12} = 0$, $b_{11} = b_{22} \neq 0$. Furthermore $u_2 v_2' - u_2' v_2 = b_{11} u_1 (b_{21} u_1' + b_{22} v_1') - b_{11} u_1' (b_{21} u_1 + b_{22} v_1) = b_{11} b_{22} (u_1 v_1' - u_1' v_1)$. Thus sign $(u_2 v_2' - u_2' v_2) =$ $=$ sign $(u_1 v_1' - u_1' v_1)$.

**Lemma 7.** *Equation* (q) *has independent solutions* $u$, $v$ *for which* (5) *holds exacty if there exist an integer* $n$ *and* $x \in \mathbf{R}$ *such that* $X(x) = \varphi_n(x)$, $X(t) \not\equiv \varphi_n(t)$ *for* $t \in \mathbf{R}$, sign $X' = 1$ *and* $\tau \,.\, (X(t) - \varphi_n(t)) \geqq 0$ *for* $t \in \mathbf{R}$, *where* $\tau = (-1)^n$ sign $(uv' - u'v)$.

Proof. ($\Rightarrow$) Let (q) have independent solutions $u$, $v$ for which (5) holds. Then $\varrho_{-1} = \varrho_1 = \varrho (= \pm 1)$ for the characteristic multipliers $\varrho_{-1}$, $\varrho_1$ of (q) relative to the dispersion $X$, sign $X' = 1$ (Lemma 4), there exist an integer $n$ and $x \in \mathbf{R}$: $X(x) = \varphi_n(x)$ (Corollary 5) and $X(t) \not\equiv \varphi_n(t)$ for $t \in \mathbf{R}$ (Lemma 5). Let $t_0$ be a number for which $X(t_0) \neq \varphi_n(t_0)$. Let $u_1$, $v_1$ be solutions of (q) satisfying the initial conditions $u_1(t_0) = 1$, $u_1'(t_0) = 0$, $v_1(t_0) = 0$, $v_1'(t_0) = 1$. Putting

$$u_2(t) := \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\, u_1(t) + \left(\varrho - \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\right) v_1(t),$$

$$v_2(t) := v_1(t), \qquad t \in \mathbf{R},$$

then $u_2$, $v_2$ are independent solutions of (q). Since $\varrho_{-1} = \varrho_1 = \varrho (= \pm 1)$, it follows from Lemma 4 and its proof:

$$\frac{u_1[X(t)]}{\sqrt{X'(t)}} = \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\, u_1(t) + \left(\sqrt{X'(t_0)}\, u_1'[X(t_0)] - \frac{1}{2}\,\frac{X''(t_0)\, u_1[X(t_0)]}{X'(t_0)\sqrt{X'(t_0)}}\right) v_1(t),$$

$$\frac{v_1[X(t)]}{\sqrt{X'(t)}} = \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\, u_1(t) + \left(\sqrt{X'(t_0)}\, v_1'[X(t_0)] - \frac{1}{2}\,\frac{X''(t_0)\, v_1[X(t_0)]}{X'(t_0)\sqrt{X'(t_0)}}\right) v_1(t)$$

and

$$\frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}} + \sqrt{X'(t_0)}\, v_1'[X(t_0)] - \frac{1}{2}\,\frac{X''(t_0)\, v_1[X(t_0)]}{X'(t_0)\sqrt{X'(t_0)}} = 2\varrho$$

from which

$$\frac{u_2[X(t)]}{\sqrt{X'(t)}} = \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\,\frac{u_1[X(t)]}{\sqrt{X'(t)}} + \left(\varrho - \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\right)\frac{v_1[X(t)]}{\sqrt{X'(t)}} =$$

$$= \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\left[\frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\, u_1(t) + \left(\sqrt{X'(t_0)}\, u_1'[X(t_0)] - \frac{1}{2}\,\frac{X''(t_0)\, u_1[X(t_0)]}{X'(t_0)\sqrt{X'(t_0)}}\right) v_1(t)\right] +$$

$$+ \left(\varrho - \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\right) \times$$

$$\times \left[\frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\, u_1(t) + \left(\sqrt{X'(t_0)}\, v_1'[X(t_0)] - \frac{1}{2}\,\frac{X''(t_0)\, v_1[X(t_0)]}{X'(t_0)\sqrt{X'(t_0)}}\right) v_1(t)\right] =$$

$$= \varrho\, \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\, u_1(t) + (v_1[X(t)]\, u_1'[X(t_0)] - v_1'[X(t_0)]\, u_1[X(t_0)])\, v_1(t) +$$

$$+ \varrho\left(2\varrho - \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\right) v_1(t) = \varrho\, \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\, u_1(t) - v_1(t) + 2v_1(t) - \varrho\, \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\, v_1(t) =$$

$$= \varrho\left[\frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\, u_1(t) + \left(\varrho - \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\right) v_1(t)\right] = \varrho u_2(t),$$

$$\frac{v_2[X(t)]}{\sqrt{X'(t)}} = \frac{v_1[X(t)]}{\sqrt{X'(t)}} = \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\, u_1(t) +$$

$$+\left(\sqrt{X'(t_0)}\,v_1'[X(t_0)] - \frac{1}{2}\frac{X''(t_0)\,v_1[X(t_0)]}{X'(t_0)\sqrt{X'(t_0)}}\right)v_1(t) =$$

$$= \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\,u_1(t) + \left(2\varrho - \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\right)v_1(t) =$$

$$= \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\,u_1(t) + \left(\varrho - \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\right)v_1(t) + \varrho v_1(t) = u_2(t) + \varrho v_2(t).$$

Thus, it holds (5) where we write $u_2$ and $v_2$ in place of $u$ and $v$. Further we have

$$u_2(t)\,v_2'(t) - u_2'(t)\,v_2(t) = \left[\frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\,u_1(t) + \left(\varrho - \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\right)v_1(t)\right]v_1'(t) -$$

$$-\left[\frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\,u_1'(t) + \left(\varrho - \frac{u_1[X(t_0)]}{\sqrt{X'(t_0)}}\right)v_1'(t)\right]v_1(t) =$$

$$= \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\left(u_1(t)\,v_1'(t) - u_1'(t)\,v_1(t)\right) = \frac{v_1[X(t_0)]}{\sqrt{X'(t_0)}}\,.$$

By Lemma 5 sign $(uv' - u'v) = \text{sign}\,(u_2v_2' - u_2'v_2) = \text{sign}\,v_1[X(t_0)]$. So, we have proved that $v_1[X(t_0)]$ is always of the same signs for every $t_0 \in \mathbf{R}$ for which $X(t_0) \neq \neq \varphi_n(t_0)$ and for the solution $v_1$ of (q) satisfying the initial conditions $v_1(t_0) = 0$, $v_1'(t_0) = 1$. By Corollary 2 $\varphi_{n-1}(t) < X(t) < \varphi_{n+1}(t)$ for $t \in \mathbf{R}$. Therefore $\tau.(X(t) - - \varphi_n(t)) \geqq 0$ for $t \in \mathbf{R}$, where $\tau := (-1)^n\,\text{sign}\,(uv' - u'v)$.

($\Leftarrow$) Let there exist an integer $n$ and $x \in \mathbf{R}$, such that $X(x) = \varphi_n(t)$, $X(t) \not\equiv \varphi_n(t)$ for $t \in \mathbf{R}$, sign $X' = 1$ and let $\tau\,.\,(X(t) - \varphi_n(t)) \geqq 0$, where $\tau = \pm 1$. The function $X(t) - \varphi_n(t)$ has at the point $t = x$ a local extreme, thus $X'(x) = \varphi_n'(x)$. Therefore by Theorem 3 $\varrho_{-1} = \varrho_1 = (-1)^n$ are the characteristic multipliers of (q) relative to the dispersion $X$ and from Lemma 5 follows the existence of independent solutions $u$, $v$ of (q) for which (5) holds. We proceed in the same manner as we did in proving ($\Rightarrow$) to prove $\tau = (-1)^n\,\text{sign}\,(uv' - u'v)$.

From Lemma 7 we obtain

**Corollary 7.** *Equation* (q) *possesses independent solutions $u$, $v$ for which* (5) *holds iff for any integer $n$ $X(t) \not\equiv \varphi_n(t)$ for $t \in \mathbf{R}$,* sign $X' = 1$ *and*

$$\min_{t\in\mathbf{R}} \tau\,.\,(X(t) - \varphi_n(t)) = 0 \qquad (\tau = \pm 1).$$

**Theorem 4.** It holds:

a) *Equation* (q) *possesses complex characteristic multipliers relative to the dispersion $X$ precisely if for any integer $n$ $\varphi_{n-1}(t) < X(t) < \varphi_n(t)$ for $t \in \mathbf{R}$.*

b) *Equation* (q) *possesses two different real characteristic multipliers relative to the dispersion* $X$ *exactly if either* $\operatorname{sign} X' = -1$, *or* $\operatorname{sign} X' = 1$ *and if there exists an integer* $n$ *such that the function* $X(t) - \varphi_n(t)$ *changes its sign on* $\mathbf{R}$.

c) *Equation* (q) *possesses two equal (real) characteristic multipliers relative to the dispersion* $X$ *and there exist independent solutions* $u, v$ of (q) *for which* (5) *holds exactly if there exists an integer* $n$ *such that* $X(t) \not\equiv \varphi_n(t)$ *for* $t \in \mathbf{R}$, $\operatorname{sign} X' = 1$ *and* $\min_{t\in\mathbf{R}} \tau \times (X(t) - \varphi_n(t)) = 0$, *where* $\tau = \pm 1$.

d) *Equation* (q) *possesses two equal (real) characteristic multipliers relative to the dispersion* $X$ *and there exist independent solutions* $u, v$ of (q) *for which* (4) *holds exactly if there exists an integer* $n$ *such that* $X(t) = \varphi_n(t)$ *for* $t \in \mathbf{R}$.

Proof. a) According to Theorem 1 equation (q) relative to the dispersion $X$ possesses complex characteristic multipliers iff there exists a phase $\alpha$ of (q), an integer $m$ and a number $a$, $0 < a < 1$: $\alpha[X(t)] = \alpha(t) + (a + 2m)\pi$. It holds further $\alpha[\varphi_{2m.\operatorname{sign}\alpha'}(t)] = \alpha(t) + 2m\pi$, $\alpha[\varphi_{(2m+1)\operatorname{sign}\alpha'}(t)] = \alpha(t) + (2m + 1)\pi$. Therefore $\alpha[\varphi_{2m.\operatorname{sign}\alpha'}(t)] < \alpha[X(t)] < \alpha[\varphi_{(2m+1)\operatorname{sign}\alpha'}(t)]$. If $\operatorname{sign}\alpha' = 1$, then $\varphi_{2m}(t) < X(t) < \varphi_{2m+1}(t)$. If $\operatorname{sign}\alpha' = -1$, then $\varphi_{-2m-1}(t) < X(t) < \varphi_{-2m}(t)$. Suppose now that there exists an integer $n$ such that $\varphi_{n-1}(t) < X(t) < \varphi_n(t)$ for $t \in \mathbf{R}$. Then (q) relative to the dispersion $X$ has no determined number and by Corollary 5 the characteristic multipliers of (q) relative to the dispersion $X$ are complex.

b) It follows from Theorems 2 and 3, from Lemma 3, from Corollary 2 and from the fact that (q) relative to the dispersion $X$, $\operatorname{sign} X' = 1$ possesses two different real characteristic multipliers exactly if $X'(x) \neq \varphi_n'(x)$ in numbers $x$ (of type $n$) of (q) relative to the dispersion $X$.

c) It follows from Corollary 7.

d) It has been proved in Lemma 5.

**Corollary 8.** *Let* $\alpha$ *be a phase of* (q) *and* $X = \alpha^{-1}\varepsilon\alpha (\varepsilon \in \mathfrak{E})$ *be a dispersion of* (q). *Then:*

a) *Equation* (q) *possesses complex characteristic multipliers relative to the dispersion* $X$ *iff there exists an integer* $n$ *such that* $(n - 1)\pi < \operatorname{sign}\alpha' . (\varepsilon(t) - t) < n\pi$ *for* $t \in \mathbf{R}$.

b) *Equation* (q) *possesses two equal (real) characteristic multipliers relative to the dispersion* $X$ *and there exist independent solutions* $u, v$ *of* (q) *for which* (5) *holds iff there exists an integer* $n$ *such that* $\varepsilon(t) \not\equiv t + n\pi . \operatorname{sign}\alpha'$, $\operatorname{sign}\varepsilon' = 1$ *and* $\min_{t\in\mathbf{R}} \tau \times (\varepsilon(t) - t - n\pi . \operatorname{sign}\alpha') = 0$, *where* $\tau = \pm 1$.

c) *Equation* (q) *possesses two equal (real) characteristic multipliers relative to the dispersion* $X$ *and there exist independent solutions* $u, v$ of (q) *for which* (4) *holds iff there exists an integer* $n$ *such that* $\varepsilon(t) = t + n\pi . \operatorname{sign}\alpha'$ *for* $t \in \mathbf{R}$.

d) *Equation* (q) *possesses two different real characteristic multipliers relative to the dispersion* $X$ *iff either* $\operatorname{sign}\varepsilon' = -1$ *or* $\operatorname{sign}\varepsilon' = 1$ *and if there exists an integer* $n$ *such that the function* $\varepsilon(t) - t - n\pi . \operatorname{sign}\alpha'$ *changes its sign on* $\mathbf{R}$.

Proof. Let $\alpha$ be a phase of (q) and $X = \alpha^{-1}\varepsilon\alpha$ ($\varepsilon \in \mathfrak{E}$). Then $\varphi(t) = \alpha^{-1}(\alpha(t) + \pi . \operatorname{sign} \alpha')$ is the basic central dispersion of (q) and $\varphi_n(t) = \alpha^{-1}(\alpha(t) + n\pi . \operatorname{sign} \alpha')$. We have next sign $\varepsilon'$ = sign $X'$ and $X(x) = \varphi_n(x)$ iff $\varepsilon(x_1) = x_1 + n\pi . \operatorname{sign} \alpha'$, where $x_1 = \alpha(x)$. Corollary 8 immediately follows from Theorem 4.

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*S. Staněk,*
*771 46 Olomouc, Leninova 26*
*Czechoslovakia*