Archivum Mathematicum

Po Fang Hsieh

On asymptotic integrations of $x^2y'' - P(x)y = 0$

Archivum Mathematicum, Vol. 14 (1978), No. 2, 75--83

Persistent URL: `http://dml.cz/dmlcz/106994`

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIV: 75—84, 1978

ON ASYMPTOTIC INTEGRATIONS OF $x^2y'' - P(x)y = 0$

By PO-FANG HSIEH

(Received December 15, 1976)

1. Introduction. In the past decade since the publication of the paper by P. F. Hsieh and Y. Sibuya [2], substantial achievements have been made in the global study of the second order equation of the form

$$(1.1)\qquad y'' - Q(x)\,y = 0, \qquad \left('' = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)$$

where x is a complex variable, and

$$(1.2)\qquad Q(x) = x^m + a_1x^{m-1} + \ldots + a_m, \qquad m\text{: positive integer.}$$

A good collection of results in this direction can be found in the recent book of Y. Sibuay [5]. A similar study on an n-th order equation is done by B. L. J. Braaksma [1].

In this paper, we shall study the asymptotic integrations of

$$(\mathrm{E})\qquad x^2y'' - P(x)\,y = 0,$$

where $P(x)$ is an m-th degree polynomial

$$(1.3)\qquad P(x) = x^m + a_1x^{m-1} + \ldots + a_m, \qquad m\text{: positive integer,}$$

with $a_1, a_2, \ldots, a_m$ complex parameters. First, let

$$(1.4)\qquad \{x^{-m}P(x)\}^{\frac{1}{2}} = \{1 + \sum_{h=1}^{m} a_hx^{-h}\}^{\frac{1}{2}} = 1 + \sum_{h=1}^{\infty} b_hx^{-h}.$$

Then, b_h are polynomials of $a_1, a_2, \ldots, a_m$. We shall prove the following

Theorem 1. *The differential equation (E) has a solution*

$$(1.5)\qquad y = y_m(x, a_1, a_2, \ldots, a_m)$$

This work is partially supported by a Faculty Research Fellowship at Western Michigan University.

such that

(i) y_m *is entire in* $(a_1, a_2, \dots, a_m)$ *and holomorphic in* x *for*

$$(1.6) \qquad |x| > 0, \qquad |\arg x| < \pi;$$

(ii) y_m *and* y'_m *admit respectively the asymptotic representations*

$$(1.7) \qquad y_m \cong x^{r_m}\{1 + \sum_{n=1}^{\infty} B_{mn}x^{-\frac{1}{2}n}\} \exp\{-2m^{-1}x^{\frac{1}{2}m} + \sum_{n=1}^{m-1} A_{mn}x^{\frac{1}{2}(m-n)}\},$$

$$(1.8) \qquad y'_m \cong -x^{\frac{1}{2}m-1+r_m}\{1 + \sum_{n=1}^{\infty} C_{mn}x^{-\frac{1}{2}n}\} \exp\{-2m^{-1}x^{\frac{1}{2}m} + \sum_{n=1}^{m-1} A_{mn}x^{\frac{1}{2}(m-n)}\}$$

uniformly on each compact set in $(a_1, a_2, \dots, a_m)$ – *space as* x *tends to infinity in any closed sector which is contained in*

$$(1.9) \qquad |x| > 0, \qquad |\arg x| < 3m^{-1}\pi$$

where

$$(1.10) \qquad r_m = \begin{cases} -\dfrac{m}{4} + \dfrac{1}{2}, & m;\ \text{odd}, \\ -\dfrac{m}{4} + \dfrac{1}{2} - b_{\frac{1}{2}m}, & m;\ \text{even}, \end{cases}$$

with $b_{\frac{1}{2}m}$, $A_{mn}B_{mn}$ *and* C_{mn} *polynomials of* $a_1, a_2, \dots, a_m$.

A similar problem has been proved by F. E. Mullin [3]. However, the quantily r_m was not given as explicitly there. The case of $m = 2$ and $m = 3$ are studied recently by T. Okada [4]. It is noteworthy that a Bessel differential equation

$$x^2w'' + xw' + (x^2 - n^2)w = 0$$

can be transformed by $w = x^{-\frac{1}{2}}y$ to

$$x^2y'' + \left(x^2 - n^2 + \frac{1}{4}\right)y = 0,$$

which is a type of (E) with $m = 2$.

2. Solutions in other sectors.

Put

$$(2.1) \qquad \hat{x} = e^{i\theta}x,$$

then (E) is reduced to

$$(2.2) \qquad \hat{x}^2\frac{d^2y}{d\hat{x}^2} - e^{im\theta}(\hat{x}^m + a_1e^{i\theta}\hat{x}^{m-1} + \dots + a_me^{im\theta})\,y = 0.$$

If we choose Θ satisfying $e^{im\Theta} = 1$, then $y_m(\hat{x}, e^{i\Theta}a_1, \dots, e^{im\Theta}a_m)$ is also a solution of (E). Let

$$\Theta_k = 2km^{-1}\pi, \qquad k = 0, 1, 2, \dots, m-1, \tag{2.3}$$

and

$$y_{m,k}(x, a_1, \dots, a_m) = y_m(e^{i\Theta_k}x, e^{i\Theta_k}a_1, e^{2i\Theta_k}a_2, \dots, e^{mi\Theta_k}a_m). \tag{2.4}$$

Denote the right hand side of (1.7) by $Y_m(x, a_1, \dots, a_m)$. Then we have the following.

Theorem 2. *The differential equation* (E) *has a solution $y_{m,k}$ satisfies the following conditions:*

(i) *$y_{m,k}$ is entire in $(a_1, a_2, \dots, a_m)$ and holomorphic in x for*

$$|x| > 0, \qquad |\arg x + \Theta_k| < 3m^{-1}\pi; \tag{2.5}$$

(ii) *$y_{m,k}$ and $y'_{m,k}$ admit respectively the asymptotic representation*

$$y_{m,k} \cong Y_m(e^{i\Theta_k}x, e^{i\Theta_k}a_1, e^{2i\Theta_k}a_2, \dots, e^{mi\Theta_k}a_m) \tag{2.6}$$

$$y'_{m,k} \cong e^{i\Theta_k}Y'_m(e^{i\Theta_k}x, e^{i\Theta_k}a_1, e^{2i\Theta_k}a_2, \dots, e^{mi\Theta_k}a_m) \tag{2.7}$$

uniformly on each compact set in $(a_1, a_2, \dots, a_m)$ − space as x tends to infinity in any closed sector which is contained in (2.5).

3. Preliminary transformations and a nonlinear equation. We shall prove Theorem 1 similar to the method in [2], as the regular singular point at $x = 0$ does not affect the asymptotic solutions at $x = \infty$. Same approach was used also in [3].

First, we shall write (E) as a system of equations. Let

$$u = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A(x) = \begin{pmatrix} 0 & 1 \\ x^{-2}P(x) & 0 \end{pmatrix}. \tag{3.1}$$

Then (E) becomes

$$u' = A(x)\,u. \tag{3.2}$$

Put

$$x = \xi^2 \quad \text{and} \quad u = \begin{pmatrix} 1 & 0 \\ 0 & \xi^{m-2} \end{pmatrix} z. \tag{3.3}$$

Then, (3.2) becomes

$$\frac{dz}{d\xi} = \{\xi^{m-1}\sum_{k=0}^{2m} A_k\xi^{-k}\}\,z. \tag{3.4}$$

where A_k are 2 by 2 matrices linear in $a_1, a_2, \dots, a_m$. In particular

$$A_0 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Let

$$z = Vw, \qquad V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{3.5}$$

Then, (3.4) becomes

$$\frac{dw}{d\xi} = \xi^{m-1} B(\xi)\, w, \tag{3.6}$$

where

$$B(\xi) = \sum_{k=0}^{2m} B_k \xi^{-k}, \qquad B_k = V^{-1} A_k V$$

and, in particular,

$$B_0 = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Put

$$B(\xi) = \begin{pmatrix} \alpha_1(\xi) & \beta_1(\xi) \\ \beta_2(\xi) & \alpha_2(\xi) \end{pmatrix}. \tag{3.7}$$

Then, $\alpha_i(\xi)$, $\beta_i(\xi)$ are linear in $a_1, a_2, \ldots, a_m$ and polynomials in ξ^{-1}. Furthermore, we have

$$\begin{cases} \alpha_1(\xi) = -2 + 0(\xi^{-1}), & \beta_1(\xi) = 0(\xi^{-1}), \\ \alpha_2(\xi) = 2 + 0(\xi^{-1}), & \beta_2(\xi) = 0(\xi^{-1}). \end{cases} \tag{3.8}$$

Now, put

$$w = \begin{pmatrix} 1 \\ p \end{pmatrix} \exp \{\int^{\xi} \eta^{m-1} \gamma(\eta)\, d\eta\} \tag{3.9}$$

into (3.6). Then we have

$$\gamma = \alpha_1 + \beta_1 p \tag{3.10}$$

and

$$\frac{dp}{d\xi} = \xi^{m-1}\{\beta_2 + \alpha_2 p - \gamma p\}. \tag{3.11}$$

Substitute (3.10) into (3.11), we obtain a nonlinear equation

$$\frac{dp}{d\xi} = \xi^{m-1}\{\beta_2 + (\alpha_2 - \alpha_1)\, p - \beta_1 p^2\}. \tag{3.12}$$

If we determine $p(\xi)$ by (3.12) and then use (3.10) to determine $\gamma(\xi)$, the quantity $w(\xi)$ in (3.9) is a solution of (3.6).

4. Existence and uniqueness of solution (3.12).

The equation (3.12) has the following form:

$$\frac{dp}{d\xi} = \xi^{m-1}\{f(\xi) + g(\xi)\, p + h(\xi)\, p^2\},$$

where f, g, h are linear functions of $a_1, a_2, \ldots, a_m$ and polynomials of ξ^{-1} such that

$$f(\xi) = 0(\xi^{-1}), \qquad g(\xi) = g_0 + 0(\xi^{-1}); \qquad h(\xi) = 0(\xi^{-1}).$$

and g_0 is a nonzero constant. Here $g_0 = 4$.

We shall state a fundamental lemma concerning such a nonlinear differential equations whose proof may be found in detail in [2].

Lemma. *Let f, g and h be polynomials in ξ^{-1} whose coefficients are linear in $a_1, a_2, \ldots, a_n$. Suppose that*

$$f(\xi) = 0(\xi^{-1}), \qquad g(\xi) = g_0 + 0(\xi^{-1}), \qquad h(\xi) = 0(\xi^{-1})$$

where g_0 is a nonzero constant independent of $a_1, a_2, \ldots, a_m$. Then the differential equation

$$\frac{dp}{d\xi} = \xi^{m-1}\{f(\xi) + g(\xi)\, p + h(\xi)\, p^2\}, \tag{4.1}$$

has the unique formal solution

$$\hat{p}(\xi) \sim \sum_{n=1}^{\infty} p_n \xi^{-n}, \tag{4.2}$$

where the quantities p_n are polynomial of $a_1, a_2, \ldots, a_m$ and independent of ξ.

Let δ be a sufficiently small positive constant. Then there exists a unique solution $p(\xi)$ of (4.1) *which satisfies*:

(i) *for each positive constant r, there exists a positive constant N_r such that $p(\xi)$ is holomorphic with respect to $(\xi, a_1, \ldots, a_m)$ in the domain defined by*

$$\begin{gathered} |\,\xi\,| > N_r, \qquad |\,a_1\,| + |\,a_2\,| + \ldots + |\,a_m\,| < N_r, \qquad (0 < r < \infty), \\ |\arg g_0 + m \arg \xi\,| \leqq \frac{3\pi}{2} - \delta; \end{gathered} \tag{4.3}$$

(ii) *$p(\xi) \cong \hat{p}(\xi)$ uniformly on each compact set in $(a_1, a_2, \ldots, a_m)$ − space as ξ tends to infinity in the sector*

$$|\arg g_0 + m \arg \xi\,| \leqq \frac{3\pi}{2} - \delta. \tag{4.4}$$

Applying this lemma, we find that equation (3.12) admits a solution $p(\xi)$ such that (i) for each $r > 0$ and each δ sufficiently small, there exists a positive number $N_{r,\delta}$ such that $p(\xi)$ is holomorphic with respect to $(\xi, a_1, \ldots, a_m)$ in the domain defined by

$$(4.5)\qquad \begin{cases} |\xi| > N_{r,\delta}, & |\arg\xi| \leqq \dfrac{3\pi}{2m} - \delta, \\ |a_1|^2 + \ldots + |a_m|^2 < r, & (0 < r < \infty), \end{cases}$$

(ii) we have

$$p(\xi) \cong \sum_{n=1}^{\infty} p_n \xi^{-n}$$

uniformly on each compact subset in $(a_1, \ldots, a_m)$-space as ξ tends to infinity in the sector

$$(4.6)\qquad |\arg\xi| \leqq \frac{3\pi}{2m} - \delta,$$

where p_n are polynomials of $a_1, \ldots, a_m$ and independent of ξ.

But $\gamma(\xi)$ is now found from (3.10) to be

$$\gamma(\xi) = \alpha_1(\xi) + p(\xi)\,\beta_1(\xi).$$

Hence $\gamma(\xi)$ is holomorphic in (4.5), and we have

$$(4.7)\qquad \gamma(\xi) \cong -2 + \sum_{n=1}^{\infty} \gamma_n \xi^{-n}$$

uniformly on each compact set in the $(a_1, \ldots, a_m)$-space as ξ tends to infinity in (4.6), where γ_n are polynomials of $a_1, \ldots, a_m$ and independent of ξ

Let

$$(4.8)\qquad \tilde{\gamma}(\xi) = \gamma(\xi) - [-2 + \sum_{n=1}^{m} \gamma_n \xi^{-n}]$$

and

$$(4.9)\qquad E(\xi) = \xi^{\gamma m} \exp\left\{ -\frac{2}{m}\xi^m + \sum_{n=1}^{m-1} \frac{\gamma_n}{m-n}\xi^{m-n} \right\}.$$

Then

$$(4.10)\qquad w(\xi) = \left(\frac{1}{p(\xi)}\right) E(\xi) \exp\{\int_{\infty}^{\xi} \eta^{m-1}\tilde{\gamma}(\eta)\,d\eta\}$$

is a solution of (3.6), where the path of integration lies in the sector (4.6). Clearly, this is holomorphic with respect to $(\xi, a_1, \ldots, a_m)$ in (4.5) and

$$(4.11)\qquad w \cong \{w_0 + \sum_{n=1}^{\infty} w_n \xi^{-n}\}\, E(\xi),$$

uniformly in each compact subset of the $(a_1, \ldots, a_m)$-space as ξ tends to infinity in (4.6) where w_n are two dimensional vectors whose elements are polynomials of $a_1, \ldots, a_m$ and independent of ξ and

(4.12)
$$w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If we now let

(4.13)
$$u(x) = \begin{pmatrix} 1 & 0 \\ 0 & \xi^{m-2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ p(\xi) \end{pmatrix} E(\xi) \exp \{\int_{\infty}^{\xi} \eta^{m-1} \gamma(\eta)\, d\eta\}.$$

then $u(x)$ is a solution of (3.2). If we can prove that $u(t)$ is an entire function of $a_1, \ldots, a_m$, then we have proved (i) of Theorem 1. To do this, let x_0 be an arbitrary point such that $| x_0 | > 0$. Let $\Phi(x)$ be the two by two matrix such that

$$\frac{d\Phi}{dx} = A(x)\, \Phi, \qquad \Phi(x_0) = I_2,$$

where I_2 is the two by two identity matrix. The elements of the vertices $\Phi(x)$ and $\Phi^{-1}(x)$ are entire functions of $a_1, \ldots, a_m$ for $| x | > 0$, and analytic in x for $| x | > 0$, $| \arg x | < \pi$, and

$$u(x) = \Phi(x)\, u(x_0).$$

Now let $(a_1^0, a_2^0, \ldots, a_m^0)$ be fixed and consider a small neighborhood U of this point. Then, if x_0 is chosen so that $(\xi_0, a_1, \ldots, a_m)$ lies in the domain (4.3), for every $(a_1, \ldots, a_m)$ in U, where $\xi_0^2 = x_0$, it follows from (4.12) that $u(x_0)$ is analytic in U. This proves that $u(x)$ is an entire function of $(a_1, \ldots, a_m)$ for $| x | > 0$ and analytic in x for $| x | > 0$, $| \arg x | < \pi$.

5. Determination of r_m, A_{mn}, B_{mn} *and* C_{mn}.

To complete the proof of Theorem 1, it remains to show the coefficients of the right hand sides of (1.6) and (1.7) are polynomials of $a_1, a_2, \ldots, a_m$.

First, by computing the entries of (3.7) from (3.2), (3.3) and (3.6), we have

$$\alpha_1(\xi) = -\{1 + x^{-m}P(x)\} - \frac{1}{2}(m-2)\,\xi^{-m},$$

$$\alpha_2(\xi) = \{1 + x^{-m}P(x)\} - \frac{1}{2}(m-2)\,\xi^{-m},$$

$$\beta_1(\xi) = \{1 - x^{-m}P(x)\} + \frac{1}{2}(m-2)\,\xi^{-m},$$

$$\beta_2(\xi) = -\{1 - x^{-m}P(x)\} + \frac{1}{2}(m-2)\,\xi^{-m}.$$

Thus,

$$\alpha_2(\xi) - \alpha_1(\xi) = 2\{1 + x^{-m}P(x)\},$$

$$\beta_1(\xi)\,\beta_2(\xi) = \frac{1}{4}(m-2)^2\,\xi^{-2m} - \{1 - x^{-m}P(x)\}^2,$$

and, consequently,

$$\{\alpha_2(\xi) - \alpha_1)\}^2 + 4\beta_1(\xi)\,\beta_2(\xi) = 16x^{-m}P(x) + (m-2)^2\,\xi^{-2m}.$$

On the other hand, from (3.12), we have

$$2\beta_1(\xi)\,p(\xi) =$$

$$= \alpha_2(\xi) - \alpha_1(\xi) - \left[\{\alpha_2(\xi) - \alpha_1(\xi)\}^2 + 4\beta_1(\xi)\,\beta_2(\xi) - 4\xi^{-(m-1)}\beta_1(\xi)\frac{dp}{d\xi}\right]^{\frac{1}{2}} =$$

$$= 2\{1 + x^{-m}P(x)\} - \left[16x^{-m}P(x) + (m-2)^2\,\xi^{-2m} - 4\xi^{-(m-1)}\beta_1(\xi)\frac{dp}{d\xi}\right]^{\frac{1}{2}}.$$

Hence

(5.1) $$\gamma(\xi) = \alpha_1(\xi) + p(\xi)\,\beta_1(\xi) = -2\sqrt{x^{-m}P(x)} + 0(\xi^{-m-2}).$$

From the expressions of $E(\xi)$ and $\gamma(\xi)$ in (4.7) and (4.9), we have

$$\sum_{n=1}^{m-1} A_{mn}x^{\frac{1}{2}m-n} = \sum_{n=1}^{m-1}\frac{\gamma_n}{m-n}\,\xi^{m-n} =$$

$$= -2\int_0^{\xi}\eta^{m-1}\sum_{1\leqq h<\frac{1}{2}m} b_h\eta^{-\frac{1}{2}h}\,dt = -\int_0^{x} t^{\frac{1}{2}m-1}\sum_{1\leqq h<\frac{1}{2}m} b_h t^h\,dt =$$

$$= -\sum_{1\leqq h<\frac{1}{2}m}\frac{2}{m-2h}\,b_h x^{\frac{1}{2}m-h}.$$

Put

(5.2) $$q(x) = -x^{\frac{1}{2}m-1}\{1 + \sum_{1\leqq h<\frac{1}{2}m} b_h x^{-h}\}$$

and

(5.3) $$y = z\exp\{\int_0^x q(t)\,dt\}.$$

Then (E) becomes

(5.4) $$z'' + 2qz' + \{q' + q^2 - x^{-2}P(x)\}\,z = 0$$

It is easy to see that

$$2q = x^{\frac{1}{2}m-1}\{-2 + 0(x^{-1})\}$$

and

$$q' + q^2 - x^{-2}P(x) = x^{\frac{1}{2}m-2}\{s_m + 0(x^{-K_m})\},$$

where

(5.5) $$s_m = \begin{cases} -\dfrac{m}{2} + 1, & m\text{: even},\\[2mm] -\dfrac{m}{2} + 1 - 2b_{\frac{1}{2}m}, & m\text{: odd},\end{cases} \qquad K_m = \begin{cases} 1, & m\text{: even},\\[2mm] \dfrac{1}{2}, & m\text{: odd}.\end{cases}$$

By putting

$$z = x^{r_m}\{1 + 0(x^{-\frac{1}{2}})\}$$

we get

(5.6)
$$r_m = \frac{1}{2} s_m = \begin{cases} -\dfrac{m}{4} + \dfrac{1}{2}, & m\colon \text{odd}, \\ -\dfrac{m}{4} + \dfrac{1}{2} - b_{\frac{1}{2}m}, & m\colon \text{even}. \end{cases}$$

Put

(5.7)
$$z = x^{r_m} w.$$

Then (5.4) becomes

$$w'' + 2(r_m x^{-1} q)\, w' + \{(r_m x^{-1})^2 - r_m x^{-2} + 2q r_m x^{-1} + q' + q^2 - x^{-2}P(x)\}\, w = 0.$$

The coefficients of w' is $x^{\frac{1}{2}m-1}\{-2 + 0(x^{-1})\}$, while that of w is $0(x^{\frac{1}{2}m-2-K_m})$. Hence, by putting

$$w = 1 + \sum_{n=1}^{\infty} B_{mn} x^{-\frac{1}{2}n},$$

we can determine B_{mn} successively as polynomials of $a_1, \ldots, a_m$.

By differentiating the right hand side of (1.6) we can get C_{mn}. Thus, Theorem 1 is proved.

REFERENCES

[1] B. L. J. Braaksma: *Recessive solutions of linear differential equations with polynomial coefficients*, Pp. 1—15, Lecture Notes in Math., Vol. 280, Springer, Berlin, 1972.

[2] P. F. Hsieh and Y. Sibuya: *On the asymptotic integration of second order linear ordinary differential equations with polynomial coefficients*, J. Math. Anal. Appl., 16 (1966), 84—103.

[3] F. E. Mullin: *On the regular perturbation of the subdominant solution to second order linear ordinary differential equations with polynomial coefficients*, Funk. Ekv., 11 (1968), 1—38.

[4] T. Okada: *A study of asymptotic solutions of second order linear differential equations*, Specialist Project, Western Mich. Univ., 1976, 53 pp.

[5] Y. Sibuya: *Global Theory of a Second Order Linear Ordinary Differential Equations With a Polynomial Coefficient*, Math. Studies 18, North-Holland, 1975, xv + 290.

Po-Fang Hsieh
Department of Mathematics
Western Michigan University
Kalamazoo, Michigan 49008
U.S.A.