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A GENERALIZATION OF RIESZ—FISCHER THEOREM

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1. Let $\{\Phi_k(x)\}$ be an orthonormal system in $[a, b]$. The expression

$$\sigma(x) = \sum_{k=0}^{\infty} a_k \Phi_k(x),$$

where $\{a_k\}$ is an arbitrary sequence of real numbers, is called an orthogonal series. If for some $f(x)$ we have $f(x)\, \Phi_k(x) \in L[a, b]$, $k = 0, 1, 2, \ldots$ and

$$a_k = \int_a^b f(x)\, \Phi_k(x)\, dx, \qquad k = 0, 1, 2, \ldots,$$

then $\sigma(x)$ is called orthogonal expansion of $f(x)$ in the system $\{\Phi_k(x)\}$ and the numbers a_k, $k = 0, 1, 2, \ldots$ are called coefficients of the expansion of $f(x)$ in $\{\Phi_k(x)\}$.

The Riesz – Fischer theorem asserts that if the coefficients of $\sigma(x)$ satisfy the condition

$$\sum_{k=0}^{\infty} a_k^2 < \infty\,, \tag{1.1}$$

then $\sigma(x)$ is the orthogonal expansion of some function $f(x) \in L^2[a, b]$.

Recently Fomin [1] observed that for (1.1) to hold, it is necessary and sufficient that there exists an increasing sequence of positive numbers $\{v_k\}$, $v_k \to \infty$, such that

$$\sum_{k=0}^{\infty} \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right) \int_a^b \Big| \sum_{m=0}^{k} a_m v_m \Phi_m(x) \Big|^2 dx < \infty\,. \tag{1.2}$$

This led him to formulate an analogue of Riesz – Fischer theorem for $L^p[a, b]$, $p \geqq 1$. He proved the following theorem with the assumption that $f(x) \in L^p[a, b] \Rightarrow f(x) \times \Phi_k(x) \in L[a, b]$, $k = 0, 1, 2, \ldots$

Theorem A. Let $\{v_k\}$ be an increasing sequence of positive numbers tending to infinity with k. If

$$\sum_{k=0}^{\infty}\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)\int_a^b\left|\sum_{m=0}^{k}a_m v_m \Phi_m(x)\right|^p dx<\infty, \tag{1.3}$$

$p \geqq 1$ then the series $\sigma(x)$ is the orthogonal expansion of some function $f(x) \in L^p[a, b]$.

The main object of this note is to obtain a generalization of Theorem A.

2. Let $F(u)$ be a non-negative function defined for $u \geqq 0$. We say that a function $f(x)$ defined in $[a, b]$ belongs to class $L_F[a, b]$ if $F(|f(x)|)$ is integrable over $[a, b]$.

We assume that $f(x) \in L_F[a, b] \Rightarrow f(x)\, \Phi_k(x) \in L[a, b]$, $k = 0, 1, 2, \ldots$

Theorem. *Let $\{v_k\}$ be an increasing sequence of positive numbers such that $v_k \to \infty$ as $k \to \infty$. If $F(u)$ is convex and non-decreasing function, but not constant, such that*

$$\sum_{k=0}^{\infty}\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)\int_a^b F\left(\left|\sum_{m=0}^{k}a_m v_m \Phi_m(x)\right|\right) dx<\infty, \tag{2.1}$$

then $\sigma(x)$ is the orthogonal expansion of some function $f(x) \in L_F[a, b]$.

Proof. The hypothesis (2.1) shows that

$$\sum_{k=0}^{\infty}\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)F\left(\left|\sum_{m=0}^{k}a_m v_m \Phi_m(x)\right|\right)<\infty \tag{2.2}$$

almost everywhere. Consider the function

$$g(x)=v_0\sum_{k=0}^{\infty}\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)\left|\sum_{m=0}^{k}a_m v_m \Phi_m(x)\right|. \tag{2.3}$$

We shall show that $g(x) \in L[a, b]$. Using Jensen's inequality for convex function we have

$$F(g(x)) \leqq v_0\sum_{k=0}^{\infty}\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)F\left(\left|\sum_{m=0}^{k}a_m v_m \Phi_m(x)\right|\right)$$

so that

$$\int_a^b F(g(x))\,dx \leqq v_0\sum_{k=0}^{\infty}\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)\int_a^b F\left(\left|\sum_{m=0}^{k}a_m v_m \Phi_m(x)\right|\right)dx<\infty.$$

Thus $g(x) \in L_F[a, b]$ and hence because of [2], $g(x) \in L[a, b]$. From this it follows. that the series in (2.3) converges almost everywhere and therefore, the series

$$\sum_{k=0}^{\infty}\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)\sum_{m=0}^{k}a_m v_m \Phi_m(x) \tag{2.4}$$

converges almost every-where to a function $f(x)$ which belongs to $L_F[a, b]$.

Let $S_n(x)$ denote the n-th partial sum of (2.4), then

$$f(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right) \sum_{m=0}^{k} a_m v_m \Phi_m(x)$$

$$= \lim_{n\to\infty} \frac{1}{v_{n+1}} \sum_{m=0}^{n} (v_{n+1} - v_m)\, a_m \Phi_m(x)\,.$$

Now $| S_n(\chi)\, \Phi_k(x) | \leqq Cg(x)\, | \Phi_k(x) |$, $k = 0, 1, \ldots$, where C is a positive ccostant. By the hypothesis $g(x)\, \Phi_k(x) \in L[a, b]$ and so

$$\int_a^b f(x)\, \Phi_k(x)\, \mathrm{d}x = \lim_{n\to\infty} \frac{1}{v_{n+1}} \int_a^b \Phi_k(x) \sum_{m=0}^{n} (v_{n+1} - v_m)\, a_m \Phi_m(x)\, \mathrm{d}x$$

$$= \lim_{n\to\infty} (v_{n+1} - v_k)\, v_{n+1}^{-1} a_k = a_k, \qquad k = 0, 1, 2, \ldots$$

Thus $\sigma(x)$ is the orthogonal expansion of $f(x) \in L_F[a, b]$.

REFERENCES

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