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HOMOMORPHISMS OF MACHINES (Part I)

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1. CONSTRUCTION OF ALL MACHINE HOMOMORPHISMS

1.0. Notation. We denote by Ord the class of all ordinals and by N the set of all finite ordinals. If $\alpha \in \mathrm{Ord}$, then we put $W_\alpha = \{\beta \in \mathrm{Ord}; \beta < \alpha\}$.

If A is a set, we denote by $|A|$ the cardinal number of A. Let φ be a partial map from A into a set B. We put $\mathrm{dom}\,\varphi = \{x \in A;$ there exists $y \in B$ such that $(x, y) \in \varphi\}$. If $\mathrm{dom}\,\varphi = A$, then we write $\varphi: A \to B$. Finally, if $C \subseteq A$, then we denote by $\varphi \mid C$ the restriction $\varphi \cap (C \times B)$ of φ.

1.1. Definition. Let A be a non empty set, f a partial map from A into A. Then the partial unary algebra $\boldsymbol{A} = (A, f)$ is called a *machine*. (See, for example, [3]).

1.2. Definition. Let $\boldsymbol{A} = (A, f)$ be a machine. Then we put $D\boldsymbol{A} = A - \mathrm{dom} f$.

1.3. Definition. Let $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ be machines and $F: A \to B$ a map. Then F is called a *homomorphism* of $\boldsymbol{A}$ into $\boldsymbol{B}$ if $x \in \mathrm{dom} f$ implies $F(x) \in \mathrm{dom}\, g$ and $F(f(x)) = g(F(x))$ for each $x \in A$. We write $F: \boldsymbol{A} \to \boldsymbol{B}$.

1.4. Definition. Let $\boldsymbol{A}$, $\boldsymbol{B}$ be machines. Then a homomorphism $F: \boldsymbol{A} \to \boldsymbol{B}$ is called a *machine homomorphism* (abbreviation an *m-homomorphism*) if $F(D\boldsymbol{A}) \subseteq D\boldsymbol{B}$. (See [3].)

1.5. Remark. (a) We see that a map $F: \boldsymbol{A} \to \boldsymbol{B}$ is an m-homomorphism of $\boldsymbol{A} = (A, f)$ into $\boldsymbol{B} = (B, g)$ if, for each $x \in A$, the following conditions are satisfied: (1) $x \in \mathrm{dom} f$ iff $F(x) \in \mathrm{dom}\, g$, (2) $x \in \mathrm{dom} f$ implies $F(f(x)) = g(F(x))$.

(b) Clearly, if $\boldsymbol{A}$, $\boldsymbol{B}$ are machines such that $D\boldsymbol{A} = \emptyset$, then a map F is an m-homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ if F is a homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$.

1.6. Problem. Let $\boldsymbol{A}$, $\boldsymbol{B}$ be machines. Find necessary and sufficient conditions for the existence of an m-homomorphism $F: \boldsymbol{A} \to \boldsymbol{B}$.

1.7. Problem. Let A, $\boldsymbol{B}$ be machines. Construct all m-homomorphisms F: $A \to \boldsymbol{B}$.

If we solve Problem 1.7, we shall get an answer to Problem 1.6. Thus, we want to solve Problem 1.7 first. When solving this problem we can directly apply the results of the paper [5]. Let us recall the main notions and assertions of this paper.

1.8. Definition. Let $A = (A, f)$ be a machine.

(a) We put $f^0 = \mathrm{id}_A$. Suppose that we have defined a partial map f^{n-1} from A into A for $n \in N - \{0\}$. We denote by f^n the following partial map from A into A: if $x \in \mathrm{dom}\, f^{n-1}$ and $f^{n-1}(x) \in \mathrm{dom}\, f$ then we put $f^n(x) = f(f^{n-1}(x))$.

(b) Let $x \in A$ be arbitrary. Then we define $[x]_A = \{y;$ there exists $n \in N$ with $x \in \mathrm{dom}\, f^n$ and $y = f^n(x)\}$. (Compare [5], 1.5 and 1.7.)

1.9. Definition. Let $A = (A, f)$ be a machine. For arbitrary $x, y \in A$, we put $(x, y) \in \varrho A$ if there exist $m, n \in N$ such that $x \in \mathrm{dom}\, f^m$, $y \in \mathrm{dom}\, f^n$ and $f^m(x) = f^n(y)$. If $\varrho A = A \times A$ then A is called a *connected machine* (abbreviation *c-machine*). (Compare [5], 1.9.)

(i) *If A is a c-machine, then* $|DA| \leqq 1$. (See [5], 2.1)

1.10. Definition. Let A be a c-machine such that $DA \neq \emptyset$. Then we denote by dA the only point with the property $\{dA\} = DA$.

(ii) *If $A = (A, f)$ is a c-machine with $DA \neq \emptyset$, then, for each $x \in A$, there is precisely one $n \in N$ such that $x \in \mathrm{dom}\, f^n$ and $f^n(x) = dA$.* (See [5], 2.3.)

1.11. Definition. Let $A = (A, f)$ be a c-machine. Then we put $ZA = \{x \in A;$ there is $n \in N - \{0\}$ such that $f^n(x) = x\}$, $RA = |ZA|$. ZA is called the *cycle* and RA the *rank* of A. (See [5], 2.4–2.8 (a).)

(iii) *Let A be a c-machine. Then $DA \neq \emptyset$ if the following conditions are satisfied:* (1) $RA = 0$, (2) *there exists $x \in A$ such that* $|[x]_A| < \aleph_0$. (See [5], 2.9.)

(iv) *If A is a c-machine then $RA < \aleph_0$.* (See [5], 2.11.)

Let $\infty_1, \infty_2 \notin \mathrm{Ord}$; suppose that, for each $\alpha \in \mathrm{Ord}$, $\alpha < \infty_1 < \infty_2$.

1.12. Definition. Let $A = (A, f)$ be a c-machine. Then we put $A^{\infty_2} = ZA$, $A^{\infty_1} = \{x \in A - ZA;$ there is a sequence $(x_i)_{i \in N}$ such that $x_i \in \mathrm{dom}\, f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N\}$, $A^0 = \{x \in A; f^{-1}(x) = \emptyset\}$. Let $\alpha \in \mathrm{Ord}$, $\alpha > 0$ and suppose that the sets $A^\varkappa$ have been defined for all $\varkappa \in W_\alpha$. Then we put $A^\alpha = \{x \in A - \bigcup_{\varkappa \in W_\alpha} A^\varkappa;\ f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\alpha} A^\varkappa\}$. Finally, we put $\vartheta A = \min\{\varkappa \in \mathrm{Ord}; A^\varkappa = \emptyset\}$. (See [5], 2.13–2.16 and 2.19, 2.20.)

(v) *Let A be a c-machine; we put $W^* = W_{\vartheta A} \cup \{\infty_1, \infty_2\}$. Then $A = \bigcup_{\varkappa \in W^*} A^\varkappa$ with disjoint summands.* (See [5], 2.22.)

1.13. Definition. Let A be a c-machine. We define a map SA: $A \to \mathrm{Ord} \cup \{\infty_1, \infty_2\}$ by the condition $SA(x) = \varkappa$ for each $x \in A^\varkappa$, $\varkappa \in W_{\vartheta A} \cup \{\infty_1, \infty_2\}$. $SA(x)$ is called the *degree* of x. (Compare [5], 2.23.)

(vi) *Let A be a c-machine. If $DA \neq \emptyset$, $A^{\infty 1} = \emptyset$, then ϑA is isolated and $SA(dA) = \vartheta A - 1$.* (See [5], 2.26 (c).)
(vii) *Let $A = (A, f)$ be a c-machine. If $DA \neq \emptyset$, $A^{\infty 1} \neq \emptyset$, then $SA(dA) = \infty_1$.* (See [5], 2.26 (d).)

1.14. Definition. Let $A = (A, f)$, $B = (B, g)$ be c-machines. Then $x \in A$, $x' \in B$ are said to form a *pair of h-elements* of A and B if, for each $n \in N$, $x \in \operatorname{dom} f^n$ implies $x' \in \operatorname{dom} g^n$ and $SA(f^n(x)) \leqq SB(g^n(x'))$. (Compare [5], 3.4.)

1.15. Definition. Let $A = (A, f)$, $B = (B, g)$ be c-machines. Then a pair $x \in A$, $x' \in B$ of h-elements of A and B is said to form a *pair of m-h-elements* of A and B if, for each $n \in N$, $x' \in \operatorname{dom} g^n$ implies $x \in \operatorname{dom} f^n$. Further, we put $H(A, B) = \{(x, x') \in A \times B;\ x, x'$ is a pair of m-h-elements of A and $B\}$.

1.16. Remark. Clearly, if $A = (A, f)$, $B = (B, g)$ be c-machines, then a pair $x \in A$, $x' \in B$ of h-elements is a pair of m-h-elements if $DA \neq \emptyset$ implies $DB \neq \emptyset$ and $x \in \operatorname{dom} f^n$, $f^n(x) = dA$, $x' \in \operatorname{dom} g^m$, $g^m(x') = dB$ imply $m = n$. (See (ii).)

1.17. Definition. Let A, B be c-machines. Then B is said to be *admissible* for A if the following conditions hold:

(a) if $RB \neq 0$, then $RB \mid RA$;*)
(b) if $RB = 0$, then there exists a pair of h-elements of A and B. (Compare [5], 3.5.)

1.18. Definition. Let A, B be c-machines. Then B is said to be *m-admissible* for A if the following conditions hold:

(a) if $RB \neq 0$, then $RB \mid RA$ and $DA = \emptyset$;
(b) if $RB = 0$, then $H(A, B) \neq \emptyset$.

(viii) *If $A = (A, f)$ is a c-machine such that $A^{\infty 2} \neq \emptyset$ and $x \in A$ arbitrary, then there is $n \in N$ such that $x \in \operatorname{dom} f^n$, $f^n(x) \in A^{\infty 2}$.* (See [4], 1.4 and 1.17 (a).)

1.19. Lemma. *Let $A = (A, f)$, $B = (B, g)$ be c-machines. Then the following assertions hold:*

(a) *If B is admissible for A, then $RB = 0$ implies $RA = 0$.* (Compare [5], 3.5.)
(b) *$H(A, B) \neq \emptyset$ iff there exists a pair of h-elements of A and B and $|DA| = |DB|$.*
(c) *B is m-admissible for A iff B is admissible for A and $|DA| = |DB|$.*

Proof of (a). Let $RB = 0$; then there is a pair $x \in A$, $x' \in B$ of h-elements of A and B by 1.17. Further, $B^{\infty 2} = \emptyset$ which implies $A^{\infty 2} = \emptyset$. Indeed, if we had, on the contrary, $A^{\infty 2} \neq \emptyset$, then, for some $n \in N$, we should have $x \in \operatorname{dom} f^n$ and $f^n(x) \in A^{\infty 2}$ by (viii) which is a contradiction to $B^{\infty 2} = \emptyset$ because $x' \in \operatorname{dom} g^n$, $SA(f^n(x)) \leqq SB(g^n(x'))$ and we have $g^n(x') \in B^{\infty 2}$. Thus, $RA = 0$.

*) $m \mid n$ means that m is a divisor of n.

Proof of (b). (1) Let $H(A, B) \neq \emptyset$. Since $H(A, B)$ is a subset of the set of all pairs of h-elements of A and B this set is not empty. Further, let $(x, x') \in H(A, B)$. Then, for each $n \in N$, $x \in \operatorname{dom} f^n$ if $x' \in \operatorname{dom} g^n$ by 1.15. Thus, $DA \neq \emptyset$ if $DB \neq \emptyset$ by (ii). It follows $| DA | = | DB |$ by (i).

(2) Let there exists a pair of h-elements $x \in A$, $x' \in B$ of A and B and let $| DA | = | DB |$. Thus, for each $n \in N$, $x \in \operatorname{dom} f^n$ implies $x' \in \operatorname{dom} g^n$ and $SA(f^n(x)) \leqq SB(g^n(x'))$. If $| DA | = | DB | = 0$, then we have finished because $x' \in \operatorname{dom} g^n$ implies $x \in \operatorname{dom} f^n$ for each $n \in N$. If $| DA | = | DB | = 1$, then there is precisely one $p \in N$ and one $q \in N$ such that $f^p(x) = dA$, $g^q(x') = dB$ by (ii). Further, $p \leqq q$ because x, x' is a pair of h-elements of A and B. We obtain $(x, g^{q-p}(x')) \in H(A, B)$ because, for each $n \in N$, $x \in \operatorname{dom} f^n$ if $g^{q-p}(x') \in \operatorname{dom} g^n$ and $SA(f^n(x)) \leqq SB(g^n(x')) \leqq SBg(^{n+q-p}(x')) = SB(g^n(g^{q-p}(x')))$. Thus, $(x, x') \in H(A, B)$.

Proof of (c). Let B be m-admissible for A. If $RB \neq 0$, then B is admissible for A by 1.17 and 1.18. Further, $DB = \emptyset$ by (iii) and thus, $| DB | = 0 = | DA |$ by 1.18. If $RB = 0$, then B is admissible for A and $| DA | = | DB |$ by 1.18, 1.17 and (b).

Let, on the other hand, B be admissible for A and $| DA | = | DB |$. Thus, if $RB \neq 0$, then $RB \mid RA$ and $DA = \emptyset$ because $DB = \emptyset$ by (iii). If $RB = 0$, then $H(A, B) \neq \emptyset$ by 1.17 and (b) and thus, B is m-admissible for A.

(ix) *Let A, B be c-machines. If B is admissible for A, then there is a pair of h-elements of A and B.* (See [5], 3.7.)

1.20. Lemma. *Let A, B be c-machines. If B is m-admissible for A, then $H(A, B) \neq \emptyset$.*

Proof. The assertion follows directly from (ix). Indeed, if $RB \neq 0$, then $DB = \emptyset$ and $DA = \emptyset$ by (iii) and 1.18. Thus, $| DA | = | DB |$ and $H(A, B) \neq \emptyset$ by (ix), 1.19 (a) and (b).

1.21. Definition. Let $A = (A, f)$ be a c-machine, $x \in A$ arbitrary. We put $P_0(x) = [x]_A$, $P_1(x) = f^{-1}(P_0(x)) - P_0(x)$. Let $n \in N - \{0\}$ and suppose that the sets $P_0(x)$, $P_1(x)$, $\ldots$, $P_n(x)$ have been defined. Then we put $P_{n+1}(x) = f^{-1}(P_n(x))$ (Compare [5], 3.8.)

(x) *If $A = (A, f)$ is a c-machine, $x \in A$ arbitrary, then $A = \bigcup_{n=0}^{\infty} P_n(x)$ with disjoint summands.* (See [5], 3.9 (c).)

(xi) *Let $A = (A, f)$, $B = (B, g)$ be c-machines and $y \in A$, $y' \in B$ such that $SA(y) \leqq SB(y')$. Then, for each $x \in f^{-1}(y)$, there exists $x' \in g^{-1}(y')$ such that $SA(x) \leqq SB(x')$.* (See [5], 3.10.)

1.22. Definition. Let $A = (A, f)$, $B = (B, g)$ be c-machines such that B is admissible for A. We define a map F: $A \to B$ in this way:

(1) We take a pair of h-elements $x_0 \in A$, $x_0 \in B$ of A and B (see (ix)). Then we put, for each $f^n(x_0) \in P_0(x_0)$, $F(f^n(x_0)) = g^n(x'_0)$.

(2) Let $n \in N - \{0\}$. Suppose that, for each $x \in \bigcup_{k=0}^{n-1} P_k(x_0)$, we have defined $F(x)$ in such a way that $SA(x) \leqq SB(F(x))$. Let $x \in P_n(x_0)$ be arbitrary. We take $x' \in g^{-1}(F(f(x)))$ such that $SA(x) \leqq SB(x')$ (see (xi)). Then we put $F(x) = x'$.

Then we say that the map F: $A \to B$ has been defined by the *construction c-K* (with respect to A and $\boldsymbol{B}$). (Compare [5], 3.11.)

1.23. Definition. Let $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ be c-machines such that $\boldsymbol{B}$ is m-admissible for $\boldsymbol{A}$. If we define a map F: $A \to B$ by the construction *c-K* with respect to $\boldsymbol{A}$ and $\boldsymbol{B}$ such that we take $(x_0, x_0') \in H(\boldsymbol{A}, \boldsymbol{B})$ (see 1.22 (1) and 1.20), then we say that the map F has been defined by the *construction m-c-K* (with respect to $\boldsymbol{A}$ and $\boldsymbol{B}$).

(xii) *Let* $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ *be c-machines*, F: $\boldsymbol{A} \to \boldsymbol{B}$ *a map. Then* F: $\boldsymbol{A} \to \boldsymbol{B}$ *is a homomorphism iff F is defined by the construction c-K with respect to* $\boldsymbol{A}$ *and* $\boldsymbol{B}$. (See [5], 3.14.)

1.24. Theorem. *Let* $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ *be c-machines*, F: $A \to B$ *a map. Then* F: $\boldsymbol{A} \to \boldsymbol{B}$ *is an m-homomorphism if and only if F is defined by the construction m-c-K with respect to* $\boldsymbol{A}$ *and* $\boldsymbol{B}$.

Proof. The assertion is clear for $DA = \emptyset$ by 1.5 (b) and (xii). Thus, suppose $DA \neq \emptyset$.

If F: $\boldsymbol{A} \to \boldsymbol{B}$ is an m-homomorphism then it is a homomorphism which implies that F: $\boldsymbol{A} \to \boldsymbol{B}$ is defined by the construction *c-K* by (xii). Let $x_0 \in A$, $x_0' \in B$ be the pair of h-elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ that we have taken by the construction *c-K* (see 1.22 (1)). Since $F(dA) = d\boldsymbol{B}$, we obtain that, for each $n \in N$, $x_0' \in \operatorname{dom} g^n$ implies $x_0 \in \operatorname{dom} f^n$ by (ii) because $F(f^n(x_0)) = g^n(x_0')$. Thus, $(x_0, x_0') \in H(\boldsymbol{A}, \boldsymbol{B})$ and the used construction is the *m-c-K* construction.

If, on the other hand, F: $A \to \boldsymbol{B}$ is a map defined by the construction *m-c-K*, then it is defined by the construction *c-K* which implies that F: $\boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism by (xii). Let $(x_0, x_0') \in H(\boldsymbol{A}, \boldsymbol{B})$ be the pair of m-h-elements that we have taken by the construction *m-c-K* (see 1.23 and 1.22 (1)). Since $F(f^n(x_0)) = g^n(x_0')$ for each $n \in N$ such that $x_0 \in \operatorname{dom} f^n$, we obtain $F(dA) = d\boldsymbol{B}$ by (ii) and 1.15. Thus, F: $\boldsymbol{A} \to \boldsymbol{B}$ is an m-homomorphism.

(xiii) *If* $\boldsymbol{A}$ *is a machine and* ϱA *is defined by 1.9, then* ϱA *is an equivalence on* A. (See [5], 4.1.)

1.25. Definition. Let $\boldsymbol{A} = (A, f)$ be a machine. Then we denote $\Theta A = A/\varrho A$.

1.26. Definition. Let $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ be machines. We define a map F: $A \to B$ in this way:

(1) We take a map Φ: $\Theta A \to \Theta B$ such that, for each $T \in \Theta A$, $(\Phi(T), g \mid \Phi(T))$ is admissible (m-admissible respectively) for $(T, f \mid T)$. For each $T \in \Theta A$, we define a map F_T: $T \to \Phi(T)$ by the construction *c-K* (*m-c-K* respectively).

(2) We put $F = \bigcup_{T \in \Theta A} F_T$.

Then we say that the map $F: A \to B$ has been defined by the *construction* K (m-K respectively) (with respect to $\boldsymbol{A}$ and $\boldsymbol{B}$). (Compare [5], 4.5.)

1.27. Lemma. *Let* $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ *be machines,* $F: A \to B$ *a map. If* F *is defined by the construction* m-K, *then is defined by the construction* K.

Indeed, it follows from 1.18 and 1.23.

(xiv) *Let* $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ *be machines,* $F : A \to B$ *a map. Then* $F : \boldsymbol{A} \to \boldsymbol{B}$ *is a homomorphism iff* F *is defined by the construction* K. (See [5], 4.8.)

1.28. Theorem. *Let* $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ *be machines,* $F : A \to B$ *a map. Then* $F : \boldsymbol{A} \to \boldsymbol{B}$ *is an m-homomorphism if and only if* F *is defined by the construction* m-K *with respect to* $\boldsymbol{A}$ *and* $\boldsymbol{B}$.

Proof. The assertion follows directly from (xiv), 1.23, 1.26 and 1.27.

Theorem 1.28 is a solution of Problem 1.7. Problem 1.6 will be dealt with in the next paragraph (Arch. Math. XIV, 2 (1978)).

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