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ON THE BOUNDEDNESS OF SOLUTIONS OF HIGHER ORDER DIFFERENTIAL EQUATIONS

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In the paper the existence and the boundedness of solutions of a $2n$-th order differential equation is proved, by means of a generalization of Gronwall-Bellman inequality. The results generalize some theorems proved by M. Ráb [2].

Lemma 1. *Let h, p, $g \in c_0(I)$, $I = \langle a, \infty)$, $p >, 0$ $h \geqq 0$, $g \geqq 0$ for all $t \in I$ and h be a nondecreasing function on I. If $u \in c_0(I)$, $u(t) \geqq 0$ and for all $t \in I$*

$$u(t) \leqq h(t) + \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \dots \int_a^{t_1} g(t_0)\, u(t_0)\, dt_0 \dots dt_{2n-1}, \tag{1}$$

then

$$u(t) \leqq h(t) \exp \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \dots \int_a^{t_1} g(t_0)\, dt_0 \dots dt_{2n-1} \qquad (t \in I). \tag{2}$$

Proof: Suppose first that $h(t) > 0$ in I. Denote

$$U(t) = \max_{a \leqq \tau \leqq t} u(\tau) \qquad (t \in I).$$

The function U is nondecreasing continuous function and $u(t(\leqq U(t)$ for all $t \in I$, hence from (1) it follows that

$$\begin{aligned} u(t) &\leqq h(t) + \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \dots \int_a^{t_1} g(t_0)\, U(t_0)\, dt_0 \dots dt_{2n-1} \leqq \\ &\leqq h(t) + \int_a^t U(t_{2n-1}) \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \dots \int_a^{t_1} g(t_0)\, dt_0 \dots dt_{2n-1} = \\ &= h(t) + \int_a^t U(t_{2n-1}) \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} G(t_n)\, dt_n \dots dt_{2n-1}, \end{aligned} \tag{3}$$

where

$$G(t_n) = \int_a^{t_n} \dots \int_a^{t_1} g(t_0)\,\mathrm{d}t_0 \dots \mathrm{d}t_{n-1}.$$

Since (3) is true for all $\tau \in \langle 0, t\rangle$, from the monotonicity of h and nonnegativity of all functions under the sign of integral we get

$$u(\tau) \leqq h(\tau) + \int_a^{\tau} U(t_{2n-1}) \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} G(t_n)\,\mathrm{d}t_n \dots \mathrm{d}t_{2n-1} \leqq$$

$$\leqq h(t) + \int_a^{t} U(t_{2n-1}) \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} G(t_n)\,\mathrm{d}t_n \dots \mathrm{d}t_{2n-1}$$

and also

(4) $$U(t) = \max_{a \leqq \tau \leqq t} u(\tau) \leqq h(t) + \int_a^{t} U(t_{2n-1})\, H(t_{2n-1})\,\mathrm{d}t_{2n-1}$$

where

$$H(t_{2n-1}) = \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} G(t_n)\,\mathrm{d}t_n \dots \mathrm{d}t_{2n-2}.$$

With respect to the positivity of the function h, we can divide (4) by the right-hand side and we get

$$\frac{U(t)}{h(t) + \int_a^t U(t_{2n-1})\, H(t_{2n-1})\,\mathrm{d}t_{2n-1}} \leqq 1, \qquad (t \in I).$$

Let $T \geqq a$ be an arbitrary number. Since $h(t)$ is a nondecreasing function for all t: $a \leqq t \leqq T$, we have that $h(t) \leqq h(T)$ and hence

$$\frac{U(t)}{h(T) + \int_a^t U(t_{2n-1})\, H(t_{2n-1})\,\mathrm{d}t_{2n-1}} \leqq 1.$$

Multiplying the last inequality by $H(t)$ and integrating from a to T we get

$$\int_a^T \frac{U(t)\, H(t)}{h(T) + \int_a^t U(t_{2n-1})\, H(t_{2n-1})\,\mathrm{d}t_{2n-1}}\,\mathrm{d}t \leqq \int_a^T H(t)\,\mathrm{d}t$$

$$\ln\left[h(T) + \int_a^T U(t_{2n-1})\, H(t_{2n-1})\,\mathrm{d}t_{2n-1}\right] - \ln h(T) \leqq \int_a^T H(t)\,\mathrm{d}t,$$

hence

$$(5)\qquad h(T) + \int_a^T U(t_{2n-1}) H(t_{2n-1}) \, dt_{2n-1} \leqq h(T) \exp \int_a^T H(t) \, dt$$

From the formulae (4), (5) we get

$$u(T) \leqq U(T) \leqq h(T) \exp \int_a^T H(t) \, dt.$$

Since $T \geqq a$ is an arbitrary number, the inequality (2) is proved on the whole interval I.

If $h(t) \geqq 0$ on the interval I, we take the functions $h_m(t) = h(t) + \frac{1}{m}$. For these functions Lemma 1 is true and we get

$$u(t) \leqq \left[h(t) + \frac{1}{m}\right] \exp \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \dots \int_a^{t_1} G(t_0) \, dt_0 \dots dt_{2n-1}, \qquad (t \in I).$$

By a limit process for $m \to \infty$ (for t fixed) we get (2) again.

Theorem 1. *Let t_0 be a real number, $j = (t_0, \infty)$, $a \in j$, $x_0^*, x_1^*, \dots, x_{2n-1}$ be real numbers. Let $p(t) > 0$, $g(t) \geqq 0$ be continuous functions on j and let $f(t, x)$ be continuous on $j \times (-\infty, \infty)$ and $|f(t, x)| \leqq g(t) . |x|$, for each point $(t, x) \in j \times (-\infty, \infty)$. Then every solution of the initialvalue problem*

$$(6)\qquad [p(t) x^{(n)}]^{(n)} = f(t, x)$$

$$(6a)\qquad x(a) = x_0^*, x'(a) = x_1^*, \dots, p(a) x^{(n)}(a) = x_n^*, \dots, [p(t) x^{(n)}(t)]^{(n-1)}(a) = x_{2n-1}^*$$

exists on the whole interval j and for $t \geqq a$ the following estimation is true

$$|x(t)| \leqq h(t) \exp \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \dots \int_a^{t_1} g(t_0) \, dt_0 \dots dt_{2n-1},$$

where

$$(7)\quad h(t) = |x_0^*| + |x_1^*| (t - a) + \dots + |x_{n-2}^*| \frac{(t - a)^{n-2}}{(n - 2)!} + |x_{n-1}^*| \frac{(t - a)^{n-1}}{(n - 1)!} +$$

$$+ |x_n^*| \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{1}{p(t_n)} dt_n \dots dt_{2n-1} + |x_{n+1}^*| \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{t_n - a}{p(t_n)} dt_n \dots dt_{2n-1} +$$

$$+ \dots + \frac{|x_{2n-1}^*|}{(n - 1)!} \int_a^t \int_a^{t_{2n-1}} \dots \int_a^{t_{n+1}} \frac{(t_n - a)^{n-1}}{p(t_n)} dt_n \dots dt_{2n-1}.$$

Proof. The initial value problem (6), (6a) is equivalent to the integral equation

$$(8)\qquad x(t) = x_0^* + x_1^*(t-a) + \ldots + x_{n-2}^* \frac{(t-a)^{n-2}}{(n-2)!} + x_{n-1}^* \frac{(t-a)^{n-1}}{(n-1)!} +$$

$$+ x_n^* \int_a^t \int_a^{t_{2n-1}} \ldots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \mathrm{d}t_n \ldots \mathrm{d}t_{2n-1} + x_{n+1}^* \int_a^t \int_a^{t_{2n-1}} \ldots \int_a^{t_{n+1}} \frac{(t_n - a)}{p(t_n)} \mathrm{d}t_n \ldots \mathrm{d}t_{2n-1} +$$

$$+ \ldots + \frac{x_{2n-1}^*}{(n-1)!} \int_a^t \int_a^{t_{2n-1}} \ldots \int_a^{t_{n+1}} \frac{(t_n - a)^{n-1}}{p(t_n)} \mathrm{d}t_n \ldots \mathrm{d}t_{2n-1} +$$

$$+ \int_a^t \int_a^{t_{2n-1}} \ldots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \ldots \int_a^{t_1} f[t_0, x(t_0)] \, \mathrm{d}t_0 \ldots \mathrm{d}t_n \ldots \mathrm{d}t_{2n-1}.$$

Let us suppose now that a solution $x(t)$ of (6), (6a) exists on the interval $\langle a, T)$ and that this is the greatest interval of existence of that solution. Then on this interval $|x(t)|$ satisfies the inequality

$$|x(t)| \leqq h(t) + \int_a^t \int_a^{t_{2n-1}} \ldots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \ldots \int_a^{t_1} g(t_0) \, |x(t_0)| \, \mathrm{d}t_0 \ldots \mathrm{d}t_{2n-1}.$$

Since all the assumptions of Lemma 1 are fulfilled, we have

$$|x(t)| \leqq h(t) \exp \int_a^t \int_a^{t_{2n-1}} \ldots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \ldots \int_a^{t_1} g(t_0) \, \mathrm{d}t_0 \ldots \mathrm{d}t_{2n-1}.$$

This function is bounded on every finite interval, hence $|x(t)| \leqq K$ on the interval $\langle a, T)$.

Let us further estimate $|x'(t)|$, $|x''(t)|$, ..., $|p(t)\, x^{(n)}(t)|$, ..., $|[p(t)\, x^{(n)}(t)]^{(n-1)}|$. From (8) we get the following system of equations:

$$x'(t) = x_1^* + \ldots + \frac{x_{2n-1}^*}{(n-1)!} \int_a^t \int_a^{t_{2n-2}} \ldots \int_a^{t_{n+1}} \frac{(t_n - a)^{n-1}}{p(t_n)} \mathrm{d}t_n \ldots \mathrm{d}t_{2n-2} +$$

$$+ \int_a^t \int_a^{t_{2n-2}} \ldots \int_a^{t_{n+1}} \frac{1}{p(t_n)} \int_a^{t_n} \ldots \int_a^{t_1} f[t_0, x(t_0)] \, \mathrm{d}t_0 \ldots \mathrm{d}t_{2n-2},$$

$$x''(t) = x_2^* + \dots + \frac{x_{2n-1}^*}{(n-1)!}\int_a^t\int_a^{t_{2n-3}}\dots\int_a^{t_{n+1}}\frac{(t_n - a)^{n-1}}{p(t_n)}\,\mathrm{d}t_n \dots \mathrm{d}t_{2n-3} +$$

$$+ \int_a^t\int_a^{t_{2n-3}}\dots\int_a^{t_{n+1}}\frac{1}{p(t_n)}\int_a^{t_n}\dots\int_a^{t_1} f[t_0, x(t_0)]\,\mathrm{d}t_0 \dots \mathrm{d}t_{2n-3},$$

$$\vdots$$

$$x^{(n)}(t) = \frac{1}{p(t)}\left[x_n^* + x_{n+1}^*(t-a) + \dots + x_{2n-1}^*\frac{(t-a)^{n-1}}{(n-1)!} + \right.$$

$$\left. + \int_a^t\int_a^{t_{n-1}}\dots\int_a^{t_1} f[t_0, x(t_0)]\,\mathrm{d}t_0 \dots \mathrm{d}t_{n-1}\right],$$

$$p(t)\,x^{(n)}(t) = x_n^* + \dots + x_{2n-1}^*\frac{(t-a)^{n-1}}{(n-1)!} + \int_a^t\int_a^{t_{n-1}}\dots\int_a^{t_1} f[t_0, x(t_0)]\,\mathrm{d}t_0 \dots \mathrm{d}t_{n-1},$$

$$\vdots$$

$$[p(t)\,x^{(n)}(t)]^{(n-1)} = x_{2n-1}^* + \int_a^t f[t_0, x(t_0)]\,\mathrm{d}t_0.$$

Thus we come to the inequality

$$|\,x'(t)\,| \leqq h_1(t) + \int_a^t\int_a^{t_{2n-2}}\dots\int_a^{t_{n+1}}\frac{1}{p(t_n)}\int_a^{t_n}\dots\int_a^{t_1} g(t_0)\,.\,|\,x(t_0)\,|\,\mathrm{d}t_0 \dots \mathrm{d}t_{2n-2},$$

where

$$h_1(t) = |\,x_1^*\,| + |\,x_2^*\,|\,.\,(t-a) + \dots +$$

$$+ \frac{|\,x_{2n-1}^*\,|}{(n-1)!}\int_a^t\int_a^{t_{2n-2}}\dots\int_a^{t_{n+1}}\frac{(t_n-a)^{n-1}}{p(t_n)}\,\mathrm{d}t_n \dots \mathrm{d}t_{2n-2}.$$

Since $|\,x(t_0)\,| \leqq K$, we deduce the existence of a $K_1 > 0$ such, that $|\,x'(t)\,| \leqq K_1$ for all $t \in \langle a, T)$. In the same way we can show that $|\,x''(t)\,|, \dots, |\,x^{(n-1)}(t)\,|$, $|\,p(t)\,x^{(n)}(t)\,|, \dots, |\,[p(t)\,x^{(n)}(t)]^{(n-1)}\,|$ are bounded. Hence from the corollary of Theorem 1.1.1 (see (1) page 14) the interval $\langle a, T)$ cannot be the greatest interval of existence of the solution of the given initial value problem.

There still remains to show the existence of the solution of the problem (6) on the interval $(t_0, a\rangle$.

If $t_0 < \tau \leqq a$, then the substitution $t = 2a - \tau$ transforms the differential equation

$$[p(\tau)\,x^{(n)}(\tau)]^{(n)} = f[\tau, x(\tau)]$$

into the differential equation

(6′) $$[p_1(t)\,x_1^{(n)}(t)]^{(n)} = f_1[t, x_1(t)],$$

where

(9) $$p_1(t) = p(2a - t) = p(\tau), f_1[t, x_1(t)] = f[2a - t, x_1(t)] = f[\tau, x(\tau)]$$

and

$$x_1(t) = x(2a - t) = x(\tau).$$

The initial conditions (6a) are transformed into the form

(6a′) $$x_1(a) = x_0^*, x_1^*(a) = -x_1^* \ldots, p_1(a)\,x_1^{(n)}(a) = (-1)^n x_n^*, \ldots,$$
$$[p_1(t)\,x_1^{(n)}(t)]^{(n-1)}(a) = (-1)^n \quad x_{2n-1}^*.$$

Since the problem (6′), (6a′) is essentially the same as the problem (6), (6a) and any solution of the latter problem exists on every finite interval, the same is true for the solution of the problem (6′), (6a′), too. From the transformation (9) we get any solution of (6), (6a) exists on $(t_0, a\rangle$. Q.E.D.

Theorem 2. *Let $p(t) > 0$, $q(t)$ be continuous functions on $I = \langle a, \infty)$. Let $A(t)$ be determined by* (12) *where the function $a_i(t)$, $i = 0, \ldots, 2n - 2$, are defined by* (11). *Let $f(t, x)$ be continuous on $I \times (-\infty, \infty)$ and such that $|f(t, x)| \leqq F(t, |x|)$, where $F(t, u)$ is continuous, nonnegative and nondecreasing in variable u on $I \times (-\infty, \infty)$ and such that the maximal solution $\Phi(t)$ of the equation*

$$B(t) = c_0 + \frac{1}{c}\int_a^t A^{2n-1}(s)\,F[s, A(s)\,B(s)]\,\mathrm{d}s$$

exists on I for all values c_0, $c > 0$. Then any solution of the problem

(10) $$[p(t)\,x^{(n)}]^{(n)} + q(t)\,x = f(t, x)$$

(6a) $$x(a) = x_0^*, x'(a) = x_1^*, \ldots, p(a)\,x^{(n)}(a) = x_n^*, \ldots, [p(t)\,x^{(n)}(t)]^{(n-1)}(a) = x_{2n-1}^*$$

exists in I.

Proof. By the method of variation of parameters we get the solution of the differential equation (10) on the interval of existence $\langle a, T)$ in the form

$$x(t) = y(t) + \int_a^t \frac{W(t, s)}{W(s)}\,f[s, x(s)]\,\mathrm{d}s,$$

where $y(t) = c_1u_1(t) + c_2u_2(t) + \ldots + c_{2n}u_{2n}(t)$, and $u_1(t)$, $u_2(t), \ldots, u_{2n}(t)$ is a fundamental system of solutions of the differential equation

$$[p(t)\,y^{(n)}]^{(n)} + q(t)\,y = 0$$

in I, $c_1, \ldots, c_{2n}$ are real numbers,

$$W(s) = \begin{vmatrix} u_1(s) & \ldots & u_{2n}(s) \\ u_1'(s) & \ldots & u_{2n}'(s) \\ \vdots & & \vdots \\ p(s)\, u_1^{(n)}(s) & \ldots & p(s)\, u_{2n}^{(n)}(s) \\ \vdots & & \vdots \\ [p(s)\, u_1^{(n)}(s)]^{(n-1)} & \ldots & [p(s)\, u_{2n}^{(n)}(s)]^{n-1} \end{vmatrix}$$

and

$$W(t,s) = \begin{vmatrix} u_1(s) & \ldots & u_{2n}(s) \\ u_1'(s) & \ldots & u_{2n}'(s) \\ \vdots & & \vdots \\ p(s)\, u_1^{(n)}(s) & \ldots & p(s)\, u_{2n}^{(n)}(s) \\ \vdots & & \vdots \\ [p(s)\, u_1^{(n)}(s)]^{(n-2)} & \ldots & [p(s)\, u_{2n}^{(n)}(s)]^{(n-2)} \\ u_1(t) & \ldots & u_{2n}(t) \end{vmatrix}, \qquad \begin{matrix} a \leqq s \leqq t, \\ t \in I. \end{matrix}$$

Let

$$c_0 = \max\{|c_1|, |c_2|, \ldots |c_{2n}|\},$$

(11)

$$a_0(t) = |u_1(t)| + \ldots + |u_{2n}(t)|,$$

$$a_1(t) = |u_1'(t)| + \ldots + |u_{2n}'(t)|,$$

$$\vdots$$

$$a_n(t) = |p(t)\, u_1^{(n)}(t)| + \ldots + |p(t)\, u_{2n}^{(n)}(t)|,$$

$$\vdots$$

$$a_{2n-2}(t) = |[p(t)\, u_1^{(n)}(t)]^{(n-2)}| + \ldots + |[p(t)\, u_{2n}^{(n)}(t)]^{(n-2)}|, \; t \in I$$

(12) $$A(t) = \max\{a_0(t), a_1(t), \ldots, a_{2n-2}(t)\}, \quad t \in I$$

$W(s)$ is a constant function different from zero and we put $|W(s)| = c$. Then

$$|x(t)| \leqq |y(t)| + \frac{1}{c}\int_a^t |W(t,s)| \,.\, |f[t, x(s)]| \,\mathrm{d}s \leqq$$

$$\leqq c_0 A(t) + \frac{1}{c}\int_a^t A^{2n-1}(s) \,.\, A(t) \,.\, F[s, |x(s)|] \,\mathrm{d}s =$$

$$= A(t)\left[c_0 + \frac{1}{c}\int_a^t A^{2n-1}(s) \,.\, F[s, |x(s)|] \,\mathrm{d}s\right], \qquad t \in \langle a, T).$$

Denote

$$B(t) = c_0 + \frac{1}{c} \int_a^t A^{(2n-1)}(s) \,.\, F[s, |\, x(s) \,|] \,\mathrm{d}s, \qquad t \in \langle a, T).$$

Then

$$|\, x(t) \,| \leqq A(t) \,.\, B(t) \leqq A(t) \,.\, \left\{ c_0 + \frac{1}{c} \int_a^t A^{2n-1}(s) \, F[s, A(s) \,.\, B(s)] \,\mathrm{d}s \right\},$$

$$t \in \langle a, T).$$

On the basis of the lemma from paper (3) we get $B(t) \leqq \Phi(t)$ if $t \in I$ and hence $|\, x(t) \,| \leqq$ $\leqq A(t) . \Phi(t)$ for all $t \in \langle a, T)$. Similarly we get that $|\, x'(t) \,| \leqq A(t) \,.\, \Phi(t), t \in \langle a, T), \ldots,$ $|\, x^{(n)}(t) \,| \leqq A(t) \,.\, \Phi(t), \quad t \in \langle a, T), \ldots, [p(t)\, x^{(n)}(t)]^{(n-1)} \leqq A(t) \,.\, \Phi(t), \quad t \in \langle a, \; T)$. If $T < \infty$, then the function $A(t) \,.\, \Phi(t)$ is bounded in $\langle a, T)$ and we come to the contradiction, similarly as in Theorem 1. Hence $T = \infty$ and the theorem is proved.

REFERENCES

[1] Cesari L.: *Asimptotičeskoje povedenije i ustojčivosť rešenij obyknovennych differenciaľnych uravnenij*, Moskva 1964.

[2] Ráb M.: *On the Boundedness of Solutions of Second-order Differential Equations*, (Lecture held on Winter-School of Differential Equations in 1974).

[3] Wiswanatham B.: *A generalization of Bellman's lemma*, Proc. Amer. Math. Soc. 14 (1963), 15—18.

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