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AN OSCILLATION CRITERION FOR A CANONICAL FORM OF THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction

A linear differential equation of the third order of the form

$$(R) \qquad y''' + p(t)\,y'' + q(t)\,y' + r(t)\,y = 0,$$

where $p(t)$, $q'(t)$, $r(t)$ are continuous on $[a, \infty)$ was studied by several authors, namely Hanan [3], Lazer [4], Ráb, Singh [9], [10], Švec and Zlámal [12] in the case $p \equiv 0$. This equation (R) in the form

$$(S) \qquad y''' + p(t)\,y'' + 2A(t)\,y' + (A'(t) + b(t))\,y = 0,$$

where $2A = q$, $A' + b = r$ for $A \leqq 0$, $p \equiv 0$ was investigated by Greguš [1], [2] and Moravský [5]. Some new results were obtained by Regenda [8].

A new canonical form was derived by F. Neuman [6], [7] for a linear differential equation of the n-th order of the form

$$(T) \qquad y^{(n)} + a_1(t)\,y^{(n-1)} + \ldots + a_n(t)\,y = 0,$$

$a_i \in C^0(I)$ for $i = 1, 2, \ldots, n$; I is an open interval (bounded or unbounded). Here $C^n(I)$ denotes for $n \geqq 0$ the class of all continuous functions on I having here continuous derivative up and including the n-th order. This canonical form is global, i.e. each linear differential equation of the n-th order can be transformed into the form on the whole interval of definition, on the contrary to local canonical forms due to Laguerre – Forsyth characterized by $a_1 \equiv 0$ and $a_2 \equiv 0$.

This general canonical form depends on an interval of definition and $n - 2$ positive functions $\alpha_i \in C^{n-i}$, $i = 1, 2, \ldots, n - 1$.

For $n = 3$ the canonical form (see [6]) is

$$(U) \qquad u''' - \alpha'(x)/\alpha(x)\,u'' + (1 + \alpha^2(x))\,u' - \alpha'(x)/\alpha(x)\,u = 0,$$

$\alpha \in C^1(J)$ and $\alpha(x) > 0$ for $x \in J$.

In this paper oscillation properties of solutions of the linear differential equation of the form (U) for $J = [a, \infty)$ are studied and some generalizations of the results in [11] are obtained.

2. Basic relations

It can be verified through differentiation that for (S) on J the following identities are satisfied. If we denote $L(t, a) = \exp\{\int_a^t p(s)\,\mathrm{d}s\} > 0$, $F[y(t), a] = (y'^2(t) - 2y(t)\,y''(t) - 2A(t)\,y^2(t))\,L(t, a)$ and $G[y(t), a] = (y''(t) + A(t)\,y(t))\,L(t, a)$, then

$$(F) \qquad F[y, a] = F[y(a), a] + \int_a^t (py'^2 + 2(b - Ap)\,y^2)\,L(s, a)\,\mathrm{d}s,$$

$$(G) \qquad G[y, a] = G[y(a), a] - \int_a^t (Ay' + (b - Ap)\,y)\,L(s, a)\,\mathrm{d}s,$$

$$(H) \qquad y''(t)\,L(t, a) = y''(a) - \int_a^t (2Ay' + (A' + b)\,y)\,L(s, a)\,\mathrm{d}s.$$

In the proofs of some theorems in the papers [4], [9] there is used the procedure given in the form of the following.

Lemma 1. *Let $u_i(t) \in C^r[a, \infty)$ be functions, c_{in} constants, $i = 1, 2, \ldots, s$. Let the sequence $\{y_n\}$ be defined by the relations*

$$y_n = \sum_{i=1}^{s} c_{in} u_i, \qquad \sum_{i=1}^{s} c_{in}^2 = 1.$$

Then there exists a subsequence $\{n_j\}$ such that $c_{in_j} \to c_i$ and $\{y_{n_j}\}$ converges on every finite subinterval of $[a, \infty)$ uniformly to the function

$$y = \sum_{i=1}^{s} c_i u_i, \qquad \sum_{i=1}^{s} c_i^2 = 1$$

as $n_j \to \infty$ such that

$$y^{(z)} = \sum_{i=1}^{s} c_i u_i^{(z)}, \qquad \sum_{i=1}^{s} c_i^2 = 1, \qquad z = 0, 1, 2, \ldots, m \leqq r.$$

The next two results were proved in [8].

Lemma 2. (Lemma 2.1.) *If $p(t) \geqq 0$, $A(t) \geqq 0$, $A'(t) + b(t) \geqq 0$, and $b(t) - A(t)\,p(t) \geqq 0$ and not identically zero on any subinterval of $[a, \infty)$, $\int_a^\infty p(t)\,\mathrm{d}t < \infty$ and $y(t) \not\equiv 0$ is a nonoscillatory solution of (S), which is eventually nonnegative with*

$$0 \leqq F[y(c), c] = y'^2(c) - 2y(c)\,y''(c) - 2A(c)\,y^2(c)$$

($c \in [a, \infty)$ arbitrary), then there exists a number $d \geqq c$ such that

$$y(t) > 0, y'(t) > 0, y''(t) \geqq 0 \quad \text{and} \quad y'''(t) \leqq 0 \quad \text{for } t \geqq d.$$

Lemma 3. (Theorem 3.3.) *If $p(t) \geqq 0$ and $b(t) - A(t)p(t) \geqq 0$, and not identically zero in any interval, then (S) has a nonoscillatory solution.*

3. Further relations

Theorem 1. *Let $p(t) \geqq 0$, $A(t) \geqq m > 0$, $A'(t) + b(t) \geqq 0$ and $b(t) - A(t)p(t) \geqq \geqq 0$ be not identically zero on any subinterval of $[a, \infty)$. If $\int_a^\infty p(t)\,dt < \infty$ then any solution which vanishes at some point is oscillatory.*

Proof: Let c be a zero of the nontrivial nonoscillatory solution $y(t)$. Then $F[y(c), c] = y'^2(c) > 0$ and from Lemma 2 there exists a number $d \geqq c$ such that $y(t) > 0$, $y'(t) > 0$, $y''(t) \geqq 0$ and $y'''(t) \leqq 0$ on $[d, \infty)$. Let $t_0 \in [d, \infty)$ be a zero of the function $y''(t)$. From (H) we have

$$y''(t)\,L(t, t_0) = -\int_{t_0}^{t} (2Ay' + (A' + b)\,y)\,L(s, t_0)\,ds < 0,$$

thus $y''(t) < 0$ on (t_0, ∞) The function y'' must be positive for all $t \geqq d$. Then $\lim y(t) = \infty$ as $t \to \infty$ and $G[y(t), d] = (y'' + Ay)\,L(t, d) \geqq my$ is the unbounded function according to (G). But we have also

$$G'[y, d] = -(Ay' + (b - Ap)\,y)\,L(t, d) < 0$$

on $[d, \infty)$ which is a contradiction, and the solution $y(t)$ is oscillatory.

Lemma 4. *Let $p(t) \leqq 0$, $A(t) \geqq 0$, $A'(t) + b(t) \leqq 0$ not identically zero on any subinterval of $[a, \infty)$ and $y(t) \not\equiv 0$ be nonoscillatory solution of the equation (S) satisfying the inequality $F[y, a] > 0$. Then $c \in [a, \infty)$ exists such that for all $t \geqq c$ there holds $y(t)\,y'(t) > 0$.*

Proof: Let $y(t)$ be a nontrivial nonoscillatory solution of the equation (S). Let t_0 be its last zero. If y is nonvanishing on $[a, \infty)$, let t_0 be arbitrary. We can suppose without loss of generality that $y > 0$ for all $t > t_0$.

We assert that the function $y'(t)$ has at most one zero on (t_0, ∞). Indeed, if $t_1 \in (t_0, \infty)$ is a zero of y', then

$$F[y(t_1), a] = (-2y(t_1)\,y''(t_1) - 2A(t_1)\,y^2(t_1)) \exp\{\int_a^{t_1} p(t)\,dt\} > 0$$

and hence $y''(t_1) < 0$. Consequently t_1 is the unique zero.

Let $c > t_1 > t_0$. Then $y(t)\,y'(t) \neq 0$ holds on $[c, \infty)$. Now we will show that

$y' > 0$. Suppose on the contrary that $y' < 0$ for $t \geqq c$. If $t_2 \in [c, \infty)$ is a zero of y'', then from (H) we have

$$y''(t)\,L(t, t_2) = -\int_{t_2}^{t} (2Ay' + (A' + b)\,y)\,L(s, t_2)\,\mathrm{d}s > 0,$$

and on $[d, \infty)$, $d > t_2 \geqq c$, it must be $y'' \neq 0$. Let $y'' < 0$. Then y' is a negative and decreasing function and $y(t) \leqq y'(d)\,(t - d) + y(d)$ holds on $[d, \infty)$ which is a contradiction with $y > 0$. If $y'' > 0$ for all $t > d$, we have from (S)

$$y'''(t) = -p(t)\,y''(t) - 2A(t)\,y'(t) - (A'(t) + b(t))\,y(t) > 0,$$

thus $y''(t) \geqq y''(d)$ and by integration of this inequality from d to t we obtain $y'(t) = y''(d)\,(t - d) + y'(d)$ which is a contradiction for $y' < 0$ on $[d, \infty)$.

Thus we proved that $y(t)\,y'(t) > 0$ on $[c, \infty)$.

Lemma 5. *Let $A(t) \geqq 0$, $p(t) \leqq 0$, $A'(t) + b(t) \leqq 0$ and $b(t) - A(t)\,p(t) \leqq 0$. If $\int_a^\infty (A(t)\,p(t) - b(t))\,L(t, a)\,\mathrm{d}t = \infty$ and $y(t)$ is a nontrivial solution of the equation (S) satisfying the inequality $F[y, a] > 0$, then $y(t)$ is an oscillatory solution.*

Proof: Let $y \not\equiv 0$ be a nonoscillatory solution of the equation (S) and $F[y, a] > 0$ on $[a, \infty)$. By Lemma 4 there exists $c \in [a, \infty)$ such that $y(t)\,y'(t) > 0$ on $[c, \infty)$. We can suppose without loss of generality that $y > 0$. Then for abritrary $d \geqq c$ there exists a positive constant K such that we can put $y(t) \geqq K$ on $[d, \infty)$. From (F) we have

$$F[y(c), c] \geqq K^2 \int_c^t (A(s)\,p(s) - b(s))\,L(s, c)\,\mathrm{d}s \to \infty \qquad \text{as } t \to \infty$$

which is a contradiction and $y(t)$ cannot be nonoscillatory.

Lemma 6. *Let $A(t) \geqq 0$, $p(t) \leqq 0$, $A'(t) + b(t) \leqq 0$, $b(t) - A(t)\,p(t) \leqq 0$. If $\int_a^\infty (Ap - b)\,L(t, a)\,\mathrm{d}t = \infty$ then a nontrivial solution $y(t)$ of the equation (S) is nonoscillatory iff $c \in [a, \infty)$ exists such that $F[y(c), c] \leqq 0$.*

Proof: The necessity follows from Lemma 5.

Under the given supposition the function $F[y, c]$ is strictly decreasing, thus $F[y, c] < 0$ on $[d, \infty)$, $d \geqq c$.

Let $y(t_0) = 0$ for $t_0 \in [d, \infty)$. Then $F[y(t_0), c] = y'^2(t_0)\,L(t_0, c) \geqq 0$, which is a contradiction. The solution y must be nonoscillatory. Thus the assertion is proved.

Theorem 2. *Let $A(t) \geqq 0$, $p(t) \leqq 0$, $A'(t) + b(t) \leqq 0$, $b(t) - A(t)\,p(t) \leqq 0$. If $\int_a^\infty (Ap - b)\,L(t, a)\,\mathrm{d}t = \infty$ then the equation (S) has two linearly independent oscillatory solutions.*

Proof: Let the solutions $y_1(t)$, $y_2(t)$, $y_3(t)$ of the equation (S) be determined by the initial conditions

$$y_i^{(j)}(a) = \delta_{i,j+1} = \begin{cases} 0 & i \neq j+1 \\ 1 & i = j+1 \end{cases} \quad \begin{matrix} i = 1, 2, 3, \\ j = 0, 1, 2. \end{matrix}$$

Let $n > a$ be positive integers, b_{1n}, b_{3n} and c_{2n}, c_{3n} constants such that the solutions v_n and w_n of the equation (S) defined by

$$v_n(t) = b_{1n}y_1(t) + b_{3n}y_3(t), \qquad b_{1n}^2 + b_{3n}^2 = 1,$$
$$w_n(t) = c_{2n}y_2(t) + c_{3n}y_3(t), \qquad c_{2n}^2 + c_{3n}^2 = 1,$$

satisfy $v_n(n) = w_n(n) = 0$. Then $F[v_n(n), a] \geqq 0$, $F[w_n(n), a] \geqq 0$ and since $F[y, a]$ is a decreasing function, there holds

$$(1) \qquad F[v_n(t), a] > 0, F[w_n(t), a] > 0 \quad \text{on } [a, n).$$

By Lemma 1 the sequence $\{n_k\}$ exists such that $\{v_{n_k}(t)\}$ converges for $n_k \to \infty$ on every finite subinterval from $[a, \infty)$ uniformly to a function $v(t)$ and there holds $v^{(s)} = b_1u_1^{(s)} + b_3u_3^{(s)}$, $s = 0, 1, 2$; $b_1^2 + b_3^2 = 1$. From (1) it follows that $F[v, a] \geqq 0$ on $[a, \infty)$. As $F[y, a]$ is a decreasing function, there must be $F[v, a] > 0$ on $[a, \infty)$. Otherwise $F[v, a]$ obtains negative values which is a contradiction. We can prove similarly that $F[w, a] > 0$ and $c_2^2 + c_3^2 = 1$ on $[a, \infty)$.

Solutions $v(t)$, $w(t)$ are oscillatory by Lemma 5. Let the solutions v, w be dependent. As $b_1^2 + b_3^2 = c_2^2 + c_3^2 = 1$ is satisfied, there holds $v(t) = Ky_3(t)$ for some $K \neq 0$. Then $v(t)$ is nonoscillatory by Lemma 6, because $F[y_3(a), a] = 0$ by definition of y_3, which is a contradiction. We have proved that $v(t)$, $w(t)$ are linearly independent solutions.

This completes the proof.

4. Applications to the canonical form

Now we consider the equation (U) on $J = [a, \infty)$ where $A(t) = (1 + \alpha^2(t))/2 > 1/2$ and $p(t) = A'(t) + b(t) = -\alpha'(t)/\alpha(t)$. Then $b(t) = 2A(t)p(t)$.

Lemma 7. *If $\alpha'(t) \leqq 0$ and not identically zero on any subinterval of $[a, \infty)$, then the equation (U) has a nonoscillatory solution.*

Proof: If $\alpha'(t) \leqq 0$, then we obtain $p \geqq 0$ and $b - Ap = Ap \geqq 0$, and not identically zero on any subinterval of $[a, \infty)$. The equation (U) has a nonoscillatory solution by Lemma 3.

We shall prove similarly

Theorem 3. *Let $\alpha'(t) \leqq 0$ be not identically zero on any subinterval of $[a, \infty)$. If $\lim \alpha(t) = \text{const} > 0$ as $t \to \infty$, then any solution which vanishes at some point is oscillatory.*

Proof: It is $\alpha(t) > 0$, $\alpha'(t) \leqq 0$ and $\lim \alpha(t) = \text{const} \geqq 0$ there exists as $t \to \infty$.

Then we have $\int_a^\infty p(t)\,dt = \lim \ln(\alpha(a)/\alpha(t)) < \infty$ if $\lim \alpha(t) > 0$, as $t \to \infty$. The assertion follows from Theorem 1.

Theorem 4. *If $\alpha'(t) \geqq 0$ and $\lim \alpha(t) = \infty$ as $t \to \infty$, then*
(i) *a nontrivial solution $y(t)$ of the equation (U) is nonoscillatory iff $c \in [a, \infty)$ exists such that $F[y(c), c] \leqq 0$,*
(ii) *the equation (U) has two linearly independent oscillatory solutions.*

Proof: It must be $A'(t) + b(t) = p(t) \leqq 0$ and $b(t) - A(t)\,p(t) = A(t)\,p(t) \leqq 0$ for $\alpha'(t) \geqq 0$, $A(t) = (1 + \alpha^2(t))/2 > 1/2$. Then we obtain (i) from Lemma 6, (ii) using Theorem 2 with $\int_a^\infty (Ap - b)\,L(t, a)\,dt = \int_a^\infty A(-p)\,L(t, a)\,dt = (\alpha(a)/2) \times$ $\times \int_a^\infty (1 + \alpha^2)\,\alpha'/\alpha^2\,dt > (\alpha(a)/2) \lim (\alpha(t) - \alpha(a)) = \infty$ as $t \to \infty$ and this completes the proof.

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